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Quadratic spline collocation method for the time fractional subdiffusion equation

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ABSTRACT

In this paper, exploiting the quadratic spline collocation (QSC) method, we numerically solve the time fractional subdiffusion equation with Dirichlet boundary value conditions. The coefficient matrix of the discretized linear system is investigated in detail. Theoretical analyses and numerical examples demonstrate the proposed technique can enjoy the global error bound with $O(\tau^3 + h^3)$ under the L_∞ norm provided that the solution $v(x, t)$ has four-order continual derivative with respects to x and t , and it can achieve the accuracy of $O(\tau^4 + h^4)$ at collocation points, where τ, h are the step sizes in time and space, respectively.

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1. Introduction

In the continuous time random walk (CTRW) theory, combining with some initial and boundary conditions, some researchers have presented the governing mathematical model

$${}_C D_{0,t}^\beta u = k \frac{\partial^\alpha u}{\partial |x|^\alpha} + f(x, t) \quad (1)$$

to describe the diffusion process in different mediums [1,2], where ${}_C D_{0,t}^\beta u$ and $\frac{\partial^{2\alpha} u}{\partial |x|^{2\alpha}}$ are respectively the Caputo and Riesz fractional derivatives [3].

When $\alpha = 2, \beta = 1$, Eq. (1) is the traditional integer-order differential model problem. As we know, in a highly non-homogeneous medium, the corresponding probability density of the concentration field obtained by traditional model may have a heavier tail than the Gaussian density [1,4]. Nevertheless, it finds that when one takes the condition $\frac{2\beta}{\alpha} < 1$, which corresponds to this case called the subdiffusion motion, may be more adequate to describe this phenomenon, refer to [5,6] for details.

Due to the nonlocal property of fractional derivatives, it is usually difficult to obtain the analytical solutions [3]. As an alternative, more and more researchers have began to increasingly focus on the efficient numerical solutions of fractional differential equations.

For the cases of space and space-time fractional subdiffusion models, many prominent numerical approaches have recently been presented, including the finite volume method [5], the fast semi-implicit difference method [7], the implicit Euler scheme [8], the alternating direct method [9], and other numerical methods (see e.g. [10,11]).

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In this paper, we are interested in the following time-fractional subdiffusion equation

$$\begin{cases} {}^C D_{0,t}^\beta u = k \frac{\partial^2 u}{\partial x^2} + f(x, t), & (x, t) \in I \times (0, T], \quad I = (a, b), T > 0; \\ u(x, 0) = \phi(x), & x \in I; \\ u(a, t) = \varphi(t), u(b, t) = \psi(t), & t \in (0, T], \end{cases} \quad (2)$$

where $0 < \beta < 1$, and ${}^C D_{0,t}^\beta$ is the β th-order Caputo derivative operator of the form

$${}^C D_{0,t}^\beta u = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial u}{\partial s} ds, \quad (3)$$

$k > 0$ is the diffusion constant, $\Gamma(\cdot)$ is the gamma function, $\phi(x)$, $\varphi(t)$, $\psi(t)$, $f(x, t)$ are all specified smooth functions, and $u(x, t)$ is the unknown function to be solved.

For the numerical solution of (2), one popular idea in the existed work is employing the relation between the Caputo derivative and the Riemann-Liouville derivative [3] to transform it into the following equivalent system

$$\begin{cases} \frac{\partial u}{\partial t} = {}_0 D_t^{1-\beta} \left[k \frac{\partial^2 u}{\partial x^2} \right] + g(x, t), & (x, t) \in I \times (0, T]; \\ u(x, 0) = \phi(x), & x \in I; \\ u(a, t) = \varphi(t), u(b, t) = \psi(t), & t \in (0, T], \end{cases} \quad (4)$$

where $g(x, t) = {}_0 D_t^{1-\beta} f(x, t)$, ${}_0 D_t^\beta s(x, t)$ is the Riemann-Liouville derivative of function $s(x, t)$ [3], and then designing some schemes to discretize Eq. (4) [12–15].

There are also some approaches which were developed to directly solve Eq. (2). Gao and Sun [16] introduced a compact finite difference scheme with $O(\tau^{2-\beta} + h^4)$ order accuracy. Jiang and Ma [17] presented an $O(\tau^{2-\beta} + h^r)$ order finite element method with r the degree of the polynomial test function space. Based on piecewise linear functions, Jin et al. [18] established a lumped mass Galerkin FEM. Azizi [19], Chen [20], and Lin [21] respectively introduced some spectral methods. Using FEM in space, and fractional linear multistep method (FLMM) in time, Zeng et al. [22,23] successively proposed two discretized approaches with accuracy of $O(\tau^{2-\beta} + h^{r+1})$ and $O(\tau^2 + h^{r+1})$. For $\beta \in (0, 0.9569347]$, Cui and Sun [24] derived a compact finite difference scheme with $O(\tau^3 + h^4)$ order accuracy. Recently, employing the Jacobi polynomials and Fourier-like basis functions, Zheng [4] also proposed a high order space-time spectral method for a time fractional Fokker-Planck equation which is a generalization of subdiffusion Eq. (2), proved theoretically an exponential convergence may be achieved when the exact solution is sufficiently smooth. Due to the lack of orthogonality for the employed basis functions, this spectral discretization results in a full stiffness matrix, which requires an efficient solver and large computational memory for handling the resulting linear system. Additionally, some numerical schemes have been introduced for high-dimensional and/or variable order models, refer to [25–27] for details.

From a lot of early established numerical methods, it notes that the resulted accuracy seems to be considerably susceptible to the order β . The contribution of this paper is to establish a novel collocation method for (2) via taking quadratic spline polynomials as basic functions. The key idea of the technique is that, by the initial value in time, we transform (2) into an equivalent system, where the new unknown function $v(x, t) = \frac{\partial u}{\partial t}$, and then, use two interpolation operators successively to approximate $v(x, t)$ and $\frac{\partial^2 v}{\partial x^2}$. Later, in the theoretical analyses and numerical examples, it can find the accuracy of the proposed collocation method is independent of β .

The outline of this paper is as follows. In Section 2, some preliminaries are provided, which are useful to construct the QSC method. In Section 3, based on the quadratic spline function, the QSC method is constructed, the corresponding collocation equations and the coefficient matrix are also given. Meanwhile, the existence, uniqueness, convergence and stability of the proposed numerical scheme are studied. In Section 4, by comparing with another recently presented scheme, some numerical examples are given to illustrate the effectiveness of the proposed technique. Finally, some conclusions about the established method are drawn.

2. Preliminaries

Define respectively $\rho_h = \{x_i\}_{i=1}^{N_h+1}$ and $\rho_t = \{t_i\}_{i=1}^{N_t+1}$ as uniform partitions of the interval $\bar{I} = [a, b]$ and $[0, T]$ with

$$x_i = a + (i-1)h, t_j = (j-1)\tau, \quad i = 1, \dots, N_h+1, j = 1, \dots, N_t+1, \quad h = \frac{b-a}{N_h}, \quad \tau = \frac{T}{N_t}.$$

Let

$$\eta_i = \frac{x_i + x_{i-1}}{2}, \quad i = 2, 3, \dots, N_h+1, \tau_j = \frac{t_j + t_{j-1}}{2}, \quad j = 2, 3, \dots, N_t+1,$$

then the collocation points in $(a, b) \times (0, T)$ can be defined by $\{(\eta_i, \tau_j)\}$, $i = 2, 3, \dots, N_h+1$, $j = 2, 3, \dots, N_t+1$, the center of each gridding cell. Moreover, with parameter $\theta \in (0, \frac{1}{2})$, we take

$$\eta_1 = a, \quad \eta_{N_h+2} = b, \quad \tau_1 = \theta\tau, \quad \tau_{N_t+2} = T$$

for conveniently dealing with the boundary conditions.

Let

$$S_2 = \{v : v \in C^1[0, 1], v|_{\delta_i} \in P_2, v|_{T_j} \in P_2, 1 \leq i \leq N_h, 1 \leq j \leq N_t\}$$

be the space of quadratic splines where P_2 is the set of polynomials of degree ≤ 2 and $\delta_i = [x_i, x_{i+1}]$, $T_j = [t_j, t_{j+1}]$.

Along the time direction, similar to [28], we take the basis of S_2 to be $\{\mathcal{D}_m\}_{m=1}^{N_t+2}$, where

$$\mathcal{D}_m(x) = \frac{1}{2}\xi\left(\frac{x}{\tau} - m + 3\right), \quad (5)$$

with ξ the quadratic spline function:

$$\xi(x) = \begin{cases} x^2, & x \in [0, 1]; \\ -3 + 6x - 2x^2, & x \in [1, 2]; \\ 9 - 6x + x^2, & x \in [2, 3]; \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Along the space direction, we take the basis of S_2 to be $\{\mathcal{B}_m\}_{m=1}^{N_h+2}$ with

$$\begin{cases} \mathcal{B}_m(x) = \frac{1}{2}\xi\left(\frac{x-a}{h} - m + 3\right), & m = 1, 3, 4, \dots, N_h, N_h + 2; \\ \mathcal{B}_2(x) = \frac{1}{2}\xi\left(\frac{x-a}{h} + 1\right) - \frac{1}{2}\xi\left(\frac{x-a}{h} + 2\right); \\ \mathcal{B}_{N_h+1}(x) = \frac{1}{2}\xi\left(\frac{x-a}{h} - N_h + 2\right) - \frac{1}{2}\xi\left(\frac{x-a}{h} - N_h + 1\right). \end{cases} \quad (7)$$

For the quadratic spline interpolations, some conclusions about the errors are given as follows. Detailed analyses about these results can be found in [28,29], which we will not pursue here.

Let $v_s(x, t) \in S_2$ be the biquadratic spline interpolant of a function $v(x, t)$ such that

$$\begin{aligned} v_s(\eta_i, \tau_j) &= v(\eta_i, \tau_j), \quad i = 2, 3, \dots, N_h + 1, \quad j = 2, 3, \dots, N_t + 1; \\ v_s(\eta_i, \tau_j) &= v(\eta_i, \tau_j) - \frac{h^4}{128} \frac{\partial^4 v}{\partial x^4}(\eta_i, \tau_j), \quad i = 1, N_h + 2, j = 2, 3, \dots, N_t + 1; \\ v_s(\eta_i, \tau_j) &= v(\eta_i, \tau_j) - \frac{\tau^4}{128} \frac{\partial^4 v}{\partial t^4}(\eta_i, \tau_j), \quad i = 2, 3, \dots, N_h + 1, j = 1, N_t + 2; \\ v_s(\eta_i, \tau_j) &= v(\eta_i, \tau_j) - \frac{h^4}{128} \frac{\partial^4 v}{\partial x^4}(\eta_i, \tau_j), \quad i = 1, N_h + 2, j = 1, N_t + 2. \end{aligned} \quad (8)$$

Then, supposing $v(x, t)$ has four-order continual derivative with respects to x and t , this interpolant satisfies

$$\begin{aligned} \frac{\partial^2 v_s}{\partial x^2}(\eta_i, \tau_j) &= \frac{\partial^2 v}{\partial x^2}(\eta_i, \tau_j) - \frac{h^2}{24} \frac{\partial^4 v}{\partial x^4}(\eta_i, \tau_j) + O(h^4), \\ i &= 2, 3, \dots, N_h + 1, j = 2, 3, \dots, N_t + 1. \end{aligned} \quad (9)$$

3. Quadratic spline collocation method

In this section, based on Eqs. (7)–(9), we deduce the collocation equations for Eq. (2), give the explicit expression of the linear system, and analyze the convergence and stability of the proposed QSC method.

In order to reduce the errors in time, we replace the derivative $\frac{\partial u}{\partial t}$ with a new unknown function $v(x, t)$ such that (8) can be utilized.

Now, let $\frac{\partial u}{\partial t} = v(x, t)$, then, the use of initial value condition $u(x, 0) = \phi(x)$ in Eq. (2) yields

$$u = \int_0^t v(x, s) ds + \phi(x), \quad x \in I. \quad (10)$$

Substituting (10) into the first equation in (2), we obtain the equivalent system

$$\frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v(x, s) ds = k \int_0^t \frac{\partial^2 v(x, s)}{\partial x^2} ds + k \frac{\partial^2 \phi}{\partial x^2} + f(x, t). \quad (11)$$

Using (8) and (9), (11) reads

$$\frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v_s(\eta_i, s) ds = k \int_0^t \frac{\partial^2 v_s}{\partial x^2}(\eta_i, s) ds + \frac{kh^2}{24} \int_0^t \frac{\partial^4 v}{\partial x^4}(\eta_i, s) ds + k \frac{\partial^2 \phi}{\partial x^2}(\eta_i) + f(\eta_i, \tau_j) + O(h^4). \quad (12)$$

Noticing

$$\begin{aligned} k \int_0^t \frac{\partial^4 v}{\partial x^4}(\eta_i, s) ds &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial^2 v}{\partial x^2}(\eta_i, s) ds - k \frac{\partial^4 \phi}{\partial x^4}(\eta_i) - \frac{\partial^2 f}{\partial x^2}(\eta_i, t) \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial^2 v_s}{\partial x^2}(\eta_i, s) ds - k \frac{\partial^4 \phi}{\partial x^4}(\eta_i) - \frac{\partial^2 f}{\partial x^2}(\eta_i, t) + O(h^2), \end{aligned} \quad (13)$$

from (12) we have

$$\begin{aligned} &\frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v_s(\eta_i, s) ds - k \int_0^t \frac{\partial^2 v_s}{\partial x^2}(\eta_i, s) ds - \frac{h^2}{24\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial^2 v_s}{\partial x^2}(\eta_i, s) ds \\ &= -\frac{kh^2}{24} \frac{\partial^4 \phi}{\partial x^4}(\eta_i) - \frac{h^2}{24} \frac{\partial^2 f}{\partial x^2}(\eta_i, t) + k \frac{\partial^2 \phi}{\partial x^2}(\eta_i) + f(\eta_i, \tau_j) + O(h^4). \end{aligned} \quad (14)$$

Dropping the $O(h^4)$ term, we obtain the collocation equations

$$\begin{aligned} \Delta_1 + \Delta_2 + \Delta_3 &= -\frac{kh^2}{24} \frac{\partial^4 \phi}{\partial x^4}(\eta_i) - \frac{h^2}{24} \frac{\partial^2 f}{\partial x^2}(\eta_i, \tau_j) + k \frac{\partial^2 \phi}{\partial x^2}(\eta_i) + f(\eta_i, \tau_j), \\ i &= 2, 3, \dots, N_h + 1, j = 2, 3, \dots, N_t + 2, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \Delta_1 &= \frac{1}{\Gamma(1-\beta)} \int_0^{\tau_j} (\tau_j - s)^{-\beta} v_h(\eta_i, s) ds, \\ \Delta_2 &= -k \int_0^{\tau_j} \frac{\partial^2 v_h}{\partial x^2}(\eta_i, s) ds, \\ \Delta_3 &= \frac{h^2}{24\Gamma(1-\beta)} \int_0^{\tau_j} (\tau_j - s)^{-\beta} \frac{\partial^2 v_h}{\partial x^2}(\eta_i, s) ds, \end{aligned}$$

with $v_h \in S_2 \times S_2$, which is the approximate solution function to be solved.

Let

$$v_h(x, t) = \sum_{m=1}^{N_h+2} \sum_{n=1}^{N_t+2} v_{m,n} \mathcal{B}_m(x) \mathcal{D}_n(t), \quad (16)$$

using the boundary value conditions

$$v(a, t) = \frac{\partial \varphi(t)}{\partial t}, \quad v(b, t) = \frac{\partial \psi(t)}{\partial t},$$

we can first compute $v_{1,j}$, $v_{N_h+2,j}$, $j = 1, 2, \dots, N_t + 2$ by

$$\frac{1}{2} G_1 \mathbf{v}_1 = \frac{\partial \varphi}{\partial t}, \quad (17)$$

and

$$\frac{1}{2} G_1 \mathbf{v}_{N_h+2} = \frac{\partial \psi}{\partial t}, \quad (18)$$

where

$$\begin{aligned} G_1 &= \begin{pmatrix} \mathcal{D}_1(\tau_1) & \mathcal{D}_2(\tau_1) & \cdots & \mathcal{D}_{N_t+2}(\tau_1) \\ \mathcal{D}_1(\tau_2) & \mathcal{D}_2(\tau_2) & \cdots & \mathcal{D}_{N_t+2}(\tau_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{D}_1(\tau_{N_t+2}) & \mathcal{D}_2(\tau_{N_t+2}) & \cdots & \mathcal{D}_{N_t+2}(\tau_{N_t+2}) \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \\ 1 & 6 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 6 & 1 \\ & & & 4 & 4 \end{pmatrix} \end{aligned}$$

with $\delta_1 = 4(\theta - 1)^2$, $\delta_2 = 4(-2\theta^2 + 2\theta + 1)$, $\delta_3 = 4\theta^2$, and

$$\frac{\partial \varphi}{\partial t} = \left[\frac{\partial \varphi}{\partial t}(\tau_1), \frac{\partial \varphi}{\partial t}(\tau_2), \dots, \frac{\partial \varphi}{\partial t}(\tau_{N_t+2}) \right]^T,$$

$$\frac{\partial \psi}{\partial t} = \left[\frac{\partial \psi}{\partial t}(\tau_1), \frac{\partial \psi}{\partial t}(\tau_2), \dots, \frac{\partial \psi}{\partial t}(\tau_{N_t+2}) \right]^T,$$

$$\mathbf{v}_1 = [v_{1,1}, v_{1,2}, \dots, v_{1,N_t+2}]^T,$$

$$\mathbf{v}_{N_h+2} = [v_{N_h+2,1}, v_{N_h+2,2}, \dots, v_{N_h+2,N_t+2}]^T.$$

Next, by (16), (17) and (18), $\Delta_1, \Delta_2, \Delta_3$ can be re-written as

$$\Delta_1 = \frac{1}{\Gamma(1-\beta)} \sum_{m=1}^{N_h+2} \mathcal{B}_m(\eta_i) \sum_{n=1}^{N_t+2} \left\{ \int_0^{\tau_j} (\tau_j - s)^{-\beta} \mathcal{D}_n(s) ds \right\} v_{m,n} = \frac{1}{\Gamma(1-\beta)} \{ [A_1 \otimes P] \mathbf{v} + [A_2 \otimes P] \mathbf{v}_1 + [A_3 \otimes P] \mathbf{v}_{N_h+2} \},$$

$$\begin{aligned} \Delta_2 &= -k \sum_{m=1}^{N_h+2} \frac{\partial^2 \mathcal{B}_m}{\partial x^2}(\eta_i) \sum_{n=1}^{N_t+2} \left\{ \int_0^{\tau_j} \mathcal{D}_n(s) ds \right\} v_{m,n} \\ &= -k \{ [A_4 \otimes Q] \mathbf{v} + [A_5 \otimes Q] \mathbf{v}_1 + [A_6 \otimes Q] \mathbf{v}_{N_h+2} \}, \end{aligned}$$

$$\begin{aligned} \Delta_3 &= \frac{h^2}{24\Gamma(1-\beta)} \sum_{m=1}^{N_h+2} \frac{\partial^2 \mathcal{B}_m}{\partial x^2}(\eta_i) \sum_{n=1}^{N_t+2} \left\{ \int_0^{\tau_j} (\tau_j - s)^{-\beta} \mathcal{D}_n(s) ds \right\} v_{m,n} \\ &= \frac{h^2}{24\Gamma(1-\beta)} \{ [A_4 \otimes P] \mathbf{v} + [A_5 \otimes P] \mathbf{v}_1 + [A_6 \otimes P] \mathbf{v}_{N_h+2} \}, \end{aligned}$$

consequently, (15) reads

$$H\mathbf{v} = \mathbf{g} \quad (19)$$

where

$$\begin{aligned} H &= \frac{1}{\Gamma(1-\beta)} A_1 \otimes P - k A_4 \otimes Q + \frac{h^2}{24\Gamma(1-\beta)} A_4 \otimes P, \\ \mathbf{v} &= (\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_{N_h+1})^T, \mathbf{v}_k = (v_{k,1}, v_{k,2}, \dots, v_{k,N_t+2}), \quad k = 2, 3, \dots, N_h + 1, \\ \mathbf{g} &= \mathbf{g}_1 + \mathbf{g}_2 \end{aligned}$$

with

$$\mathbf{g}_1((N_t + 2)(i - 1) + j) = -\frac{kh^2}{24} \frac{\partial^4 \phi}{\partial x^4}(\eta_i) - \frac{h^2}{24} \frac{\partial^2 f}{\partial x^2}(\eta_i, \tau_j) + k \frac{\partial^2 \phi}{\partial x^2}(\eta_i) + f(\eta_i, \tau_j),$$

$$i = 1, 2, \dots, N_h, j = 1, 2, \dots, N_t + 2,$$

$$\begin{aligned} \mathbf{g}_2 &= \frac{-1}{\Gamma(1-\beta)} [(A_2 \otimes P) \mathbf{v}_1 + (A_3 \otimes P) \mathbf{v}_{N_h+2}] + k [(A_5 \otimes Q) \mathbf{v}_1 + (A_6 \otimes Q) \mathbf{v}_{N_h+2}] \\ &\quad - \frac{h^2}{24\Gamma(1-\beta)} [(A_5 \otimes P) \mathbf{v}_1 + (A_6 \otimes P) \mathbf{v}_{N_h+2}], \end{aligned}$$

$$A_1 = \frac{1}{8} \begin{pmatrix} 5 & 1 & & & \\ 1 & 6 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6 & 1 \\ & & & 1 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{8} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{8} \end{pmatrix},$$

$$A_4 = \frac{1}{h^2} \begin{pmatrix} -3 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -3 \end{pmatrix}, \quad A_5 = \begin{pmatrix} \frac{1}{h^2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{h^2} \end{pmatrix},$$

and

$$P_{i,j} = \int_0^{\tau_i} (\tau_i - s)^{-\beta} \mathcal{D}_j(s) ds, \quad i, j = 1, 2, \dots, N_t + 2,$$

$$Q_{i,j} = \int_0^{\tau_i} \mathcal{D}_j(s) ds, \quad i, j = 1, 2, \dots, N_t + 2.$$

It is worth noting that taking parameter $\theta = 0$ will lead to $P_{1,j} = 0, Q_{1,j} = 0$ for all $j = 1, 2, \dots, N_t + 2$, and consequently cause the coefficient matrix H being singular.

By the features of the basis function $\mathcal{D}_n(x)$, for the matrices P, Q , the following conclusions are available.

Property 1. For $1 \leq i, j \leq N_t + 2$, denoting by $t_0 = 0$, then

$$P_{i,j} = \int_0^{t_{i-1}} (\tau_i - s)^{-\beta} \mathcal{D}_j(s) ds + \frac{1}{1-\beta} \left[(\tau_i - t_{i-1})^{1-\beta} \mathcal{D}_j(t_{i-1}) + \int_{t_{i-1}}^{\tau_i} (\tau_i - s)^{1-\beta} \mathcal{D}'_j(s) ds \right],$$

where the two integrals can be computed by numerical quadrature formula such as Gauss integral formula.

Proof. This can be easily proved using integration by parts. $\square$

From the expression of $P_{i,j}$ in [Property 1](#), we know $P_{i,j}$ can be enough accurately obtained for any given β . It means that, for the specified basis $\mathcal{D}_m(t)$, $t = 1, 2, N_t + 2$, the error caused by approximating the solution function $v_h(x, t)$ in [\(16\)](#) will be independent of the order β .

Property 2. Both P and Q will be the Hessenberg matrices if substituting $P_{1,3}$ and $Q_{1,3}$ with 0, respectively.

Proof. Since

$$\frac{s}{\tau} - j + 3 \leq 0, \quad i = 1, \quad 4 \leq j \leq N_t + 2, \quad 0 \leq s \leq \tau_1 = \theta \tau$$

and

$$\frac{s}{\tau} - j + 3 \leq -\frac{1}{2}, \quad 2 \leq i \leq N_t, \quad i + 2 \leq j \leq N_t + 2, \quad 0 \leq s \leq \tau_i = \frac{2i-3}{2} \tau,$$

by [\(6\)](#), we have

$$\mathcal{D}_j(s) = \frac{1}{2} \xi \left(\frac{s}{\tau} - j + 3 \right) = 0, \quad 4 \leq j \leq N_t + 2, \quad 0 \leq s \leq \tau_1$$

and

$$\mathcal{D}_j(s) = \frac{1}{2} \xi \left(\frac{s}{\tau} - j + 3 \right) = 0, \quad 2 \leq i \leq N_t, \quad i + 2 \leq j \leq N_t + 2, \quad 0 \leq s \leq \tau_i.$$

Consequently,

$$P_{1,j} = \int_0^{\tau_1} (\tau_1 - s)^{-\beta} \mathcal{D}_j(s) ds = 0, \quad 4 \leq j \leq N_t + 2,$$

and

$$P_{i,j} = \int_0^{\tau_i} (\tau_i - s)^{-\beta} \mathcal{D}_j(s) ds = 0, \quad 2 \leq i \leq N_t, \quad i + 2 \leq j \leq N_t + 2.$$

The case of Q can be similarly proved. This completes the proof of [Property 2](#). $\square$

For the sake of clarity, [Fig. 1](#) is an example of Q corresponding to $h = \tau = \frac{1}{32}$.

Property 3. $P_{2:N_t+1,3:N_t+2}, Q_{2:N_t+1,3:N_t+2}$ are Toeplitz matrices.

Proof. For $2 \leq j \leq N_t, 3 \leq n \leq N_t + 1$, let $s = z + \tau$,

$$\begin{aligned} P_{j+1,n+1} &= \int_0^{\tau_{j+1}} (\tau_{j+1} - s)^{-\beta} \mathcal{D}_{n+1}(s) ds \\ &= \int_{-\tau}^{\tau_j} (\tau_j - z)^{-\beta} \mathcal{D}_{n+1}(z + \tau) dz. \end{aligned}$$

Since

$$\mathcal{D}_{n+1}(z + \tau) = \frac{1}{2} \xi \left(\frac{z + \tau}{\tau} - (n + 1) + 3 \right) = \frac{1}{2} \xi \left(\frac{z}{\tau} - n + 3 \right) = \mathcal{D}_n(z),$$

we have

$$\begin{aligned} P_{j+1,n+1} &= \int_{-\tau}^{\tau_j} (\tau_j - z)^{-\beta} \mathcal{D}_n(z) dz \\ &= \int_{-\tau}^0 (\tau_j - z)^{-\beta} \mathcal{D}_n(z) dz + \int_0^{\tau_j} (\tau_j - z)^{-\beta} \mathcal{D}_n(z) dz = P_{j,n}. \end{aligned}$$

The case of Q can be similarly proved. This completes the proof of [Property 3](#). $\square$

From [Property 2](#) and [3](#) we can see the computational complexity for generating the linear system [\(19\)](#) is $O(N)$ with $N = \max\{N_t, N_h\}$.

According to the above discretized process and properties, we can implement the proposed scheme as follows:

Fig. 1. Pattern of Q with $h = \frac{1}{32}$, $\tau = \frac{1}{32}$.

- Step 1. Choose the time step τ , the space step h , parameter θ , a , b , β , give the mesh points (x_i, t_j) and collocation points (η_i, τ_j) .
- Step 2. Give the initial value condition ϕ , the right-hand term $f(\eta_i, \tau_j)$, the boundary value conditions $\frac{\partial \phi}{\partial t}$, $\frac{\partial \psi}{\partial t}$, and generate the matrix G_1 .
- Step 3. Compute the unknowns $\mathbf{v}_1, \mathbf{v}_{N_h+2}$ in boundary by (17)–(18).
- Step 4. Give $A_1, A_2, A_3, A_4, A_5, A_6$. Compute $P(1 : Nt + 2, 1 : 3)$, $P(Nt + 2, 4 : Nt + 2)$, $Q(1 : Nt + 2, 1 : 3)$, $Q(Nt + 2, 4 : Nt + 2)$ by Property 1. Construct P, Q by Property 2,3.
- Step 5. Generate the coefficient matrix H , the right-hand-side $\mathbf{g}$, and solve the unknown solution $\mathbf{v}$ by (19).
- Step 6. Compute the solution u by (10).

For convenience of analyzing the error of the established method, we give the following lemmas,

Lemma 1 ([30,31]). Given $v(x) \in C^4[a, b]$, then let $\wedge_x v(x)$ be the quadratic spline interpolant of function $v(x)$ in the sense of

$$\wedge_x v(\eta_i) = v(\eta_i), \quad i = 1, 2, \dots, N+2$$

with η_i the collocation points and N a positive integer number, then

$$\|v(\eta_i) - \wedge_x v(\eta_i)\| = O(h^4), \quad \|v - \wedge_x v\|_\infty = O(h^3), \quad (20)$$

where $\|v(\eta_i) - \wedge_x v(\eta_i)\| = \max\{|v(\eta_i) - \wedge_x v(\eta_i)|, i = 1, 2, \dots, N+2\}$ and $\|v - \wedge_x v\|_\infty = \max\{|v(x) - \wedge_x v(x)|, x \in [a, b]\}$.

Lemma 2. Given $v(x, y) \in C_{x,y}^{4,4}([a, b] \times [a, b])$. Let $\wedge_{xy} v(x, y)$ be the biquadratic spline interpolant of function $v(x, y)$ in the sense of

$$\wedge_{xy} v(\eta_i^x, \eta_j^y) = v(\eta_i^x, \eta_j^y), \quad i, j = 1, 2, \dots, N+2$$

with (η_i^x, η_j^y) the collocation points, then

$$\|\wedge_{xy} v(\eta_i^x, \eta_j^y) - v(\eta_i^x, \eta_j^y)\| = O(h^4), \quad \|v - \wedge_{xy} v\|_\infty = O(h^3). \quad (21)$$

Proof. Since $\wedge_{xy} v = \wedge_x(\wedge_y v)$, we have

$$\begin{aligned} \wedge_{xy} v - v &= \wedge_x(\wedge_y v - v) + (\wedge_x v - v) \\ &= (\wedge_x(\wedge_y v - v) - (\wedge_y v - v)) + (\wedge_y v - v) + (\wedge_x v - v). \end{aligned}$$

By $v(x, y) \in C_{x,y}^{4,4}([a, b] \times [a, b])$ and (20), we have

$$\|v(\eta_i^x) - \wedge_x v(\eta_i^x)\| = O(h^4), \quad \|v(\eta_i^y) - \wedge_y v(\eta_i^y)\| = O(h^4),$$

$$\|v - \wedge_x v\|_\infty = O(h^3), \quad \|v - \wedge_y v\|_\infty = O(h^3).$$

Consequently,

$$\|v(\eta_i^x, \eta_j^y) - \wedge_{xy} v(\eta_i^x, \eta_j^y)\| \leq O(h^4 h^4) + O(h^4) + O(h^4) = O(h^4),$$

and

$$\|v - \wedge_{xy} v\|_{\infty} \leq O(h^3 h^3) + O(h^3) + O(h^3) = O(h^3).$$

This completes the proof of Lemma 2. $\square$

Lemma 3. Supposing positive real parameter θ is sufficiently small, then the matrix Q is invertible.

Proof. Because

$$Q_{i,j} = \int_0^{\tau_i} \mathcal{D}_j(s) ds, \quad i, j = 1, 2, \dots, N_t + 2,$$

by some simple computations, we can obtain

$$Q_{i,j} = Q_{i-1,j}, \quad i = 4, 5, \dots, N_t + 2, \quad j = 1, 2, \dots, i - 3.$$

Multiplying Q from the left by the matrix

$$M = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix},$$

we get

$$MQ = \begin{pmatrix} Q_{1,1} & Q_{1,2} & Q_{1,3} & & & \\ Q_{2,1} & Q_{2,2} & Q_{2,3} & & & \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_2 & \varepsilon_1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \varepsilon_1 & \varepsilon_2 & \varepsilon_2 & \varepsilon_1 \\ & & & s_1 & s_2 & s_3 \end{pmatrix},$$

where

$$\begin{aligned} Q_{1,1} &= \left(\theta - \theta^2 + \frac{\theta^3}{3} \right) \frac{\tau}{2}, \quad Q_{1,2} = \left(\theta + \theta^2 - \frac{2}{3} \theta^3 \right) \frac{\tau}{2}, \quad Q_{1,3} = \frac{\theta^3 \tau}{6}, \quad \varepsilon_1 = \frac{\tau}{48}, \quad \varepsilon_2 = \frac{23\tau}{48}, \\ Q_{2,1} &= \frac{7\tau}{48}, \quad Q_{2,2} = \frac{\tau}{3}, \quad Q_{2,3} = \frac{\tau}{48}, \quad s_1 = \frac{\tau}{48}, \quad s_2 = \frac{\tau}{3}, \quad s_3 = \frac{7\tau}{48}. \end{aligned}$$

Let $MQ\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = (x_1, x_2, \dots, x_{N_t+2})^T$. Then, it is easy to get

$$x_2 = -\frac{1 - \theta - 2\theta^2}{1 + \theta - 6\theta^2} x_1, \quad x_3 = \frac{9 - 23\theta + 10\theta^2}{1 + \theta - 6\theta^2} x_1, \quad x_4 = \frac{-185 + 505\theta - 270\theta^2}{1 + \theta - 6\theta^2} x_1,$$

$$x_{k-3} + 23x_{k-2} + 23x_{k-1} + x_k = 0, \quad k = 4, 5, \dots, N_t + 2,$$

and

$$s_1 x_{N_t} + s_2 x_{N_t+1} + s_3 x_{N_t+2} = 0.$$

For sufficiently small θ , the use of the recursive thinking immediately yields $|x_k| \geq 3|x_{k-1}|$, $k = 3, 4, \dots, N_t + 2$, which results in

$$\begin{aligned} |s_1 x_{N_t} + s_2 x_{N_t+1} + s_3 x_{N_t+2}| &\geq s_3 |x_{N_t+2}| - s_2 |x_{N_t+1}| - s_1 |x_{N_t}| \\ &\geq s_3 |x_{N_t+2}| - \frac{s_2}{3} |x_{N_t+2}| - \frac{s_1}{9} |x_{N_t+2}| \\ &= \frac{7}{216} |x_{N_t+2}| \\ &\geq \frac{7}{216} 3^{N_t} |x_2| \geq 0. \end{aligned}$$

Noticing $s_1 x_{N_t} + s_2 x_{N_t+1} + s_3 x_{N_t+2} = 0$, we have $x_1 = x_2 = \dots = x_{N_t+2} = 0$, which means Q is invertible. This completes the proof of Lemma 3. $\square$

Theorem 1. Supposing the solution function $v(x, t) \in C_{x,t}^{4,4}([a, b] \times [0, T])$. Then, for sufficiently small $h, \tau, \theta > 0$, the collocation equation (15) has unique solution $v_h(x, t)$, and the caused error estimates are

$$\begin{aligned} \|v(\eta_i, \tau_j) - v_h(\eta_i, \tau_j)\| &= O(h^4 + \tau^4), \quad i = 1, \dots, N_h + 2, \quad j = 1, \dots, N_t + 2; \\ \|v - v_h\|_{\infty} &= O(h^3 + \tau^3). \end{aligned} \tag{22}$$

Proof. First, for sufficiently small h, τ, θ , the nonsingularity of the matrix $A_4 \otimes Q$ can be proved as follows.

Let $A_4 \otimes Q \mathbf{x} = \mathbf{0}$ with $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_{N_h+2}^T]^T$, $\mathbf{x}_k = [x_{k,1}, x_{k,2}, \dots, x_{k,N_t+2}]^T$. Then we have

$$\begin{aligned} & \frac{-3}{h^2} Q \mathbf{x}_1 + \frac{1}{h^2} Q \mathbf{x}_2 = \mathbf{0}; \\ & \frac{1}{h^2} Q \mathbf{x}_1 - \frac{2}{h^2} Q \mathbf{x}_2 + \frac{1}{h^2} Q \mathbf{x}_3 = \mathbf{0}; \\ & \frac{1}{h^2} Q \mathbf{x}_2 - \frac{2}{h^2} Q \mathbf{x}_3 + \frac{1}{h^2} Q \mathbf{x}_4 = \mathbf{0}; \\ & \dots\dots\dots \\ & \frac{1}{h^2} Q \mathbf{x}_{N_h} - \frac{2}{h^2} Q \mathbf{x}_{N_h+1} + \frac{1}{h^2} Q \mathbf{x}_{N_h+2} = \mathbf{0}; \\ & \frac{1}{h^2} Q \mathbf{x}_{N_h+1} - \frac{3}{h^2} Q \mathbf{x}_{N_h+2} = \mathbf{0}. \end{aligned} \quad (23)$$

Using Lemma 3, (23) reads

$$\begin{aligned} & -3\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{0}; \\ & \mathbf{x}_1 - 2\mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}; \\ & \mathbf{x}_2 - 2\mathbf{x}_3 + \mathbf{x}_4 = \mathbf{0}; \\ & \dots\dots\dots \\ & \mathbf{x}_{N_h} - 2\mathbf{x}_{N_h+1} + \mathbf{x}_{N_h+2} = \mathbf{0}; \\ & \mathbf{x}_{N_h+1} - 3\mathbf{x}_{N_h+2} = \mathbf{0}, \end{aligned}$$

where we can recursively obtain $\mathbf{x}_{N_h+2} = (2N_h + 3)\mathbf{x}_1$, $\mathbf{x}_{N_h+1} = (2N_h + 1)\mathbf{x}_1$, which means $\mathbf{x}_1 = \mathbf{x}_2 = \dots = \mathbf{x}_{N_h+2} = \mathbf{0}$ considering $\mathbf{x}_{N_h+1} - 3\mathbf{x}_{N_h+2} = \mathbf{0}$. Hence, we know $A_4 \otimes Q$ is invertible. Furthermore, it is not hard to find that

$$(A_4 \otimes Q)^{-1} \rightarrow \mathbf{0} \text{ when } h \rightarrow 0.$$

Next, denoting by Z the matrix $(-kA_4 \otimes Q)^{-1}$. Multiplying H in (19) from the left with Z , we obtain

$$ZH = I + \frac{1}{\Gamma(1-\beta)} ZA_1 \otimes P + \frac{h^2}{24\Gamma(1-\beta)} ZA_4 \otimes P,$$

which is a strictly diagonally dominant matrix for small enough h, τ . Hence, H is a nonsingular matrix, and $\|H^{-1}\|$ is bound, which indicates the solution $v_h(x, t)$ uniquely exists.

Third, subtracting (14) from (15), we get

$$L(v_s - v_h) = O(h^4),$$

where

$$Lv = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} v(\eta_i, s) ds - k \int_0^t \frac{\partial^2 v}{\partial x^2}(\eta_i, s) ds + \frac{h^2}{24\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial^2 v}{\partial x^2}(\eta_i, s) ds,$$

by the boundedness of H^{-1} , we have

$$\|v_s - v_h\|_\infty = O(h^4). \quad (24)$$

Let $v_s(x, y) = \wedge_{xy} v(x, y)$. The error bounds (22) follow from (21), (24) and the use of triangular inequality. $\square$

From Theorem 1 we see that, when the solution function $u(x, t)$ satisfies $\frac{\partial u}{\partial t} \in C_{x,t}^{4,4}([a, b] \times [0, T])$, superconvergence can be obtained at all collocation points.

For analyzing the stability of this collocation method, let $v_h(x, t), \tilde{v}_h(x, t)$ be the numerical solutions of (2) subject to initial value $\phi(x), \tilde{\phi}(x)$, respectively. Supposing the disturbed initial value $\tilde{\phi}$ satisfies $\phi(x) - \tilde{\phi}(x) = E(x)$, then by (15), we have

$$L(v_h - \tilde{v}_h) = -\frac{kh^2}{24} \frac{\partial^4 E}{\partial x^4} + k \frac{\partial^2 E}{\partial x^2},$$

which reads

$$H(\mathbf{v} - \tilde{\mathbf{v}}) = \tilde{\mathbf{g}},$$

where

$$\tilde{\mathbf{v}} = (\tilde{v}_2, \tilde{v}_3, \dots, \tilde{v}_{N_h+1})^T, \tilde{v}_k = (\tilde{v}_{k,1}, \tilde{v}_{k,2}, \dots, \tilde{v}_{k,N_t+2}), k = 2, 3, \dots, N_h + 1$$

Table 1
Results of Example 1.

β	N	Proposed method				[23]	
		E_c	Rate	E_n	Rate	E_n	Rate
0.1	8	1.4778e−6		6.7030e−7		3.3208e−5	
	16	9.6260e−8	3.9404	7.7621e−8	3.1103	8.7408e−6	1.9257
	32	6.1416e−9	3.9702	9.2809e−9	3.0641	2.2339e−6	1.9682
	64	3.8783e−10	3.9851	1.1354e−9	3.0311	5.6396e−7	1.9859
	128	2.4365e−11	3.9925	1.3984e−10	3.0214	1.4160e−7	1.9937
0.4	8	1.4677e−6		8.2039e−7		1.0464e−4	
	16	9.5926e−8	3.9355	9.5367e−8	3.1047	2.3972e−5	2.1260
	32	6.1309e−9	3.9678	1.1439e−8	3.0595	5.8013e−6	2.0469
	64	3.8749e−10	3.9839	1.4013e−9	3.0291	1.4405e−6	2.0098
	128	2.4354e−11	3.9919	1.7141e−10	3.0312	3.6047e−7	1.9986
0.6	8	1.4605e−6		9.3916e−7		9.9510e−5	
	16	9.5688e−8	3.9320	1.0948e−7	3.1007	2.1673e−5	2.1990
	32	6.1233e−9	3.9660	1.3163e−8	3.0561	5.3568e−6	2.0165
	64	3.8724e−10	3.9830	1.6134e−9	3.0283	1.3501e−6	1.9883
	128	2.4347e−11	3.9914	1.9909e−10	3.0186	3.4006e−7	1.9892
0.9	8	1.4493e−6		1.1496e−6		3.0396e−5	
	16	9.5323e−8	3.9264	1.3463e−7	3.0941	6.8103e−6	2.1581
	32	6.1116e−9	3.9632	1.6245e−8	3.0509	1.7251e−6	1.9810
	64	3.8687e−10	3.9816	1.9940e−9	3.0263	4.3649e−7	1.9827
	128	2.4334e−11	3.9908	2.4728e−10	3.0114	1.0997e−7	1.9889

Table 2
Results of Example 2.

β	N	Proposed method				[23]	
		E_c	Rate	E_n	Rate	E_n	Rate
0.1	8	4.8476e−5		7.1895e−4		2.9027e−4	
	16	3.0771e−6	3.9776	2.5272e−4	1.5084	7.5111e−5	1.9503
	32	2.2205e−7	3.7926	1.0230e−4	1.3047	1.9004e−5	1.9827
	64	2.4144e−8	3.2011	3.7099e−5	1.4634	4.7706e−6	1.9941
	128	5.9729e−9	2.0152	1.0955e−5	1.7598	1.1942e−6	1.9981
0.4	8	4.7128e−5		7.2661e−4		1.2842e−3	
	16	3.9393e−6	3.5806	1.3531e−4	2.4249	3.0577e−4	2.0704
	32	8.8628e−7	2.1521	2.2995e−5	2.5569	7.5761e−5	2.0129
	64	1.9493e−7	2.1848	3.5985e−6	2.6759	1.8956e−5	1.9988
	128	4.2041e−8	2.2131	5.6047e−7	2.6827	4.7467e−6	1.9976
0.6	8	4.2389e−5		5.1661e−4		1.5855e−3	
	16	4.3046e−6	3.2997	6.6382e−5	2.9602	3.8705e−4	2.0344
	32	8.3307e−7	2.3694	8.5945e−6	2.9493	9.7561e−5	1.9882
	64	1.5485e−7	2.4276	1.1886e−6	2.8542	2.4521e−5	1.9923
	128	2.7868e−8	2.4742	1.8943e−7	2.6495	6.1461e−6	1.9963
0.9	8	3.5854e−5		4.1660e−4		1.0811e−3	
	16	2.2136e−6	4.0177	4.7903e−5	3.1205	2.7603e−4	1.9696
	32	1.3728e−7	4.0112	5.9668e−6	3.0051	6.9679e−5	1.9860
	64	1.7426e−8	2.9778	7.9856e−7	2.9015	1.7494e−5	1.9938
	128	2.3838e−9	2.8699	1.1657e−7	2.7762	4.3823e−6	1.9971

and

$$\tilde{\mathbf{g}}((Nt+2)(i-1)+j)) = -\frac{kh^2}{24} \frac{\partial^4 E}{\partial x^4}(\eta(i)) + k \frac{\partial^2 E}{\partial x^2}(\eta(i)).$$

Hence, with the help of the boundedness of $\|H^{-1}\|$, we can immediately obtain the following result about the stability of our presented numerical scheme.

Theorem 2. *Supposing $\phi(x), \tilde{\phi}(x) \in C^4([a, b])$, then the collocation scheme (15) is stable, and there is*

$$\|\mathbf{v} - \tilde{\mathbf{v}}\| \leq C \|\tilde{\mathbf{g}}\|,$$

where C is a positive constant independent of the time step τ .

Table 3
Results of Example 3.

β	N	Proposed method				[23]	
		E_c	Rate	E_n	Rate	E_n	Rate
0.1	8	1.4858e-4		4.0586e-4		1.4561e-4	
	16	3.6232e-5	2.0359	1.9133e-4	1.0849	3.5453e-5	2.0382
	32	8.6377e-6	2.0685	9.0073e-5	1.0869	8.4012e-6	2.0772
	64	2.0608e-6	2.0674	4.2931e-5	1.0691	1.9609e-6	2.0991
	128	4.8829e-7	2.0774	2.0251e-5	1.0840	4.5380e-7	2.1114
0.4	8	2.2963e-4		5.7557e-4		1.7615e-4	
	16	4.5564e-5	2.3333	2.2338e-4	1.3655	3.8107e-5	2.2087
	32	8.8307e-6	2.3673	8.7181e-5	1.3574	9.2580e-6	2.0413
	64	1.6920e-6	2.3838	3.3927e-5	1.3616	2.3548e-6	1.9751
	128	3.2272e-7	2.3904	1.3102e-5	1.3726	6.0351e-7	1.9642
0.6	8	1.5358e-4		3.5592e-4		2.1684e-4	
	16	2.6561e-5	2.5316	1.2181e-4	1.5469	4.6129e-5	2.2329
	32	4.4839e-6	2.5665	4.2122e-5	1.5320	1.1579e-5	1.9942
	64	7.4809e-7	2.5835	1.4337e-5	1.5548	2.9440e-6	1.9756
	128	1.2409e-7	2.5918	4.8556e-6	1.5620	7.4433e-7	1.9838
0.9	8	3.0087e-5		5.8889e-5		9.5992e-5	
	16	4.2334e-6	2.8293	1.6872e-5	1.8034	2.0752e-5	2.2097
	32	5.8094e-7	2.8654	4.9072e-6	1.7817	5.3157e-6	1.9649
	64	7.8755e-8	2.8829	1.3810e-6	1.8292	1.3475e-6	1.9799
	128	1.0613e-8	2.8915	3.8423e-7	1.8457	3.3902e-7	1.9909

Table 4
Results of Example 4.

β	N	Proposed method				[23]	
		E_c	Rate	E_n	Rate	E_n	Rate
0.1	8	1.5066e-4		4.2439e-4		1.4108e-4	
	16	3.6475e-5	2.0463	1.9399e-4	1.1294	3.4326e-5	2.0392
	32	8.6661e-6	2.0735	9.0927e-5	1.0932	8.1264e-6	2.0786
	64	2.0707e-6	2.0653	4.2449e-5	1.0990	1.8942e-6	2.1011
	128	4.8944e-7	2.0809	1.9903e-5	1.0927	4.3765e-7	2.1137
0.4	8	2.3272e-4		5.6495e-4		2.3991e-4	
	16	4.5854e-5	2.3435	2.1252e-4	1.4105	5.8177e-5	2.0440
	32	8.8582e-6	2.3720	8.0942e-5	1.3926	1.4828e-5	1.9721
	64	1.6946e-6	2.3861	3.0874e-5	1.3905	3.8010e-6	1.9639
	128	3.2346e-7	2.3893	1.1786e-5	1.3893	9.6953e-7	1.9710
0.6	8	1.5534e-4		3.3577e-4		3.2515e-4	
	16	2.6706e-5	2.5402	1.1172e-4	1.5876	8.0781e-5	2.0090
	32	4.4959e-6	2.5705	3.7180e-5	1.5873	2.0659e-5	1.9672
	64	7.4909e-7	2.5854	1.2428e-5	1.5809	5.2376e-6	1.9798
	128	1.2418e-7	2.5927	4.1452e-6	1.5841	1.3198e-6	1.9886
0.9	8	3.0225e-5		6.4926e-5		8.8893e-5	
	16	4.2438e-6	2.8323	1.8197e-5	1.8351	2.4143e-5	1.8805
	32	5.8169e-7	2.8670	5.0089e-6	1.8611	6.2412e-6	1.9517
	64	7.8807e-8	2.8839	1.3633e-6	1.8774	1.5839e-6	1.9783
	128	1.0616e-8	2.8921	3.6856e-7	1.8871	3.9884e-7	1.9897

4. Numerical examples

In this section, taking $k = 1$, $\theta = 0.25$, $\bar{I} = [0, 1]$ and $T = 1$, a few numerical examples covered two parts are reported to illustrate the effectiveness of this QSC method for the time fractional subdiffusion Eq. (2).

The first set of test problems includes four examples. In these examples, we compare the presented QSC method with the method recently proposed by Zeng [23]. By choosing $\beta = 0.1, 0.4, 0.6, 0.9$ respectively, we investigate the effects of the order β and smoothness of unknown function $u(x, t)$ on the accuracy of the proposed method. Besides testing the errors at all nodes, we also need to examine the errors at all collocation points.

Example 1. $u(x, t) = (t^2 + 2) \sin x \in C_{x,t}^{\infty, \infty}(\bar{I} \times [0, T])$, $f(x, t) = \frac{2t^{2-\beta} \sin x}{(1-\beta)(2-\beta)\Gamma(1-\beta)} + (t^2 + 2) \sin x$.

Example 2. $c = 2 + \beta$, $u(x, t) = (t^c - \frac{\Gamma(2-\beta)}{\pi^2}) \sin(\pi x) \in C_{x,t}^{\infty, 2}(\bar{I} \times [0, T])$, $f(x, t) = \frac{\Gamma(c+1)t^{c-\beta} \sin(\pi x)}{\Gamma(c+1-\beta)} + \pi^2(t^c - \frac{\Gamma(2-\beta)}{\pi^2}) \sin(\pi x)$.

Fig. 2. Errors at $t = 1$, left: $N = 8$, right: $N = 16$.

Example 3. $c = 2$, $u(x, t) = t^c x^{2+\beta} \in C_{x,t}^{2,\infty}(\bar{I} \times [0, T])$, $f(x, t) = \frac{\Gamma(c+1)t^{c-\beta}x^{2+\beta}}{\Gamma(c+1-\beta)} - t^c(2+\beta)(1+\beta)x^\beta$.

Example 4. $c = 2 + \beta$, $u(x, t) = t^c x^c \in C_{x,t}^{2,2}(\bar{I} \times [0, T])$, $f(x, t) = \frac{\Gamma(c+1)t^{c-\beta}x^c}{\Gamma(c+1-\beta)} - c(c-1)t^c x^{c-2}$.

Since the expected error of the proposed method is $O(h^4 + \tau^4)$ at all collocation points, we take the step lengths $h = \tau = \frac{1}{N}$. In the listed tables, we compute all the errors via using L_∞ norm, and we respectively denote by “ E_n , E_c ” the errors at all nodes $\{(x_i, t_j)\}_{i,j=1}^{N+1}$ and collocation points $\{(\eta_i, \tau_j)\}_{i,j=1}^{N+2}$. The convergence rate “Rate” is defined as

$$\text{Rate} = \log(\text{Error}(N/2)/\text{Error}(N))/\log(2).$$

From the results listed in these tables we can make the following observations.

1. When the solution function $u(x, t)$ is sufficiently smooth, the accuracy of the proposed QSC method at nodes is clearly higher than that of method in [23], this has been reflected in Table 1. Alternatively, for the relatively non-smooth $u(x, t)$, comparing with [23], the QSC method still shows preferable numerical behaviors when β is relatively large, this can be seen in Tables 2–4.
2. For the QSC method itself, the accuracy at collocation points is basically higher than that at nodes.
3. The accuracy of the QSC method has something to do with the smoothness of the solution function $u(x, t)$. It shows from Tables 1–4 that better smoothness leads to higher accuracy. On the contrary, the accuracy of the existed method in [23] relies less on the smoothness of true solution $u(x, t)$.

Now, in the second part, taking $\beta = 0.9$, we test the following practical problem [1,22].

$$\begin{cases} c \mathbf{D}_{0,t}^\beta u = \frac{\partial^2 u}{\partial x^2}, (x, t) \in (0, 1) \times (0, 1); \\ u(x, 0) = 4x(1-x), x \in (0, 1); \\ u(0, t) = 0, u(1, t) = 0, t \in (0, 1], \end{cases} \quad (25)$$

the exact solution of (25) can be represented by

$$u(x, t) = \frac{16}{\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^3} E_{\beta,1}(-k^2 \pi^2 t^\beta) (1 - (-1)^k) \sin(k\pi x),$$

where $E_{\beta,\gamma}$ is the Mittag-Leffler function of the form [3]

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + \gamma)}.$$

In Figs. 2 and 3, by taking $N = 8, 16, 32, 64$ respectively, we validate the corresponding errors between exact solutions and numerical solutions at $t = 1$.

From Figs. 2 and 3, we can see the presented collocation method is still available in the case of $u(x, t) \in C_{x,t}^{\infty,0}([0, 1] \times [0, 1])$.

Additionally, for contrastively investigating the behaviors of the fractional model (25) and corresponding classical model, taking $N = 32$ and $\beta = 0.5, 0.75, 0.95, 1$, Fig. 4 is plotted to illustrate the evolutions of normal diffusion and subdiffusion at $t = \tau(5), \tau(10), 1$.

Fig. 4 also shows that, for Eq. (2), larger fractional order β leads to faster decaying process, and it seems that the decay of subdiffusion is slower than that of the normal diffusion when t increases.

Fig. 3. Errors at $t = 1$, left: $N = 32$, right: $N = 64$.

Fig. 4. Solutions of fractional/classical models.

5. Conclusions

In the present paper, we have introduced a QSC method for numerically handling the time fractional subdiffusion Eq. (2) subject to Dirichlet boundary value conditions in space. Compared with the early established methods, theoretical analyses and

numerical examples demonstrate the proposed method is efficient and highly accurate when the solution function $u(x, t)$ satisfies $\frac{\partial u}{\partial t} \in C_{x,t}^{4,4}([a, b] \times [0, T])$. Moreover, the accuracy of the proposed method does not depend on the order β .

Our future work will focus on the extension of the proposed QSC method for handling two/three-dimension fractional sub-diffusion equations subject to various boundary value conditions. Additionally, we will improve the proposed QSC method such that it relies on weaker smoothness of the solution function $u(x, t)$ in the study. At the same time, the development of efficient solutions for the discretized systems of linear equations arising from subdiffusion equations remains an interesting topic of further research.

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