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Knauf's Degree and Monodromy in Planar Potential Scattering

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Abstract—We consider Hamiltonian systems on $(T^*\mathbb{R}^2, dq \wedge dp)$ defined by a Hamiltonian function H of the “classical” form $H = p^2/2 + V(q)$. A reasonable decay assumption $V(q) \rightarrow 0, \|q\| \rightarrow \infty$, allows one to compare a given distribution of initial conditions at $t = -\infty$ with their final distribution at $t = +\infty$. To describe this Knauf introduced a topological invariant $\deg(E)$, which, for a nontrapping energy $E > 0$, is given by the degree of the scattering map. For rotationally symmetric potentials $V(q) = W(\|q\|)$, scattering monodromy has been introduced independently as another topological invariant. In the present paper we demonstrate that, in the rotationally symmetric case, Knauf's degree $\deg(E)$ and scattering monodromy are related to one another. Specifically, we show that scattering monodromy is given by the jump of the degree $\deg(E)$, which appears when the nontrapping energy E goes from low to high values.

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1. INTRODUCTION

Scattering theory is of great importance to our understanding of nature. It allows one to study small or inaccessible objects, such as atoms and molecules or distant celestial bodies, via the known dynamics of test waves and particles. Many of the most important advances in elementary particle physics, which range from Rutherford's model of the atomic nucleus to the recent discovery of the Higgs boson, come from scattering.

The first theoretical approach to *quantum* scattering goes back to Born's paper *Zur Quantenmechanik der Stoßvorgänge* [2], published in 1926. Surprisingly, the corresponding *classical* scattering problem (of particles in a Newtonian force field) was only addressed forty years later in the works of Cook [3], Hunziker [10] and Simon [14]. Since then, the framework of *classical potential scattering* has been drawing more and more attention; see [9, 11] and the more recent works [1, 6, 7, 13].

In the present paper we consider the framework of classical *planar* potential scattering. Here classical point particles are moving under the influence of a conservative force field

$$F = -DV, \quad V: \mathbb{R}^2 \rightarrow \mathbb{R}.$$

The potential function V plays the role of the scatterer and it is assumed to satisfy a reasonable decay assumption $V(q) \rightarrow 0, \|q\| \rightarrow \infty$. If the decay is sufficiently fast, then the Hamiltonian flows of

$$H = \frac{1}{2}p^2 + V(q) \quad \text{and} \quad H_0 = \frac{1}{2}p^2$$

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become similar when $\|q\| \rightarrow \infty$. This allows one to compare a given distribution of particles at $t = -\infty$ (initial data) with their final distribution at $t = +\infty$ (asymptotic data).

Fig. 1. The projection of a trajectory on the q -plane. The potential function V has compact support $\text{supp} V = K$. Outside the “interaction zone” K the dynamics is trivial and is given by the Hamiltonian flow of H_0 : $t \mapsto (q_0 + tp_0, p_0)$.

To be more specific, assume that the energy $E > 0$ is such that no trajectory on the level surface $H^{-1}(E)$ is *trapped*. Then the initial and asymptotic data can be parametrized by points on the quotient space

$$H_0^{-1}(E)/g_{H_0}^{\mathbb{R}}$$

of the level surface $H_0^{-1}(E)$ by the flow of the “reference” Hamiltonian H_0 . This quotient space has a natural smooth structure, which makes it diffeomorphic to a cotangent bundle T^*S^1 ; see Remark 3. The flow of H induces the homeomorphism

$$S_E: T^*S^1 \rightarrow T^*S^1,$$

called the *energy E scattering map*. Its topological degree $\deg(E)$ was introduced by A. Knauf in [11] as an indicator of the presence of (nontrivial) scattering.

In the context of *Liouville integrable* systems another topological invariant, which indicates the presence of scattering, appears. It is called *scattering monodromy*.

Remark 1. Scattering monodromy was introduced in [1, 6] as a topological invariant of scattering in Liouville integrable systems with cylinder fibers. The notion of scattering monodromy is different from *standard Hamiltonian monodromy* in the sense of Duistermaat [5]. Specifically, it was shown in [1] that a 2-degree-of-freedom hyperbolic oscillator has trivial standard monodromy and nontrivial scattering monodromy. This implies, in particular, that scattering monodromy, as opposed to standard monodromy, does not obstruct the existence of global action-angle coordinates. Rather, scattering monodromy obstructs the existence of a global $\mathbb{R}$ -valued *deflection angle*; see Section 2.5. In the quantum case, the nonexistence of such a deflection angle entails the nonexistence of a globally smooth *phase shift* [6].

A class of systems for which both the degree $\deg(E)$ and scattering monodromy are defined is given by rotationally symmetric potentials

$$V: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad V(q) = W(\|q\|).$$

As we shall show in Section 3, the presence of scattering monodromy in such systems is related to the jump of the degree $\deg(E)$, which appears when the nontrapping energy $E > 0$ goes from low to high values.

2. PRELIMINARIES

2.1. Classical Potential Scattering

We consider Hamiltonian systems on $P = T^*\mathbb{R}^2$ defined by a Hamiltonian function H of the form *kinetic + potential energy*: $H = \frac{1}{2}p^2 + V(q)$. The time evolution of the system is defined by Hamilton's equations $\dot{q} = p$ and $\dot{p} = -DV$. Equivalently, one has Newton's second law of motion $\ddot{q} = -DV$.

Throughout the paper and unless stated otherwise, we assume that the potential function V is regular (that is, smooth everywhere on $\mathbb{R}^2$) and satisfies the following *short-range estimate*

$$\|D^n V\| < c\langle q \rangle^{-|n|-1-\varepsilon}, \quad (2.1)$$

where $n = (n_1, n_2) \in \mathbb{N}_0^2$ is a multi-index with the norm $|n| = n_1 + n_2$, the constants c, ε are strictly positive and $\langle q \rangle = \sqrt{\|q\|^2 + 1}$.

The decay estimate (2.1) implies that, for any $E > 0$, there exists a *virial radius* $R_{vir} = R_{vir}(E) > 0$ such that

$$\|V(q)\| < E/2 \quad \text{and} \quad \langle q, DV \rangle < E/2$$

outside the *interaction zone*

$$IZ(E) = \{q \in \mathbb{R}^2 \mid \|q\| \leq R_{vir}\}.$$

A trajectory $g_H^t(x_0) = (q(t, x_0), p(t, x_0))$ with energy $H = E$, $E > 0$, that leaves the interaction zone $IZ(E)$ cannot reenter it, but goes to spatial infinity. Moreover, for such a trajectory $\lim_{t \rightarrow \infty} p(t) \in (\mathbb{R}^2)^*$ exists and $q(t)$ asymptotically approaches a straight line. This is illustrated for the example of a potential with compact support in Fig. 1. The proofs can be found in Knauf [12].

2.2. Scattering Data

The Hamiltonian flow $g_H^t: P \rightarrow P$ of H partitions the phase space P into the following invariant subsets:

$$b^\pm = \{x \in P \mid \sup_{t \in \mathbb{R}^\pm} \|g_H^t(x)\| < \infty\} \quad \text{and} \quad s^\pm = P \setminus b^\pm.$$

The invariant subsets

$$b = b^+ \cap b^-, \quad s = s^+ \cap s^- \quad \text{and} \quad trp = (b^+ \cap s^-) \cup (b^- \cap s^+)$$

are the *bound*, the *scattering* and the *trapped* states, respectively. We note that s^-, s^+ and hence $s^- \cap s^+$ are open subsets of P .

Let

$$P_0 = \{(q, p) \in T^*\mathbb{R}^2 \mid H_0(q, p) = p^2/2 > 0\}$$

be the set of states with nonzero momentum p . Then the *Møller transformations*

$$M^\pm = \lim_{t \rightarrow \pm\infty} g_H^{-t} \circ g_{H_0}^t: P_0 \rightarrow s^\pm$$

are defined.

Theorem 1 ([14]). *The Møller transformations $M^\pm$ are symplectic diffeomorphisms from P_0 onto their images $s^\pm$.*

From Theorem 1 it follows that the limits

$$\hat{p}^\pm(x) = \lim_{t \rightarrow \pm\infty} \frac{p(t, x)}{\sqrt{2E}} \quad \text{and} \quad q_\perp^\pm(x) = \lim_{t \rightarrow \pm\infty} (q(t, x) - \langle q(t, x), \hat{p}^\pm(x) \rangle \hat{p}^\pm(x)),$$

called the *asymptotic direction* and the *impact parameter* of the trajectory $g_H^t(x)$, $x \in s^\pm$, are defined and depend smoothly on x . Since the asymptotic direction is always orthogonal to the corresponding impact parameter, we get the maps

$$(\hat{p}^\pm, q_\perp^\pm): s^\pm \rightarrow T^*S^1,$$

which are smooth and g_H^t -invariant.

Remark 2. The asymptotic direction and the impact parameter of $g_{H_0}^t(x)$ are obviously well defined and smooth whenever $x \in P_0$. It follows from Theorem 1 that the same is true for $g_H^t(x)$ whenever $x \in s^\pm$.

2.3. Scattering Map

Fix an energy $E > 0$. Due to the g_H^t -invariance, the maps $(\hat{p}^\pm, q_\perp^\pm)$ descend to the maps

$$A_E^\pm = (\hat{p}^\pm, q_\perp^\pm): (H^{-1}(E) \cap s^\pm)/g_H^\mathbb{R} \rightarrow T^*S^1$$

from the space of trajectories of $X_H = (p, -DV)$ on $H^{-1}(E) \cap s^\pm$ to T^*S^1 .

Remark 3. The cotangent bundle T^*S^1 can be identified with the space of trajectories of X_{H_0} on $H_0^{-1}(E)$:

$$T^*S^1 \simeq H_0^{-1}(E)/g_{H_0}^\mathbb{R}.$$

Proof. First, we note that $H_0^{-1}(E)/g_{H_0}^\mathbb{R}$ is a smooth manifold. This follows from the fact that the $\mathbb{R}$ action

$$(t, x) \mapsto g_{H_0}^t(x), \quad g_{H_0}^t(q_0, p_0) = (q_0 + tp_0, p_0), \quad (2.2)$$

is free and proper on $H_0^{-1}(E)$. Since T^*S^1 and $T_{p_0}^*\{p \mid p^2 = 2E\}$ differ only in scaling, it is sufficient to prove that the manifolds $T_{p_0}^*\{p \mid p^2 = 2E\}$ and $H_0^{-1}(E)/g_{H_0}^\mathbb{R}$ are diffeomorphic. The latter follows from Eq. (2.2) and the fact that the line $T_{p_0}^*\{p \mid p^2 = 2E\}$ is orthogonal to the line $t \mapsto q_0 + tp_0$. $\square$

Definition 1 ([11]). An energy $E > 0$ is called *nontrapping* if $H^{-1}(E)$ has no trapped states: $H^{-1}(E) \cap \text{trp} = \emptyset$.

By the definition, for a nontrapping energy $E > 0$, unbounded trajectories go to spatial infinity in both forward and backward time:

$$s^+ \cap H^{-1}(E) = s^- \cap H^{-1}(E) = s \cap H^{-1}(E).$$

These trajectories are asymptotic to the energy E trajectories of the ‘reference’ Hamiltonian H_0 . Moreover, the maps $A_E^\pm$ are bijective homeomorphisms [4]. The composition map

$$S_E = A_E^+ \circ (A_E^-)^{-1}: T^*S^1 \rightarrow T^*S^1$$

is called *the energy $E > 0$ scattering map* [13].

Remark 4. In the case of the Hamiltonian $H_0 = \frac{1}{2}p^2$ every energy $E > 0$ is nontrapping and the scattering map S_E is the identity map. In general, for every nontrapping energy $E > 0$ the scattering map S_E is a symplectic diffeomorphism of T^*S^1 ; see [4, 11]. This result will not be used in the sequel.

2.4. The Degree of the Scattering Map

Let $\text{Pr}: T^*S^1 \rightarrow S^1$ be the canonical projection of the cotangent bundle T^*S^1 and

$$S_{p,\perp}^1 = T_p^*S^1 \cup \{\infty\}$$

be the one-point compactification of the cotangent space $T_p^*S^1$. Since

$$\lim_{q \rightarrow \infty} S_E(p, q) = p,$$

there is a unique continuous extension of $S_E|_{T_p^*S^1}$ to the point $\{\infty\}$. We denote this extension by $S_E^{p,\perp}$.

Definition 2 ([11]). *The degree of the energy E scattering map S_E (notation: $\deg(E)$) is defined as the topological degree of the map*

$$\text{Pr} \circ S_E^{p,\perp} : S_{p,\perp}^1 \rightarrow S^1.$$

Remark 5. We note that $\deg(E)$ is independent of the choice of the initial direction $p \in S^1$ since the maps $S_E^{p,\perp}$ depend continuously on p ; see [11].

The following theorem shows that $\deg(E) \in \{0, 1\}$, depending on the value of the energy; see Fig. 2.

Theorem 2 ([13]). *Let V be a short-range potential and $E > 0$ be a nontrapping energy. Then*

$$\deg(E) = \begin{cases} 0, & E \in (\sup V, \infty), \\ 1, & E \in (0, \sup V). \end{cases}$$

Fig. 2. Left figure: for “high” energies $E > \sup V \geq 0$ the topological effect of the scattering can be neglected, so $\deg(E) = 0$. Right figure: for “small” energies $0 < E < \sup V$ trajectories get reflected from the potential barrier $\{V \leq E\}$ and, as a result, $\deg(E) = 1$.

Proof. We consider the case of a rotationally symmetric potential V (a proof in the general case can be found in [13]).

Let $E \in (\sup V, \infty)$ be nontrapping. Consider the family

$$V_t = t \cdot V, \quad t \in [0, 1],$$

of regular, rotationally symmetric and short-range potentials. For each potential V_t the energy E is nontrapping. By homotopy, we can thus assume that $V = 0$. But for the “reference” Hamiltonian $H_0 = \frac{1}{2}p^2$ the scattering map S_E is the identity map and, therefore, $\deg(E) = 0$.

Let $E \in (0, \sup V)$. In this case we cannot deform V into zero since the energy E is trapping (for $t \cdot V$) when $t \cdot \sup V = E$. It is possible, however, to deform V into the step function

$$V_0 = \begin{cases} \sup V, & q \in B_1(0), \\ 0, & q \in \mathbb{R}^2 \setminus B_1(0), \end{cases}$$

where $B_1(0) = \{q \in \mathbb{R}^2 \mid \|q\| < 1\}$ is the unit ball in $\mathbb{R}^2$.

More specifically, there exists a family V_t , $t \in [0, 1]$, of rotationally-symmetric potential functions such that

- 1) the functions V_t depend continuously on $t \in [0, 1]$ and converge to V_0 in the sense of [15];
- 2) each function V_t , $t \in (0, 1]$, is smooth and short-range;
- 3) for each function V_t , $t \in (0, 1]$, the energy E is nontrapping;
- 4) $V_1 = V$ and V_0 is the step function.

The existence of such a family follows from the nontrapping assumption on the energy $E \in (0, \sup V)$. For each function V_t we let $\deg(E)(t)$ be the degree of the associated energy E scattering map.

Remark 6. The energy E scattering map of the Hamiltonian $p^2/2 + V_0$ and the degree $\deg(E)(0)$ can be defined as in the previous subsections, even though V_0 is not smooth.

From assumptions 1–4 it follows that $\deg(E)(t)$ does not depend on t . Hence we only need to consider the case of the step potential V_0 and compute the degree $\deg(E)(0)$. But the dynamics of $p^2/2 + V_0$ on the energy- E surface is that of a billiard in $\mathbb{R}^2 \setminus B_1(0)$. For such a billiard the degree $\deg(E)(0) = 1$. The result follows. $\square$

Remark 7. For singular potentials V , such as the Kepler potential, values different from 0 and 1 may appear; see [11].

2.5. Scattering Monodromy

Suppose that the potential V is rotationally symmetric: $V(q) = W(\|q\|)$. Then the angular momentum $L = q_1 p_2 - q_2 p_1$ is conserved and the Hamiltonian system $(T^*\mathbb{R}^2, dq \wedge dp, H)$ is integrable. Denote by $F = (H, L)$ the *energy-momentum map* and let

$$NT = \{(E, l) \in \text{image}(F) \mid E > 0 \text{ and } F^{-1}(E, l) \text{ is nontrapping}\}. \quad (2.3)$$

Let $\text{Pr}_q: T^*\mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the canonical projection to the configuration q -plane and φ be a polar angle in this q -plane. For each $(E, l) \in NT$, define the *total deflection angle* by

$$\Phi(E, l) = \int_{\text{Pr}_q(g_H^{\mathbb{R}}(x))} d\varphi,$$

where x is any point in the unbounded part of $F^{-1}(E, l)$. We note that the unbounded part of $F^{-1}(E, l)$ is contained in s . Since the maps $\hat{p}^\pm$ and $q_\perp^\pm$ are continuous on $s^\pm \supset s$, it follows that the map $\Phi: NT \rightarrow S^1$ is continuous.

Definition 3. Let $\gamma: S^1 \rightarrow NT$ be a continuous path. *Scattering monodromy along γ* is defined as the topological degree of the map $\Phi \circ \gamma$.

Remark 8. In [6] scattering monodromy is defined as a variation of the angle $\delta(\gamma(t)) = \Phi(\gamma(t)) - \pi$ along γ , which is the difference between the total deflection angles of outgoing *scattered* and *unscattered* trajectories. It follows from

$$\deg \Phi = \frac{1}{2\pi} \int_{\gamma} d\delta$$

that the two definitions are equivalent.

Remark 9. The assumption that $F^{-1}(\gamma)$ has no trapping states implies that γ does not intersect the critical values of $F|_{(s^+ \cap s^-)}$. It may happen, however, that γ intersects critical values of F . Such critical values necessarily come from the “compact part” of the Liouville *foliation*; see Example 1.

The following theorem shows that the critical value $(\sup V, 0) \in \text{image}(F)$ of the energy-momentum map F manifests the presence of scattering monodromy.

Theorem 3 ([6]). *Scattering monodromy along a given nontrapping path γ is given by its winding number with respect to the critical value $(\sup V, 0) \in \text{image}(F)$.*

Proof. The result has been proven in [6] for repulsive potentials. The general case can be treated similarly. For the sake of completeness we provide a proof below.

Since the potential V is rotationally symmetric, for every $l > 0$ the total deflection angle is given by

$$\Phi(E, l) = 2 \int_{r_{\min}}^{\infty} \frac{l dr}{r^2 \sqrt{E - W_l(r)}}, \quad (2.4)$$

where $W_l(r) = l^2/2r^2 + W(r)$, $W(\|q\|) = V(q)$, is the effective potential. Without loss of generality we can assume that γ is a smooth path that intersects the $l = 0$ axis at a right angle. Then it is left to show that for a nontrapping energy E

$$\lim_{l \rightarrow 0^+} \Phi(E, l) = \begin{cases} \pi, & \text{when } E \in (\sup V, \infty) \\ 0, & \text{when } E \in (0, \sup V). \end{cases}$$

Consider a trajectory $g_H^t(x) = (q(t, x), p(t, x))$ with $l = 0$ and a nontrapping energy $E > 0$. The projection $q(t, x)$ is then constrained to a straight line. We have two cases.

1. $E \in (0, \sup V)$. In this case $q(t, x)$ is reflected from the potential barrier when $t = \min\{\tau \mid p(\tau, x) = 0\}$. It follows that $\Phi(E, 0) = 0$.
2. $E \in (\sup V, \infty)$. In this case the vector $\dot{q}(t, x) = p(t, x)$ is bounded away from zero and its direction is constant. Hence $|\Phi(E, 0)| = \pi$.

Since in both cases $\lim_{l \rightarrow 0^+} \Phi(E, l) = |\Phi(E, 0)|$, the result follows. $\square$

3. A HOMOTOPY ARGUMENT

We consider Hamiltonian systems on $T^*\mathbb{R}^2$ defined by a smooth rotationally symmetric short-range potential V . We denote the angular momentum by $L = q_1 p_2 - q_2 p_1$ and the energy-momentum map by $F = (H, L)$. Let NT be defined as in (2.3).

The following lemma shows that the degree $\deg(E)$ of the nontrapping energy E scattering map is given by the variation of the total deflection angle $\Phi(E, l)$, just as scattering monodromy.

Lemma 1 ([11]). *Let $\gamma: \mathbb{R} \rightarrow NT$ be given by $\gamma(t) = (E, t)$. Then*

$$\deg(E) = \frac{1}{2\pi} \int_{\gamma} d\Phi.$$

Proof. For a sufficiently large positive number N_0 we define the following map:

$$M = \Phi \circ F \circ f: [0, 1] \times [N_0, \infty) \times (\mathbb{R} \cup \{\infty\}) \rightarrow S^1,$$

where the restriction of the map f on $[0, 1] \times [N_0, \infty) \times \mathbb{R}$ is given by

$$(\tau, N, q_1) \mapsto \left(0, \sqrt{2(E - \tau V(q_1, -N))}\right) \in T_{(q_1, -N)}^* \mathbb{R}^2.$$

Since $V(q) \rightarrow 0$, $\|q\| \rightarrow \infty$, we have that $\lim_{q_1 \rightarrow \infty} f(\tau, N, q_1) = (0, \sqrt{2E})$. It follows that f and M are well-defined continuous functions. By homotopy invariance, we have

$$\frac{1}{2\pi} \int_{\gamma} d\Phi = \deg(M(0, N)) = \deg(M(1, N)) = \lim_{N \rightarrow \infty} \deg(M(1, N)).$$

The equality

$$\lim_{N \rightarrow \infty} \deg(M(1, N)) = \deg(E)$$

completes the proof. $\square$

Let E_{low}, E_{high} be nontrapping energies. Set

$$R = \{E_{low} \leq H \leq E_{high}\} \cap (\{L \leq l_{-\alpha}\} \cup \{L \geq l_{+\alpha}\})$$

and assume that this region satisfies $R \subset NT$; see Fig. 3. Finally, let γ_α be the boundary of the ‘box region’

$$\{E_{low} \leq H \leq E_{high}\} \cap \{l_{-\alpha} \leq L \leq l_{+\alpha}\}.$$

With this notation we have the following result.

Fig. 3. Bifurcation diagram of the energy-momentum map $F = (H, L)$. The potential function $V(q) = W(\|q\|)$ is given by $W(r) = r^2/(1 + r^4)$. The curves γ and γ_α surround the transversally hyperbolic branch of critical values of F .

Theorem 4. *Scattering monodromy along γ_α is equal to the difference $\deg(E_{low}) - \deg(E_{high})$.*

Proof. Observe that

$$\deg(\Phi|_{\gamma_\alpha}) = \frac{1}{2\pi} \int_{\gamma_\alpha} d\Phi = \frac{1}{2\pi} \left(\int_{E=E_{low}} - \int_{E=E_{high}} \right) d\Phi - \frac{1}{2\pi} \int_{\partial R} d\Phi.$$

The region R consists of two connected components, which we denote by R_1 and R_2 . By homotopy invariance, we can assume that $l_\alpha > 0$ and $l_{-\alpha} < 0$, so that $l < 0$ on R_1 and $l > 0$ on R_2 . Since $\lim_{l \rightarrow \infty} \Phi(E, l) = \pi$, Eq. (2.4) implies that the variation of Φ along ∂R_1 is zero:

$$\int_{\partial R_1} d\Phi = \lim_{l \rightarrow +\infty} \Phi(E_{low}, l) - \lim_{l \rightarrow +\infty} \Phi(E_{high}, l) = 0.$$

Similarly, the variation of Φ along ∂R_2 is zero. We conclude that

$$\deg(\Phi|_{\gamma_\alpha}) = \frac{1}{2\pi} \left(\int_{E=E_{low}} - \int_{E=E_{high}} \right) d\Phi.$$

It is left to apply Lemma 1. □

Example 1. Let the potential $V(q) = W(\|q\|)$ be given by $W(r) = r^2/(1 + r^4)$. The graph of the function W is given in Fig. 4. The bifurcation diagram of the energy-momentum map $F = (H = p^2/2 + V, L)$ is illustrated in Figs. 3 and 4.

The path γ surrounds the transversally hyperbolic branch of critical values of F and intersects the transversally elliptic branches of critical values (emanating from the origin $H = L = 0$).

Fig. 4. Left figure: the graph of $W = r^2/(1 + r^4)$. Right figure: the bifurcation diagram of $F = (H = p^2/2 + V, L)$.

Corollary 1. *Scattering monodromy along γ equals 1.*

Proof. The statement follows from Theorem 3. Alternatively, one may note that, by homotopy invariance, scattering monodromy along γ is equal to scattering monodromy along γ_α . $\square$

4. DISCUSSION

We have shown that, for systems with rotationally symmetric short-range potentials $V: \mathbb{R}^2 \rightarrow \mathbb{R}$, the degree $\deg(E)$ of the energy E scattering map and scattering monodromy are related via the simple formula

$$\deg(\Phi \circ \gamma_\alpha) = \deg(E_{low}) - \deg(E_{high}).$$

The proof was based on the fact that both the degree and scattering monodromy are given by the variation of the deflection angle Φ and the fact that this variation is invariant under suitable deformations (homotopy invariance of the degree). For standard monodromy such a result is known; see Cushman's proof of monodromy in the spherical pendulum [5] and the recent work on monodromy and Chern classes [8].

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