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Uniform synchronization in multi-agent systems with switching topologies

Nima Monshizadeh^{*,†}, Harry L. Trentelman and M. Kanat Camlibel

*Johann Bernoulli Institute for Mathematics and Computer Science, Faculty of Mathematics and Natural Sciences,
University of Groningen, Groningen, The Netherlands*

SUMMARY

This paper deals with uniform synchronization analysis of multi-agent systems with switching topologies. The agents are assumed to have general, yet identical, linear dynamics. The underlying communication topology may switch arbitrarily within a finite set of admissible topologies. We establish conditions under which the network is uniformly synchronized meaning that synchronization is valid under all possible switching scenarios. The primary conditions established are in terms of a pair of Lyapunov strict inequalities. Following those conditions, small gain and passivity types of conditions are proposed under which uniform synchronization is guaranteed. The proposed results are also extended to the case of observer-based protocols. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The distributed control of multi-agent systems has gained a lot of attention during the last decade. In particular, the consensus problem has been widely investigated for networks of agents. Consensus roughly means that the agents of a network reach an agreement on the state components' values. The pioneering work in this direction has been carried out in [1–3], and [4] for the case where the agents have simple dynamics, like single or double integrators. An excellent review can be found in [5]. Results on consensus for the case where the agents have general, yet identical, linear dynamics with time-independent communication topology are reported in some recent papers (see, e.g., [6–8]).

Despite the extensive amount of research available in the context of consensus/synchronization of multi-agent systems, relatively few works have considered network of agents with general linear dynamics together with a time-dependent communication topology. This situation is considered in [9], [10], and [11] where consensus protocols are proposed for possibly time-varying communication structures. However, in these papers, the agents are *not* allowed to have exponentially unstable dynamics. The aforementioned results are further generalized in [12] to heterogeneous networks, and synchronizability of the network is characterized by an internal model principle. In [13], synchronization analysis is carried out by imposing simultaneous triangularizability condition on the Laplacian matrices of the underlying communication (directed) graphs. Assuming nonzero dwell time for the switching among admissible communication graphs, the consensus problem for multi-agent systems with general linear dynamics is studied in [14]. Sufficient conditions for achieving synchronization via fast switching are established in [15].

^{*}Correspondence to: Nima Monshizadeh, Johann Bernoulli Institute for Mathematics and Computer Science, Faculty of Mathematics and Natural Sciences, University of Groningen, Groningen, The Netherlands.

[†]E-mail: n.monshizadeh@rug.nl

In this paper, we consider a network of agents with general, yet identical, linear dynamics, and the communication topology may switch within a finite set of admissible topologies. No *a priori* relationship is assumed for the admissible communication graphs, and the agents are allowed to have exponentially unstable dynamics. We are interested in the case where the switching signal is not constrained, and no information regarding the switching rate or switching time instances is available. In particular, we seek for a synchronization property that is valid under all possible switching scenarios. We will refer to this property as *uniform synchronization* that will be formally defined later in the sequel.

Given the agents' dynamics, a coupling rule, and a set of admissible topologies, we derive conditions under which uniform synchronization is guaranteed. In particular, we show that uniform synchronization is achieved if a certain pair of linear matrix inequalities (LMI) admits a positive definite solution. The LMI obtained are simple and depend only on the agents' dynamics and the (nontrivial) extremal Laplacian eigenvalues. It will be observed that necessary and sufficient conditions for solvability of the proposed LMI are available for the special case of single-input single-output (SISO) dynamics. In addition, for the multi-input multi-output case, conditions in terms of bounded realness and also passivity of certain subsystems of the networks are established under which the proposed LMI admit a positive definite solution, thus implying uniform synchronization of the network.

In case the overall information on the relative states of the neighboring agents is not available, observer-based protocols achieving synchronization are proposed in the literature (see, e.g., [6, 9, 11], and [16]), which exploit only the relative output information of adjacent agents. We will also incorporate the case of observer-based protocols and show how the proposed conditions guaranteeing uniform synchronization carry over to this case.

The paper is organized as follows. First, in Section 2, we introduce some notation and review some basic definitions. Notion of uniform synchronization together with corresponding necessary conditions and a motivating example is provided in Section 3. Sufficient conditions under which the network is uniformly synchronized are established in Section 4. The proposed results are also extended to the case of observer-based protocols. Finally, Section 5 is dedicated to conclusions.

2. PRELIMINARIES

For $i = 1, 2, \dots, N$, let $G_i = (V, E_i)$ be a simple undirected (unweighted) graph with vertex set $V = \{1, 2, \dots, p\}$ and edge set $E_i \subseteq V \times V$ with the properties that $(v, v) \notin E_i$ for any $v \in V$ and $(v, w) \in E_i$ if and only if $(w, v) \in E_i$ for all $v, w \in V$. Corresponding to this set of graphs, a diffusively coupled multi-agent system is given by

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t) \quad (1a)$$

$$y_j(t) = Cx_j(t) \quad (1b)$$

together with the diffusive coupling rule

$$u_j(t) = \sum_{(i,j) \in E_{\sigma(t)}} (y_i(t) - y_j(t)), \quad (1c)$$

where $j \in V$, $x_j \in \mathbb{R}^n$ is the state of agent j , $u_j \in \mathbb{R}^m$ is the diffusive coupling term, and $\sigma \in \mathcal{S}$ with $\mathcal{S}$ denoting the set of all right-continuous piecewise constant switching signals from $\mathbb{R}^+$ to $\{1, 2, \dots, N\}$.

Throughout this paper, it is assumed that (A, B) is stabilizable. This is a necessary condition for synchronization of the multi-agent system (1), even in the special case of a time-independent topology, that is, $N = 1$. We introduce the following nomenclature for later use. For each $i = 1, 2, \dots, N$, let L_i denote the Laplacian matrix corresponding to the graph $G_i = (V, E_i)$. The eigenvalues of L_i are denoted by

$$0 = \lambda_1^i \leq \lambda_2^i \leq \dots \leq \lambda_p^i \quad (2)$$

for each i . We define

$$\underline{\lambda} = \min \{ \lambda_j^i \mid i \in \{1, 2, \dots, N\}, j \in \{2, 3, \dots, p\} \} \quad (3a)$$

and

$$\bar{\lambda} = \max \{ \lambda_j^i \mid i \in \{1, 2, \dots, N\}, j \in \{2, 3, \dots, p\} \}. \quad (3b)$$

Note that $\lambda_1^i = 0$ for $i = 1, 2, \dots, N$ are excluded in the aforementioned definitions.

The multi-agent system (1) can be written in a compact form as

$$\dot{x}(t) = \mathcal{A}_{\sigma(t)} x(t), \quad (4)$$

where $x = \text{col}(x_1, \dots, x_p)$ and $\mathcal{A}_{\sigma(t)} = I_p \otimes A - L_{\sigma(t)} \otimes BC$, where ‘ $\otimes$ ’ denotes the Kronecker product. The following basic properties of the Kronecker product are frequently used in the sequel: $A \otimes (B + C) = A \otimes B + A \otimes C$, $(A \otimes B)^T = A^T \otimes B^T$, and $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

For a given switching signal $\sigma \in \mathcal{S}$, we call an absolutely continuous function x a *solution of (4) with respect to σ* if (4) is satisfied for almost all $t \geq 0$. We call an absolutely continuous function x a *solution of (4)* if there exists a switching signal $\sigma \in \mathcal{S}$ such that (4) is satisfied for almost all $t \geq 0$. We call the switched linear system (4) globally uniformly exponentially stable (GUES) if there exist positive constants α and β such that for any solution x of (4), we have $\|x(t)\| \leq \beta e^{-\alpha t} \|x(0)\|$ for all $t \geq 0$.

Remark 2.1

A sufficient condition for GUES is the existence of a common quadratic Lyapunov function (CQLF), that is, the existence of a positive definite matrix X such that $\mathcal{A}_i^T X + X \mathcal{A}_i < 0$ for all $i = 1, 2, \dots, N$ (see, e.g., [17]).

3. NETWORK SYNCHRONIZATION

In this section, we discuss the synchronization problem for the network (4). A synchronized network has the property that the state trajectories of the coupled agents converge to a common trajectory. In this paper, we are interested in ‘uniform synchronization’. By that we mean no information (except $\sigma \in \mathcal{S}$) is available on the switching signal, and synchronization of the network does not depend on the rate or the choice of the switching signal. In other words, uniform synchronization is a kind of synchronization that is uniform over all switching signals $\sigma \in \mathcal{S}$. More precisely, we have the following definition.

Definition 3.1

For a given switching signal $\sigma \in \mathcal{S}$, we call the network (4) *synchronized with respect to σ* if

$$\lim_{t \rightarrow \infty} (x_j(t) - x_k(t)) = 0 \quad (5)$$

holds for every solution of (4) with respect to σ and for each $j, k = 1, 2, \dots, p$. Then, the network (4) is called *uniformly synchronized* if it is synchronized with respect to σ for all switching signal $\sigma \in \mathcal{S}$.

Note that in the context of consensus and synchronization, typically, agents do not have asymptotically stable dynamics. Hence, following the notion of uniform synchronization, our treatment in this paper particularly aims to address the following problem.

Problem 3.2

Assume that the matrix A in (1) is not Hurwitz (with possible eigenvalues in the open right-half plane). Under what conditions the network (4) is uniformly synchronized?

First, we provide necessary conditions for uniform synchronization of (4). Clearly, synchronization with respect to constant switching signals is necessary for uniform synchronization, and thus we immediately obtain the following lemma.

Lemma 3.3

The network (4) is uniformly synchronized only if $A - \lambda_j^i BC$ is Hurwitz for each $i = 1, 2, \dots, N$ and $j = 2, 3, \dots, p$, where λ_j^i s are given by (2).

Proof

Obviously, $\sigma(t) = i$ belongs to the set $\mathcal{S}$ for each $i = 1, 2, \dots, N$. Then, the result readily follows from [18, Lem. 3.1]. $\square$

Moreover, to achieve uniform synchronization for network (4), the corresponding communication graphs need to be connected. This is stated formally in the following lemma.

Lemma 3.4

Assume that the matrix A in (1) is not Hurwitz. Then, the network (4) is uniformly synchronized only if G_i is connected for each $i = 1, 2, \dots, N$.

Proof

Suppose that the network (4) is uniformly synchronized. Then, by Lemma 3.3, we obtain that $A - \underline{\lambda} BC$ is Hurwitz, where $\underline{\lambda}$ is given by (3a). As the matrix A is not Hurwitz, we obtain that $\underline{\lambda} \neq 0$. This indeed means that L_i has a distinct eigenvalue at zero for each $i = 1, 2, \dots, N$ and thus G_i is connected for each i . $\square$

Before preceding any further, we briefly compare the problem at hand with those already studied in the literature. First, note that the most common approach to deal with synchronization analysis of network (4) is to assume that (i) the agents do not possess exponentially unstable dynamics, that is, the eigenvalues of A are all in the closed left-half plane and (ii) the admissible communication graphs are jointly (or uniformly) connected (see, e.g., [9], [10], [11], and [12, Sec. 3.2]). Then, the underlying idea in the aforementioned papers is that network trajectories converge exponentially fast in the time intervals for which the underlying communication graph is connected and on top of that, they do not diverge in other time intervals due to the absence of exponentially unstable modes. This indeed results in synchronization with respect to a subclass of $\mathcal{S}$ that is identified by the joint (uniform) connectivity assumption. However, here in this paper, we allow exponentially unstable dynamics for the agents. In addition, as mentioned earlier, we are interested in a synchronization property that is uniform over all switching scenarios. Obviously, this requires different assumptions and necessary conditions as partially summarized in Lemmas 3.3 and 3.4 (see also [12, Sec. 3.4]). Next, we bring a motivating example showing that the conditions provided in Lemmas 3.3 and 3.4 are not suffices to conclude uniform synchronization of (4), even in the case where both of the aforementioned assumptions (i) and (ii) hold.

Example 3.5

Consider the agents' dynamics in (1) is given by

$$A = \begin{bmatrix} 0 & 0.6619 & -0.2194 & 0.4494 & 0.2791 & 0.4553 & 0 \\ 0 & -0.1762 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4775 & 0.7807 & 0.4849 & 0.7909 & 0 \\ 0 & 0 & 0 & -0.6957 & 0.3125 & 0.5097 & 0 \\ 0 & 0 & 0 & 0 & -1.52 & 0.3166 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.5 & 1.936 \\ 0 & 0 & 0 & 0 & 0 & -1.936 & -3.5 \end{bmatrix}, \quad (6)$$

$$B = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 9.155]^T, \quad (7)$$

$$C = [0.2018 \ 0.3328 \ -0.1103 \ 0.2259 \ 0.1403 \ 0.2289 \ 0].$$

Suppose that $N = 2$ in (1) and the corresponding communication graphs G_1 and G_2 are as depicted in Figures 1 and 2, respectively.

Figure 1. Communication graph G_1 . Figure 2. Communication graph G_2 .

Figure 3. The value of $(x_2)_1 - (x_3)_1$ in Example 3.5 depicted over time for the switching signal σ given by (8) with $T = 1$ and initial states $x_1(0) = [2 \ 2 \ 2 \ 2 \ 2 \ 2 \ 2]^T$, $x_2(0) = [-1 \ -1 \ -1 \ -1 \ -1 \ -1 \ -1]^T$, and $x_3(0) = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T$.

Let L_1 and L_2 denote the Laplacian matrices of G_1 and G_2 , respectively. The eigenvalues of L_1 are 0, 1, and 3, and the eigenvalues of L_2 are 0, 3, and 3. It is easy to verify that the matrices $A - BC$ and $A - 3BC$ are both Hurwitz. Therefore, based on [18, Lem. 3.1], synchronization is achieved with respect to constant switching signals, that is, the network (4) is synchronized for $\sigma(t) = 1$ as well as $\sigma(t) = 2$. Now, suppose that the switching signal is chosen as the $2T$ -period signal:

$$\sigma(t) = \begin{cases} 1 & \lfloor \frac{t}{T} \rfloor \text{ is odd} \\ 2 & \lfloor \frac{t}{T} \rfloor \text{ is even} \end{cases}, \quad (8)$$

where $\lfloor x \rfloor$ denotes the biggest integer k such that $k \leq x$. Then, one can verify that the network (4) may not be synchronized with respect to σ for sufficiently small T . In Figure 3, the value of $(x_2)_1 - (x_3)_1$ is depicted over time for $T = 1$ and a given initial state x_0 . Note that $(x_i)_j$ denotes the j^{th} component of x_i . As can be seen from this figure, the value of $(x_2)_1 - (x_3)_1$ does not converge

to zero; thus, the network is not synchronized with respect to σ given by (8). Obviously, this means that the network is not uniformly synchronized.

Note that the communication graphs G_1 and G_2 are both connected. Also note that the eigenvalues of A are all in the closed left-half plane. In addition, the network is synchronized for constant switching signals. However, as observed above the network is not synchronized for the particular choice of the switching signal σ given by (8) with $T = 1$; thus, the network is not uniformly synchronized in this case. This sheds light to the fact that the analysis of uniform synchronization requires additional effort and cannot be deduced solely based on the connectivity of the underlying communication graphs or synchronization of the individual modes.

4. SUFFICIENT CONDITIONS FOR UNIFORM SYNCHRONIZATION

In this section, we derive sufficient conditions guaranteeing uniform synchronization of network (4).

Observe that network synchronization requires that the differences of the agents' state components converge to zero, which can be viewed as *output stability* ([19, sec. 4.4.] or [20, exc. 4.10]). Let $z_j = x_j - x_{j+1}$ for each $j = 1, 2, \dots, p-1$. Then, we have the compact form

$$z(t) = (Q \otimes I_n)x(t), \quad (9)$$

where $z = \text{col}(z_1, z_2, \dots, z_{p-1})$ and Q is the $(p-1) \times p$ matrix given by

$$Q = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}. \quad (10)$$

Clearly, the network (4) is uniformly synchronized if and only if $\lim_{t \rightarrow \infty} z(t) = 0$ for every solution x of (4). Consequently, the network (4) is uniformly synchronized if and only if the system (4) with output variable z is output stable. We use the structure of the network and the properties of the Laplacian matrix to convert the output stability problem into an internal stability problem. This is discussed next.

Consider the network (4) together with the output given by (9). We will apply a state space transformation such that the state x is transformed to $\tilde{z} = \text{col}(z, x_p)$. In order to do this, define $\tilde{Q}$ to be the $p \times p$ matrix given by

$$\tilde{Q} = \begin{pmatrix} Q \\ e_p^\top \end{pmatrix} \quad (11)$$

with e_p denoting the p^{th} standard basis vector for $\mathbb{R}^p$, that is, $e_p = (0, 0, \dots, 1)^\top$, and where Q is given by (10). Note that $\tilde{Q}$ is nonsingular. Then, on the one hand, we indeed have $\tilde{z} = \text{col}(z, x_p) = (\tilde{Q} \otimes I_n)x(t)$. On the other hand, the transformed dynamics of (4) is given by

$$\dot{\tilde{z}}(t) = (I_p \otimes A - \tilde{Q}L_{\sigma(t)}\tilde{Q}^{-1} \otimes BC)\tilde{z}(t).$$

It is easy to observe that $\tilde{Q}^{-1}$ is an upper triangular matrix where the entries on and above its diagonal are all equal to 1. As the row sums of L_i is zero for each $i = 1, 2, \dots, N$, the matrix $\tilde{Q}L_i\tilde{Q}^{-1}$ can be partitioned as

$$\tilde{Q}L_i\tilde{Q}^{-1} = \begin{pmatrix} \tilde{L}_i & 0 \\ * & 0 \end{pmatrix}, \quad (12)$$

where $\tilde{L}_i$ is a $(p-1) \times (p-1)$ matrix for each i and $*$ denotes the values that are not of interest to us. It follows from the structure of (12) that the dynamics of the state components $z_1, z_2, \dots, z_{p-1}$ are not affected by that of x_p and thus we obtain

$$\dot{z}(t) = (I_{p-1} \otimes A - \tilde{L}_{\sigma(t)} \otimes BC)z(t). \quad (13)$$

Therefore, (uniform) synchronization analysis of (4) boils down to stability analysis of the switched linear system (13). Consequently, synchronization analysis of (4) can be carried out by dwell time, average dwell time, or Lyapunov-based arguments available in the context of switched linear systems (SLS). In this paper, however, we will investigate the existence of a CQLF as we are interested in uniform synchronization. Observe that existence of a CQLF for the SLS (13) implies global uniform exponential stability of (13) that results in uniform synchronization of the network (4).

From (13), it readily follows that (4) is synchronized if there exists a CQLF for the state matrices $I_{p-1} \otimes A - \tilde{L}_i \otimes BC$ with $i = 1, 2, \dots, N$. Similar to the case of networks with time-independent topologies (see, e.g., [6, 7], and [8]), it would be desirable to relate synchronization of (4) to the dynamics of the agents and the Laplacian eigenvalues, more specifically to the matrices $A - \lambda_j^i BC$ where $i = 1, 2, \dots, N$ and $j = 2, 3, \dots, p$. This brings us to the following theorem.

Theorem 4.1

Let $\underline{\lambda}$ and $\bar{\lambda}$ be defined as in (3a). Then, the network (4) is uniformly synchronized if there exists a positive definite matrix X satisfying both of the following LMI:

$$(A - \underline{\lambda}BC)^\top X + X(A - \underline{\lambda}BC) < 0 \quad \text{and} \quad (14a)$$

$$(A - \bar{\lambda}BC)^\top X + X(A - \bar{\lambda}BC) < 0. \quad (14b)$$

Before proving the theorem, we need the following additional result.

Lemma 4.2

Let Q , $\tilde{Q}$, L_i , and $\tilde{L}_i$ be defined as before, for $i = 1, 2, \dots, N$. For each i , let the matrix L_i admit the spectral decomposition $L_i = U_i \Lambda_i U_i^\top$ where U_i is an orthogonal matrix with its first column being the normalized vector of ones and $\Lambda_i = (0 = \lambda_1^i, \lambda_2^i, \dots, \lambda_p^i)$. Also, let the product QU_i be partitioned as

$$QU_i = \begin{pmatrix} 0_{(p-1) \times 1} & \tilde{U}_i \end{pmatrix} \quad (15)$$

for each i . Then, for each i , the matrix $\tilde{L}_i$ admits a spectral decomposition $\tilde{L}_i = \tilde{U}_i \tilde{\Lambda}_i \tilde{U}_i^{-1}$ where $\tilde{\Lambda}_i$ is the diagonal matrix obtained by deleting the first row and column of Λ_i . Moreover, $\tilde{U}_i \tilde{U}_i^\top = Q Q^\top$ for each $i = 1, 2, \dots, N$.

Proof

Because the first column of U_i is the normalized vector of ones, we have

$$\tilde{Q}U_i = \begin{pmatrix} 0_{(p-1) \times 1} & \tilde{U}_i \\ \frac{1}{\sqrt{p}} & * \end{pmatrix} \quad (16)$$

for each $i = 1, 2, \dots, N$. Therefore, we obtain

$$\tilde{Q}L_i \tilde{Q}^{-1} = \tilde{Q}U_i \Lambda_i U_i^\top \tilde{Q}^{-1} = \begin{pmatrix} 0 & \tilde{U}_i \\ \frac{1}{\sqrt{p}} & * \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\Lambda}_i \end{pmatrix} \begin{pmatrix} 0 & \tilde{U}_i \\ \frac{1}{\sqrt{p}} & * \end{pmatrix}^{-1},$$

where Λ_i is partitioned as $\text{blockdiag}(0, \tilde{\Lambda}_i)$. Hence, we have

$$\tilde{Q}L_i \tilde{Q}^{-1} = \begin{pmatrix} \tilde{U}_i \tilde{\Lambda}_i \tilde{U}_i^{-1} & 0 \\ * & 0 \end{pmatrix} \quad (17)$$

for each i . Comparing the right hand side of (17) to that of (12), we obtain $\tilde{L}_i = \tilde{U}_i \tilde{\Lambda}_i \tilde{U}_i^{-1}$ that corresponds to a spectral decomposition of $\tilde{L}_i$. The rest follows from

$$Q Q^\top = Q U_i U_i^\top Q^\top = \begin{pmatrix} 0 & \tilde{U}_i \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{U}_i^\top \end{pmatrix} = \tilde{U}_i \tilde{U}_i^\top.$$

□

Now, based on Lemma 4.2 and our treatment preceding the theorem, we have the following proof for Theorem 4.1.

Proof of Theorem 4.1

Assume that there exists $X > 0$ such that (14) holds. Then, we have

$$(A - \lambda_j^i BC)^\top X + X (A - \lambda_j^i BC) < 0 \quad (18)$$

for all $j = 2, 3, \dots, p$ and $i = 1, 2, \dots, N$. Note that we have used the fact that $A - \lambda_j^i BC$ can be written as a convex combination of the matrices $A - \underline{\lambda} BC$ and $A - \bar{\lambda} BC$ for any $j = 2, 3, \dots, p$ and $i = 1, 2, \dots, N$. The LMI (18) can be rewritten as

$$(I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC)^\top (I_{p-1} \otimes X) + (I_{p-1} \otimes X) (I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC) < 0, \quad (19)$$

where $\tilde{\Lambda}_i = \text{diag}\{\lambda_2^i, \lambda_3^i, \dots, \lambda_p^i\}$ for each i . By Lemma 4.2, there exist diagonalizing transformations $\tilde{U}_i$ such that $\tilde{L}_i = \tilde{U}_i \tilde{\Lambda}_i \tilde{U}_i^{-1}$ for $i = 1, 2, \dots, N$. Clearly, the LMI (19) can be restated as

$$\begin{aligned} & (I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC)^\top (\tilde{U}_i^\top \otimes I_n) (\tilde{U}_i^{-\top} \otimes I_n) (I_{p-1} \otimes X) \\ & + (I_{p-1} \otimes X) (\tilde{U}_i^{-1} \otimes I_n) (\tilde{U}_i \otimes I_n) (I_{p-1} \otimes A - \tilde{\Lambda}_i \otimes BC) < 0 \end{aligned} \quad (20)$$

for $i = 1, 2, \dots, N$. Multiplying (20) from the left and right by $(\tilde{U}_i^{-\top} \otimes I_n)$ and $(\tilde{U}_i^{-1} \otimes I_n)$, respectively, we obtain

$$(I_{p-1} \otimes A - \tilde{L}_i \otimes BC)^\top (\tilde{U}_i^{-\top} \tilde{U}_i^{-1} \otimes X) + (\tilde{U}_i^{-\top} \tilde{U}_i^{-1} \otimes X) (I_{p-1} \otimes A - \tilde{L}_i \otimes BC) < 0. \quad (21)$$

Now, by Lemma 4.2, the product $\tilde{U}_i \tilde{U}_i^\top$ is independent of i for $i = 1, 2, \dots, N$ and is equal to $Q Q^\top$, where Q is given by (10). Therefore, the quadratic function W defined by $W(z) = z^\top ((Q Q^\top)^{-1} \otimes X) z$ serves as a CQLF for the switched linear system (13). Hence, (13) is GUES, and the network (4) is uniformly synchronized.

Remark 4.3

Note that conditions verifying the synchronization of networks with time-independent topologies are obtained in the literature mostly by applying a certain state space transformation to obtain an appropriate decomposition of the overall network. These conditions depend on the agent dynamics together with the nontrivial Laplacian eigenvalues (see, e.g., [6] and [11]). However, obviously the same technique cannot be adopted directly for the case of networks with switching topologies, unless the admissible topologies satisfy certain constraints like commutativity or simultaneous triangularizability of the corresponding Laplacian matrices ([13]). Note that the condition of simultaneous triangularizability severely restricts the set of admissible Laplacian matrices. To see this, observe that if the Laplacian matrices are simultaneously triangularizable, then the eigenvalues of $L_i + L_r$ are the sums and the eigenvalues of $L_i L_r$ are the products of the eigenvalues of L_i and L_r , in some order, for $i, r = 1, 2, \dots, N$. As such, baring in mind that we do not assume *a priori* relationship among the admissible Laplacian matrices, the existing results in the literature do not directly apply to the uniform switching scenario we treat in this paper.

Clearly, solvability of the LMI (14) is equivalent to the existence of a CQLF for the pair of matrices $A - \underline{\lambda} BC$ and $A - \bar{\lambda} BC$. In general, there are conditions to guarantee the existence of

a CQLF based on commutativity, simultaneous triangularizability, and solvable Lie algebras ([17] and [21]). However, these conservative conditions involve direct constraints on the matrices A and BC , namely, A and BC must commute or at least be simultaneously triangularizable. A subtle point is that these conditions are basically for an SLS with arbitrary given modes whereas in our case, certain structures and system properties are present in the network dynamics. Thus, we can take advantage of these available structures and properties to derive sensible conditions guaranteeing network synchronization.

First, consider the special case of SISO, that is, $m = 1$ in (1). In this case, the difference of the matrices $A - \bar{\lambda}BC$ and $A - \underline{\lambda}BC$ is equal to $(\bar{\lambda} - \underline{\lambda})BC$ that is of rank 1. Hence, the result of [22, Thm. 4] gives a necessary and sufficient condition for solvability of the LMI (14), in the SISO case. This result leads to the following corollary.

Corollary 4.4

Consider the multi-agent system (1) and assume that $m = 1$. Let $\underline{\lambda}$ and $\bar{\lambda}$ be defined as in (3a). Let $A_1 = A - \underline{\lambda}BC$ and $A_2 = A - \bar{\lambda}BC$. Then, the network (4) is uniformly synchronized if the matrices A_1 and A_2 are both Hurwitz, and $A_1 + \eta A_2^{-1}$ is nonsingular for all $\eta \in [0, +\infty)$.

Example 4.5

Consider the multi-agent system composed of a group of N harmonic oscillators described by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (0 \quad 1). \quad (22)$$

Suppose that the communication graph switches, arbitrarily, among a given set of graphs $G_i = (V, E_i)$ with $i = 1, 2, \dots, N$. By Lemma 3.3, uniform synchronization is achieved only if the matrices $A - \underline{\lambda}BC$ and $A - \bar{\lambda}BC$ are both Hurwitz, which results in the condition $\underline{\lambda} > 0$ in this case. This means that G_i must be connected for each $i = 1, 2, \dots, N$. Now, to ensure uniform synchronization by Corollary 4.4, it remains to check whether the matrix $(A - \underline{\lambda}BC) + \eta(A - \bar{\lambda}BC)^{-1}$ is nonsingular for all $\eta \in [0, +\infty)$. This matrix is computed as

$$\begin{pmatrix} \eta\bar{\lambda} & 1 - \eta \\ -1 + \eta & \underline{\lambda} \end{pmatrix},$$

which is nonsingular for any nonnegative η if $\underline{\lambda} > 0$. Consequently, the network of harmonic oscillators given by (22) is uniformly synchronized providing that the underlying communication graphs are all connected.

Next, we return to the multi-input multi-output case. As observed earlier, the network dynamics associated with $A - \lambda_j^i BC$ for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, p$ play a crucial role in synchronization of the network. We will now show that the solvability of the LMI (14) is guaranteed by imposing certain structural properties, namely, bounded realness and passivity, to this dynamics. For similar results in the fixed topology case, see [23].

First, small gain conditions are provided to guarantee the solvability of (14) and thus uniform synchronization of the network (4).

Theorem 4.6

Let $\underline{\lambda}$ and $\bar{\lambda}$ be defined as in (3a). The network (4) is uniformly synchronized if there exists at least one pair of indices (k, ℓ) with $k \in \{1, 2, \dots, N\}$ and $\ell \in \{2, 3, \dots, p\}$ such that $A - \lambda_\ell^k BC$ is Hurwitz and

$$\max \left\{ \lambda_\ell^k - \underline{\lambda}, \bar{\lambda} - \lambda_\ell^k \right\} \|H_\ell^k\|_\infty < 1, \quad (23)$$

where

$$H_\ell^k(s) = C \left(sI - A + \lambda_\ell^k BC \right)^{-1} B.$$

Proof

Suppose that (23) holds. For simplicity, denote $\max\{\lambda_\ell^k - \underline{\lambda}, \bar{\lambda} - \lambda_\ell^k\}$ by d and λ_ℓ^k by λ . Then, there exists a positive definite matrix K that satisfies the Riccati inequality ([24])

$$(A - \lambda BC)^\top K + K(A - \lambda BC) + C^\top C + d^2 K B B^\top K < 0. \quad (24)$$

We claim that the matrix $K > 0$ in (24) satisfies

$$(A - \alpha BC)^\top K + K(A - \alpha BC) < 0 \quad (25)$$

both for $\alpha = \underline{\lambda}$ and $\alpha = \bar{\lambda}$ and, therefore the network (4) is uniformly synchronized based on Theorem 4.1. Indeed, we have

$$\begin{aligned} (A - \alpha BC)^\top K + K(A - \alpha BC) &= (A - \lambda BC)^\top K + K(A - \lambda BC) - (\alpha - \lambda)(C^\top B^\top K + KBC) \\ &= (A - \lambda BC)^\top K + K(A - \lambda BC) + C^\top C + d^2 K B B^\top K \\ &\quad + ((\alpha - \lambda)^2 - d^2) K B B^\top K - ((\alpha - \lambda) K B \\ &\quad + C^\top)((\alpha - \lambda) B^\top K + C). \end{aligned} \quad (26)$$

Now, by (24), the right hand side of (26) is negative definite. Clearly, this holds both for $\alpha = \underline{\lambda}$ and $\alpha = \bar{\lambda}$ because $d = \max\{\lambda - \underline{\lambda}, \bar{\lambda} - \lambda\}$. $\square$

Remark 4.7

By Lemma 3.3, the network (4) is uniformly synchronized only if the matrix $A - \lambda_j^i BC$ is Hurwitz for each $i = 1, 2, \dots, N$ and $j = 2, 3, \dots, p$. Therefore, there is no conservatism involved in the Hurwitzness condition provided in Theorem 4.6.

For the analysis and design of protocols that achieve synchronization, assumptions on Lyapunov stability or passivity of the agents' dynamic have been made in the literature (see, e.g., [9] and [11]). Next, we show that uniform synchronization can be guaranteed by imposing passivity conditions on certain network dynamics, namely, $(A - \underline{\lambda} BC, B, C)$. Note that in this case, the matrix A is allowed to have eigenvalues in the right half plane whereas $A - \underline{\lambda} BC$ is required to be Hurwitz, which is indeed a necessary condition for uniform synchronization of (4). The notion we use here is *strict passivity*. Consider the finite-dimensional linear time-invariant system

$$\dot{w} = Aw + Bv \quad (27a)$$

$$z = Cw + Dv. \quad (27b)$$

We will denote this system by $\Sigma(A, B, C, D)$. We call the system (27) strictly passive if there exist $\epsilon > 0$ and $X > 0$ such that

$$\begin{pmatrix} A^\top X + XA + \epsilon X & XB - C^\top \\ B^\top X - C & -(D + D^\top) \end{pmatrix} \leq 0. \quad (28)$$

For relations of the aforementioned definition to other types of passivity, positive realness, and strict positive realness, we refer to [25]. Now, we have the following result.

Theorem 4.8

Assume that $\underline{\lambda} \neq \bar{\lambda}$. Then, the network (4) is uniformly synchronized if the linear system

$$\Sigma \left(A - \underline{\lambda} BC, B, C, \frac{1}{\bar{\lambda} - \underline{\lambda}} I_m \right)$$

is strictly passive.

Proof

Suppose $\Sigma(A - \underline{\lambda}BC, B, C, \frac{1}{\bar{\lambda} - \underline{\lambda}}I_m)$ is strictly passive. Then, using an appropriate Schur complement, by (28), we have

$$(A - \underline{\lambda}BC)^\top X + X(A - \underline{\lambda}BC) + \frac{1}{2}(\bar{\lambda} - \underline{\lambda})(XB - C^\top)(XB - C^\top)^\top < 0. \quad (29)$$

Therefore, we obtain

$$(A - \bar{\lambda}BC)^\top X + X(A - \bar{\lambda}BC) + (\bar{\lambda} - \underline{\lambda})(C^\top B^\top X + XBC) + \frac{1}{2}(\bar{\lambda} - \underline{\lambda})(XB - C^\top)(XB - C^\top)^\top < 0,$$

which can be rewritten as

$$(A - \bar{\lambda}BC)^\top X + X(A - \bar{\lambda}BC) + \frac{1}{2}(\bar{\lambda} - \underline{\lambda})(XB + C^\top)(XB + C^\top)^\top < 0. \quad (30)$$

Consequently, because both (29) and (30) hold, the network (4) is uniformly synchronized by Theorem 4.1. $\square$

Remark 4.9

Note that the case $\underline{\lambda} = \bar{\lambda}$ is not of our interest because it corresponds to a network with a time-independent topology associated with a complete graph. Obviously, in this case, synchronization is achieved if and only if $A - \underline{\lambda}BC$ is Hurwitz.

Remark 4.10

Let $\Sigma_\alpha = \Sigma(A - \underline{\lambda}BC, B, C, \alpha I_m)$. It can be verified from (28) that if Σ_α is strictly passive, then so is Σ_β for any $\beta \geq \alpha$. Hence, by Theorem 4.8, the network (4) is uniformly synchronized if Σ_α is strictly passive for some $\alpha \in [0, \frac{1}{\bar{\lambda} - \underline{\lambda}}]$. Note that the smaller the value of $\bar{\lambda} - \underline{\lambda}$ is, the less conservative the proposed passivity condition is compared to the necessary conditions in Lemma 3.3.

It is worth mentioning that the condition proposed in Theorem 4.8 can be useful also for the special case where the topology is time-independent. While the available necessary and sufficient conditions (see, e.g., [18, Lem. 3]) need information on all eigenvalues of the Laplacian matrix, the proposed passivity condition merely requires knowledge of the (nontrivial) extremal eigenvalues. Moreover, one can also use lower/upper bounds for these extremal Laplacian eigenvalues (see, e.g., [26]) to conclude synchronization of network (4) from Theorem 4.8. In fact, it is easy to observe that both in Theorems 4.6 and 4.8, the values of $\underline{\lambda}$ and $\bar{\lambda}$ can be replaced by lower and upper bounds, say $\underline{\mu}$ and $\bar{\mu}$, respectively, that is, $\underline{\mu} \leq \underline{\lambda}$ and $\bar{\lambda} \leq \bar{\mu}$. Besides, as it is clear from Remark 4.10, the information on $\bar{\lambda}$ is included to obtain less restrictive condition in Theorem 4.8. Indeed, strict passivity of Σ_0 in Remark 4.10 does not depend on $\bar{\lambda}$.

As an example to show how the result of Theorem 4.8 can be employed, next, we design a protocol to achieve uniform synchronization of (4). Consider again the diffusively coupled multi-agent system

$$\dot{x}_j = Ax_j + Bu_j,$$

$j = 1, 2, \dots, p$, together with the static protocol

$$u_j = K \sum_{(i,j) \in E_{\sigma(t)}} (x_i - x_j), \quad (31)$$

for some matrix K that will be determined later. For each $i = 1, 2, \dots, N$, assume that $G_i = (V, E_i)$ is connected. In addition, assume that (A, B) is stabilizable. Recall that the aforementioned assumptions are necessary to achieve uniform synchronization. Then, obviously, there exists a positive definite matrix P satisfying the Riccati inequality

$$A^\top P + PA - 2\underline{\lambda}PBB^\top P < 0. \quad (32)$$

Observe that

$$\begin{pmatrix} (A - \underline{\lambda}BK)^\top X + X(A - \underline{\lambda}BK) + \epsilon XXB - K^\top & \\ B^\top X - K & 0 \end{pmatrix} \preceq 0 \quad (33)$$

for the choices $K = B^\top P$, $X = P$, and a sufficiency small ϵ . In other words, the system $\Sigma(A - \underline{\lambda}BK, B, K, 0)$ is strictly passive. Thus, it follows from Theorem 4.8 and Remark 4.10 that the protocol (31) with $K = B^\top P$ achieves uniform synchronization.

Recall that in the multi-agent system (1), the output information of the agents is transmitted through the network. Hence, the result obtained before, inherently, can be used to analyze uniform synchronization of the agents for a given static state feedback protocol in the form of (31). However, as the overall information on relative states of the neighboring agents is not always available, observer-based protocols are proposed in the literature (see, e.g., [6, 9], and [11]), which use only the relative output information of the adjacent agents. Next, we show how the conditions proposed before carries over to the case of observer-based protocols. We restrict ourselves to the type of protocols proposed in [6] as it is more compatible with our framework of uniform synchronization.

Consider the following observer-based protocol:

$$\dot{v}_j(t) = (A + BK)v_j(t) + GC \sum_{(i,j) \in E_{\sigma(t)}} (v_i(t) - v_j(t)) - (x_i(t) - x_j(t)) \quad (34a)$$

$$u_j(t) = Kv_j(t), \quad (34b)$$

where $j \in V$, v_j is the state of the observer protocol, K is the feedback gain, and G is the observer gain. Attaching this protocol to the agents (1) results in

$$\dot{\xi}_j(t) = A\xi_j(t) - \sum_{i=1}^N (L_{\sigma(t)})_{ij} H\xi_j, \quad (35)$$

where

$$\xi_j = \begin{pmatrix} x_j \\ v_j \end{pmatrix}, \quad A = \begin{pmatrix} A & BK \\ 0 & A + BK \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 0 & 0 \\ -GC & GC \end{pmatrix}.$$

Observe that the matrix H can be written as $H = BC$, where

$$B = \begin{pmatrix} 0 \\ G \end{pmatrix}, \quad C = \begin{pmatrix} -C & C \end{pmatrix}.$$

Hence, the multi-agent system (35) can be written in compact form as

$$\dot{\xi}(t) = (I_p \otimes A - L_{\sigma(t)} \otimes BC)\xi(t), \quad (36)$$

where $\xi = \text{col}(\xi_1, \xi_2, \dots, \xi_p)$. This coincides with the network representation (4), where the matrices A , B , and C are replaced by A , B , and C , respectively. Therefore, the proposed conditions for uniform synchronization are also applicable to the case of observer-based protocol (34). In particular, regarding the key result of Theorem 4.1, we obtain that the network (36) achieves uniform synchronization if there exists a CQLF for the pair of matrices $A - \underline{\lambda}BC$ and $A - \bar{\lambda}BC$. These matrices are computed as follows:

$$A - \underline{\lambda}BC = \begin{pmatrix} A & BK \\ \underline{\lambda}GCA + BK - \underline{\lambda}GC \end{pmatrix}, \quad A - \bar{\lambda}BC = \begin{pmatrix} A & BK \\ \bar{\lambda}GCA + BK - \bar{\lambda}GC \end{pmatrix}.$$

Now, we apply the similarity transformation $T = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}$ to obtain

$$T(\mathbf{A} - \underline{\lambda}\mathbf{B}\mathbf{C})T^{-1} = \begin{pmatrix} A - \underline{\lambda}GC & 0 \\ \underline{\lambda}GCA + BK \end{pmatrix} \text{ and } T(\mathbf{A} - \bar{\lambda}\mathbf{B}\mathbf{C})T^{-1} = \begin{pmatrix} A - \bar{\lambda}GC & 0 \\ \bar{\lambda}GC & A + BK \end{pmatrix}.$$

As the existence of a CQLF is invariant under similarity transformation, (36) is synchronized if there exists a CQLF for the pair of matrices $T(\mathbf{A} - \underline{\lambda}\mathbf{B}\mathbf{C})T^{-1}$ and $T(\mathbf{A} - \bar{\lambda}\mathbf{B}\mathbf{C})T^{-1}$. Observe that these matrices are in a block triangular form. Hence, according to [27, Thm. 5.1], they share a CQLF if and only if $A + BK$ is Hurwitz and the pair of matrices $A - \underline{\lambda}GC$ and $A - \bar{\lambda}GC$ share a CQLF. This brings us to the following proposition.

Proposition 4.11

Let $\underline{\lambda}$ and $\bar{\lambda}$ be defined as in (3a). Then, the network (36) is uniformly synchronized if $A + BK$ is Hurwitz and there exists a positive definite matrix X satisfying both of the following LMI:

$$(A - \underline{\lambda}GC)^{\top}X + X(A - \underline{\lambda}GC) < 0 \text{ and } (A - \bar{\lambda}GC)^{\top}X + X(A - \bar{\lambda}GC) < 0.$$

The proposition above can be exploited to design a feedback gain K and an observer gain G such that the network (36) is uniformly synchronized. The assumptions (necessary conditions) required are stabilizability of (A, B) , detectability of (C, A) , and connectedness of L_i for $i = 1, 2, \dots, N$. Now, choose K such that $A + BK$ is Hurwitz. Also choose $G = QC^{\top}$ where Q is a positive definite matrix satisfying the Riccati inequality

$$AQ + QA^{\top} - 2\underline{\lambda}QC^{\top}CQ < 0.$$

Clearly, these choices of K and G satisfy the conditions of Proposition 4.11 with $X = Q^{-1}$ and thus achieve uniform synchronization for network (36). This extends the result of [6] to the case of networks with arbitrary (undirected) switching topologies. Finally, note that similarly, one can modify the results of Theorems 4.6 and 4.8 to incorporate the case of observer-based protocols.

5. CONCLUSIONS

In this paper, we have studied uniform synchronization of networks with switching topologies. The agents of the network are assumed to have identical but general linear dynamics, and the underlying communication topology may switch arbitrarily within a finite set of admissible topologies. We have shown by a numerical example that connectedness of underlying communication graphs and synchronization of the individual modes of the networks, that is, synchronization with respect to constant switching signals, are not enough to ensure uniform synchronization. Therefore, we have established conditions in terms of existence of a CQLF guaranteeing uniform synchronization of the network. These conditions require that a certain pair of LMI admits a positive definite solution. The solvability of this pair of LMI is discussed for both SISO and MIMO cases. Sufficient conditions for uniform synchronization are established in terms of bounded realness and passivity of certain network dynamics. The conditions proposed depend on the agents' dynamics as well as the (nontrivial) extremal eigenvalues of the Laplacian matrices. The results established have also been extended to the case of observer-based protocols. It has been explained how static feedback protocols as well as observer-based protocols guaranteeing uniform synchronization can be designed based on the proposed conditions.

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