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Published in:
Computer aided geometric design

DOI:
[10.1016/j.cagd.2016.02.003](https://doi.org/10.1016/j.cagd.2016.02.003)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2016

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Kosinka, J., & Barton, M. (2016). Convergence of barycentric coordinates to barycentric kernels. *Computer aided geometric design*, 43, 200-210. <https://doi.org/10.1016/j.cagd.2016.02.003>

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Convergence of barycentric coordinates to barycentric kernels

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ARTICLE INFO

Article history:

Available online 12 February 2016

Keywords:

Barycentric coordinates

Barycentric kernel

Convergence

ABSTRACT

We investigate the close correspondence between barycentric coordinates and barycentric kernels from the point of view of the limit process when finer and finer polygons converge to a smooth convex domain. We show that any barycentric kernel is the limit of a set of barycentric coordinates and prove that the convergence rate is quadratic. Our convergence analysis extends naturally to barycentric interpolants and mappings induced by barycentric coordinates and kernels. We verify our theoretical convergence results numerically on several examples.

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1. Introduction

Generalised barycentric coordinates have become a key tool in many applications in geometric modelling and computer graphics, ranging from shading, deformation, and animation to colour and shape blending. Wachspress coordinates (Wachspress, 1975; Meyer et al., 2002) and mean value coordinates (Floater, 2003; Hormann and Floater, 2006) belong to the most popular generalised barycentric coordinates for polygons in the plane; see also Floater (2014). These two examples of coordinates fall into the family of three-point coordinates (Floater et al., 2006). Other barycentric coordinates for polygons that do not belong to the three-point family exist, including complex coordinates (Weber et al., 2009) and local barycentric coordinates (Zhang et al., 2014). Various applications of barycentric coordinates were discussed and summarised in Gillette et al. (2012), Rustamov (2010), Schneider et al. (2013), Hormann (2014).

In certain scenarios such as transfinite interpolation (Ju et al., 2005; Warren et al., 2007; Dyken and Floater, 2009), it is preferable to deal with smooth domains rather than polygons. Considering denser and denser polygons inscribed into a smooth domain leads to a (convergent) series of barycentric coordinates. This limit process was recently investigated in the case of Wachspress coordinates in Kosinka and Bartoň (2015). It was shown that in the limit, as the number of vertices of the polygon approaches infinity, one obtains the Wachspress kernel (Warren et al., 2007) introduced by Warren et al., and that the convergence rate is quadratic. Barycentric kernels constitute a natural generalisation of barycentric coordinates from polygons to smooth domains and their applications include deformation and transfinite interpolation (Warren et al., 2007). For more details on generalised barycentric coordinates and kernels, consult the recent survey (Floater, 2015).

In this work, we investigate the same type of convergence for the complete family of barycentric coordinates (Floater et al., 2006), which includes Wachspress and mean value coordinates as special cases. Building on the approach taken in Kosinka and Bartoň (2015), we show that any set of barycentric coordinates converges quadratically to a well-defined limit given by the corresponding barycentric kernel (Belyaev, 2006; Schaefer et al., 2007).

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Fig. 1. Distances and signed triangle areas in the definition of three-point coordinates; see (7). The i -th weight w_i of $\mathbf{x}$ in a polygon P depends only on three consecutive vertices $\mathbf{p}_{i-1}$, $\mathbf{p}_i$, and $\mathbf{p}_{i+1}$.

After recalling some basic concepts and introducing our notation for barycentric coordinates and kernels (Section 2), we study the convergence of barycentric coordinates for a sequence of convex polygons converging to a smooth convex domain (Section 3). Our convergence analysis carries over to interpolants and mappings based on barycentric coordinates and kernels (Section 4). Numerical examples based on our theoretical convergence analysis are presented in Section 5. We conclude the paper by discussing the results and their relevance in Section 6.

2. Barycentric coordinates and kernels

The classical barycentric coordinates for triangles are unique, but there exist various generalised barycentric coordinates for planar polygons with more than three vertices. Let the vertices of a strictly convex polygon P be $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$, ordered anticlockwise. (Generalised) barycentric coordinates $\lambda_i : \bar{P} \rightarrow \mathbb{R}$, $i = 1, \dots, n$ for P are given by the following three properties:

(P1) Non-negativity

$$\lambda_i(\mathbf{x}) \geq 0, \quad i = 1, \dots, n; \quad (1)$$

(P2) Partition of unity

$$\sum_{i=1}^n \lambda_i(\mathbf{x}) = 1; \quad (2)$$

(P3) Barycentric property

$$\sum_{i=1}^n \lambda_i(\mathbf{x}) \mathbf{p}_i = \mathbf{x}; \quad (3)$$

for all $\mathbf{x} \in \bar{P}$, where $\bar{P}$ denotes the closure of P , which is itself understood as an open set.

Several other properties follow from (P1)–(P3), including the Lagrange property, i.e., $\lambda_i(\mathbf{p}_j) = \delta_{i,j}$, and the interpolation property of piece-wise linear functions defined on ∂P , the boundary of P , for continuous coordinates (Floater et al., 2006).

For later use, we define the three signed triangle areas

$$A_i(\mathbf{x}) = \mathcal{A}(\mathbf{x}, \mathbf{p}_i, \mathbf{p}_{i+1}), \quad B_i(\mathbf{x}) = \mathcal{A}(\mathbf{x}, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}), \quad C_i = \mathcal{A}(\mathbf{p}_{i-1}, \mathbf{p}_i, \mathbf{p}_{i+1}), \quad (4)$$

and the distance $r_i(\mathbf{x}) = \|\mathbf{x} - \mathbf{p}_i\|$; see Fig. 1. The indices of A_i , B_i , $\mathbf{p}_i$ and so on are treated cyclically with respect to n .

Barycentric coordinates $\lambda_i(\mathbf{x})$ are usually defined in terms of weights $w_i(\mathbf{x})$ via

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{W(\mathbf{x})}, \quad W(\mathbf{x}) = \sum_{j=1}^n w_j(\mathbf{x}); \quad \mathbf{x} \in \bar{P}. \quad (5)$$

In Floater et al. (2006, Corollary 3), it was shown that any set of weights (also called homogeneous coordinates) is of the form

$$w_i(\mathbf{x}) = \frac{c_{i+1}(\mathbf{x})A_{i-1} - c_i(\mathbf{x})B_i + c_{i-1}(\mathbf{x})A_i}{A_{i-1}A_i} \quad (6)$$

for some set of real bivariate functions $c_i(\mathbf{x})$ defined on P .

A particularly interesting family of barycentric coordinates for convex polygons are the three-point coordinates (Floater et al., 2006), given by the weights

$$w_i(p, \mathbf{x}) = \frac{r_{i+1}^p A_{i-1} - r_i^p B_i + r_{i-1}^p A_i}{A_{i-1} A_i}, \quad (7)$$

obtained by setting $c_i = r_i^p$, where p is a real parameter. As was shown in Floater et al. (2006), $\lambda_i(p, \mathbf{x})$ form a set of coordinates satisfying (P2) and (P3) for any value of p . On the other hand, (P1) generically holds for $p = 0$ and $p = 1$ only. Setting $p = 0$ leads to Wachspress coordinates (Wachspress, 1975) with

$$w_i(0, \mathbf{x}) = \frac{C_i}{A_{i-1} A_i} \quad (8)$$

and $p = 1$ gives mean value coordinates (Floater, 2003) with

$$w_i(1, \mathbf{x}) = \frac{r_{i+1} A_{i-1} - r_i B_i + r_{i-1} A_i}{A_{i-1} A_i}. \quad (9)$$

The coordinates for other values of p are not, in general, non-negative. However, they can still be used for interpolation or in other applications. An interesting example is given by $p = 2$, leading to discrete harmonic coordinates (Pinkall and Polthier, 1993; Eck et al., 1995).

In this paper, and in accord with the standard literature, coordinates satisfying (P2) and (P3), but not necessarily (P1), will still be called barycentric.

Three-point coordinates, among other types of coordinates, are defined for convex polygons with an arbitrary number $n \geq 3$ of vertices. One may thus ask what happens when the number of vertices approaches infinity and the polygon approaches a smooth convex domain. This view was taken in Kosinka and Bartoň (2015) and it was shown that Wachspress coordinates converge quadratically in this process to the Wachspress kernel (Warren et al., 2007).

Barycentric kernels are defined on smooth domains very similarly to their polygonal counterparts, barycentric coordinates. Let Ω be a bounded, open, strictly convex domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. Let $\mathbf{p} : [a, b] \rightarrow \mathbb{R}^2$, with $\mathbf{p}(a) = \mathbf{p}(b)$, be a continuous parametrisation of $\partial\Omega$. To avoid degenerate cases, we assume that $\mathbf{p}$ is injective in $[a, b)$. A kernel $\lambda(\mathbf{x}, t) : \Omega \rightarrow \mathbb{R}$ is called barycentric if it satisfies for all $\mathbf{x} \in \Omega$:

(Q1) Non-negativity

$$\lambda(\mathbf{x}, t) \geq 0, \quad t \in [a, b]; \quad (10)$$

(Q2) Partition of unity

$$\int_a^b \lambda(\mathbf{x}, t) dt = 1; \quad (11)$$

(Q3) Barycentric property

$$\int_a^b \lambda(\mathbf{x}, t) \mathbf{p}(t) dt = \mathbf{x}. \quad (12)$$

We emphasise that (Q2) and (Q3) are integral equivalents of (P2) and (P3), respectively. The Lagrange property is now replaced by the Dirac delta function property, namely $\lambda(\mathbf{p}(u), t) = \delta(u - t)$. The interpolation property of barycentric kernels was, for smooth functions defined on $\partial\Omega$, established in Floater and Kosinka (2010a).

Similarly to coordinates for polygons, see Eq. (5), barycentric kernels are typically defined by a weight function $w(\mathbf{x}, t)$, which is normalised to give the kernel itself

$$\lambda(\mathbf{x}, t) = w(\mathbf{x}, t) / \int_a^b w(\mathbf{x}, s) ds. \quad (13)$$

In Section 2.3 of Dyken and Floater (2009), the authors suggest three approaches to evaluating the above integrals. One of them is based on a certain angular formula, which does not concern us here. Another is based directly on the integrals presented above, using a quadrature rule. The third option is to use the corresponding discrete version of the kernel based on barycentric coordinates and evaluate these coordinates for a sufficiently dense polygonal approximation of the smooth domain Ω . We show in Section 3 that the latter two approaches are essentially the same and that the limit process involved converges quadratically.

A complete classification of barycentric kernels was presented in Belyaev (2006) and independently in Schaefer et al. (2007). We follow the approach of Schaefer et al. (2007), where it was also shown that barycentric kernels reduce to barycentric coordinates when Ω is a (correctly interpreted) convex polygon.

We denote by $\cdot$ the dot-product and by $\times$ the planar cross-product of two vectors in $\mathbb{R}^2$. Given $\mathbf{p}(t) \in C^2[a, b]$, we define $\mathbf{d}(\mathbf{x}, t) = \mathbf{p}(t) - \mathbf{x}$. Any barycentric kernel on Ω can be expressed via a function $c(\mathbf{x}, t)$ by the weight function (see Eq. (10) in Schaefer et al. (2007))

$$w(\mathbf{x}, t) = \frac{\mathbf{N}(\mathbf{D}(\mathbf{s}(\mathbf{x}, t))) \cdot \mathbf{D}(\mathbf{s}(\mathbf{x}, t))}{c(\mathbf{x}, t)}, \quad (14)$$

where $\mathbf{s}(\mathbf{x}, t) = \mathbf{d}(\mathbf{x}, t)/c(\mathbf{x}, t)$, the operator $\mathbf{D}$ returns the dual curve

$$\mathbf{D}(\mathbf{p}(t)) = \frac{\mathbf{N}(\mathbf{p}(t))}{\mathbf{N}(\mathbf{p}(t)) \cdot \mathbf{p}(t)} \quad (15)$$

of $\mathbf{p}(t)$, and $\mathbf{N}(\mathbf{p}(t))$ denotes the outward normal of $\mathbf{p}(t)$ of the same magnitude as $\mathbf{p}'(t)$, i.e.,

$$\mathbf{N}(\mathbf{p}(t)) = (p'_y(t), -p'_x(t)) \quad (16)$$

for anticlockwise orientation of $\mathbf{p}(t) = (p_x(t), p_y(t))$.

For later use, we expand (14). To simplify notation, we omit the dependence of the functions on t and $\mathbf{x}$. All derivatives are with respect to t .

With $\mathbf{D}(\mathbf{u}) = (D_x, D_y)$, $\mathbf{N}(\mathbf{D}(\mathbf{u})) = (D'_x, -D'_y)$ and thus $\mathbf{N}(\mathbf{D}(\mathbf{u})) \cdot \mathbf{D}(\mathbf{u}) = \mathbf{D}(\mathbf{u}) \times \mathbf{D}'(\mathbf{u})$ for any $\mathbf{u} \in C^2[a, b]$. We have

$$\mathbf{D}(\mathbf{s}) = \frac{(cp'_y - c'd_y, -cp'_x + c'd_x)}{\mathbf{d} \times \mathbf{p}'} \quad (17)$$

and thus

$$\mathbf{D}'(\mathbf{s}) = \frac{(\mathbf{d} \times \mathbf{p}') (cp''_y - c''d_y, -cp''_x + c''d_x) - (\mathbf{d} \times \mathbf{p}'') (cp'_y - c'd_y, -cp'_x + c'd_x)}{(\mathbf{d} \times \mathbf{p}')^2}. \quad (18)$$

After simplifications of the cross product $\mathbf{D}(\mathbf{s}) \times \mathbf{D}'(\mathbf{s})$ divided by c , we obtain

$$w = \frac{c(\mathbf{p}' \times \mathbf{p}'') - c'(\mathbf{d} \times \mathbf{p}'') + c''(\mathbf{d} \times \mathbf{p}')}{(\mathbf{d} \times \mathbf{p}')^2}. \quad (19)$$

The function $c(\mathbf{x}, t)$ is assumed to be twice differentiable with respect to t .

Similarly to the polygonal case, an interesting family of barycentric kernels $\lambda(p, \mathbf{x}, t)$ is obtained by setting $c = r^p$ with $r = \|\mathbf{d}\|$ in (19) and using (13). The most prominent examples of barycentric kernels are the Wachspress kernel $\lambda(0, \mathbf{x}, t)$ with $p = 0$ and the mean value kernel $\lambda(1, \mathbf{x}, t)$ with $p = 1$. The Wachspress kernel, proposed by Warren et al. (2007), is given by the weight function

$$w(0, \mathbf{x}, t) = \frac{\mathbf{p}'(t) \times \mathbf{p}''(t)}{((\mathbf{p}(t) - \mathbf{x}) \times \mathbf{p}'(t))^2} \quad (20)$$

for domains with $\mathbf{p} \in C^2[a, b]$. This kernel corresponds in the polygonal case to Wachspress coordinates, as shown in Kosinka and Bartoň (2015). The mean value kernel (Ju et al., 2005; Dyken and Floater, 2009) is given by

$$w(1, \mathbf{x}, t) = \frac{(\mathbf{p}(t) - \mathbf{x}) \times \mathbf{p}'(t)}{\|\mathbf{p}(t) - \mathbf{x}\|^3}, \quad (21)$$

which is defined for $\mathbf{p} \in C^1[a, b]$.

With this close correspondence between barycentric coordinates and kernels, a natural question arises: For any set of barycentric coordinates, is there a corresponding limit barycentric kernel? We now present a positive answer to this question and show that the underlying convergence rate is quadratic.

3. Convergence of barycentric coordinates

Following the approach introduced in Kosinka and Bartoň (2015), we now investigate the convergence of barycentric coordinates for denser and denser convex polygons converging to a smooth convex domain. First, we formalise this concept.

Given a strictly convex domain Ω , we approximate its boundary curve $\mathbf{p}(t)$ by a sequence of polygons P_h . For a sample of parameter values $a < t_1 < t_2 < \dots < t_n \leq b$, we set $t_0 = t_n$ and $t_{n+1} = t_1$, denote $h_j = t_{j+1} - t_j$, and define $h = \max_{j=1, \dots, n} h_j$. P_h stands for the convex polygon with vertices $\mathbf{p}_i = \mathbf{p}(t_i)$, $i = 1, \dots, n$. To simplify notation, we denote $\mathbf{p}'_i$ the first derivative of $\mathbf{p}(t)$ at t_i and similarly for higher derivatives.

In the smooth setting, $c(\mathbf{x}, t)$ gives rise to a barycentric kernel on Ω via (19) and (13). Sampling this function at t_i yields $c_i(\mathbf{x}) = c(\mathbf{x}, t_i)$. The set $\{c_i(\mathbf{x})\}_{i=1}^n$ of functions then determines a set of barycentric coordinates on P_h via (6) and (5). Note that any barycentric kernel over Ω and any set of barycentric coordinates on P_h can be expressed in terms of $c(\mathbf{x}, t)$ and $c_i(\mathbf{x})$, respectively.

Further, we assume that $\mathbf{x} \in \Omega$ and h is small enough that $\mathbf{x} \in P_h$. The dependence of $w_i(\mathbf{x})$ and $\lambda_i(\mathbf{x})$ on the sampling is not made explicit. We define $k_i = \frac{h_{i-1}+h_i}{2}$ and $l_i = \frac{h_i-h_{i-1}}{2}$. We then have the following lemma, which generalises Lemma 1 of Kosinka and Bartoň (2015).

Lemma 1. For $\mathbf{p}(t) \in C^4[a, b]$ and $c(\mathbf{x}, t)$ with $c(\mathbf{x}, \cdot) \in C^4[a, b]$, there exists a function $e(\mathbf{x}, t)$, with $e(\mathbf{x}, \cdot) \in C^1[a, b]$, and a kernel $w(\mathbf{x}, t)$ such that

$$w_i(\mathbf{x}) = k_i(w(\mathbf{x}, t_i) + l_i e(\mathbf{x}, t_i)) + R_i$$

and there exists a function $C(\mathbf{x})$, independent of i , such that $|R_i| \leq C(\mathbf{x})k_i^3$.

Proof. Where no confusion is likely to arise, we omit the dependence of some of the functions involved in the proof on their variables and parameters, and write e.g. simply $\mathbf{d}$ instead of $\mathbf{d}(\mathbf{x}, t)$. We define $\mathbf{d}_i = \mathbf{p}_i - \mathbf{x}$ and write the weight function $w_i(\mathbf{x})$ from (6) as

$$w_i(\mathbf{x}) = \frac{2a_i(\mathbf{x})}{b_i(\mathbf{x})},$$

where

$$a_i(\mathbf{x}) = c_{i+1}(\mathbf{d}_i \times (\mathbf{p}_i - \mathbf{p}_{i-1})) - c_i(\mathbf{d}_i - \mathbf{p}_i + \mathbf{p}_{i-1}) \times (\mathbf{d}_i - \mathbf{p}_i + \mathbf{p}_{i+1}) + c_{i-1}(\mathbf{d}_i \times (\mathbf{p}_{i+1} - \mathbf{p}_i))$$

and

$$b_i(\mathbf{x}) = (\mathbf{d}_i \times (\mathbf{p}_i - \mathbf{p}_{i-1}))(\mathbf{d}_i \times (\mathbf{p}_{i+1} - \mathbf{p}_i)).$$

For $\mathbf{p} \in C^4[a, b]$, its Taylor expansion gives

$$\begin{aligned} \mathbf{p}_{i-1} &= \mathbf{p}_i - h_{i-1}\mathbf{p}'_i + \frac{h_{i-1}^2}{2}\mathbf{p}''_i - \frac{h_{i-1}^3}{6}\mathbf{p}'''_i + \mathcal{O}(h_{i-1}^4), \\ \mathbf{p}_{i+1} &= \mathbf{p}_i + h_i\mathbf{p}'_i + \frac{h_i^2}{2}\mathbf{p}''_i + \frac{h_i^3}{6}\mathbf{p}'''_i + \mathcal{O}(h_i^4). \end{aligned} \quad (22)$$

Treating $b_i(\mathbf{x})$ first, we see that

$$\frac{b_i(\mathbf{x})}{h_{i-1}h_i} = (\mathbf{d}_i \times \mathbf{p}'_i)^2 + l_i(\mathbf{d}_i \times \mathbf{p}'_i)(\mathbf{d}_i \times \mathbf{p}''_i) + \mathcal{O}(k_i^2).$$

Using $(f_0 + f_1h + \mathcal{O}(h^2))(\frac{1}{f_0} - \frac{f_1}{f_0^2}h + \mathcal{O}(h^2)) = 1 + \mathcal{O}(h^2)$ for $f_0 \neq 0$, we obtain

$$\frac{h_{i-1}h_i}{b_i(\mathbf{x})} = \frac{1}{(\mathbf{d}_i \times \mathbf{p}'_i)^2} - l_i \frac{\mathbf{d}_i \times \mathbf{p}''_i}{(\mathbf{d}_i \times \mathbf{p}'_i)^3} + \mathcal{O}(k_i^2). \quad (23)$$

Similarly to (22), we have

$$\begin{aligned} c_{i-1} &= c_i - h_{i-1}c'_i + \frac{h_{i-1}^2}{2}c''_i - \frac{h_{i-1}^3}{6}c'''_i + \mathcal{O}(h_{i-1}^4), \\ c_{i+1} &= c_i + h_ic'_i + \frac{h_i^2}{2}c''_i + \frac{h_i^3}{6}c'''_i + \mathcal{O}(h_i^4), \end{aligned} \quad (24)$$

where all derivatives are again with respect to t . This then gives

$$\frac{2a_i(\mathbf{x})}{h_{i-1}h_i} = k_i(G_i + 2l_iH_i) + \mathcal{O}(k_i^3), \quad (25)$$

where $G_i = G(\mathbf{x}, t_i)$, $H_i = H(\mathbf{x}, t_i)$, and

$$\begin{aligned} G(\mathbf{x}, t) &= 2(c(\mathbf{p}' \times \mathbf{p}'') - c'(\mathbf{d} \times \mathbf{p}'') + c''(\mathbf{d} \times \mathbf{p}')), \\ H(\mathbf{x}, t) &= \frac{2}{3}(c(\mathbf{p}' \times \mathbf{p}''') - c'(\mathbf{d} \times \mathbf{p}''') + c'''(\mathbf{d} \times \mathbf{p}')). \end{aligned} \quad (26)$$

Finally, multiplying (23) by (25) gives the result

$$w_i(\mathbf{x}) = k_i(w(\mathbf{x}, t_i) + l_i e(\mathbf{x}, t_i)) + R_i \quad (27)$$

with the kernel weight function

$$w(\mathbf{x}, t) = \frac{G}{(\mathbf{d} \times \mathbf{p}')^2} \quad (28)$$

and the function

$$e(\mathbf{x}, t) = \frac{2H}{(\mathbf{d} \times \mathbf{p}')^2} - G \frac{\mathbf{d} \times \mathbf{p}''}{(\mathbf{d} \times \mathbf{p}')^3}. \quad \square \quad (29)$$

Observe that the resulting limit weight (28) is, up to the insignificant factor of 2, identical to the weight function derived in (19). This allows us to conclude that any barycentric kernel can be obtained as the limit of its corresponding set of barycentric coordinates. This correspondence is facilitated by the function $c(\mathbf{x}, t)$ and its samples at t_i .

To establish the rate of convergence of barycentric coordinates to their limit kernels, we employ Lemma 2 of Kosinka and Bartoň (2015), which we reproduce here for convenience.

Lemma 2. For $\mathbf{x} \in \Omega$ and for $u \in C^2[a, b]$ and $\mathbf{p}(t) \in C^4[a, b]$,

$$\sum_{i=1}^n w_i(\mathbf{x}) u(t_i) = \int_a^b w(\mathbf{x}, t) u(t) dt + \mathcal{O}(h^2) \quad \text{as } h \rightarrow 0.$$

A proof of this lemma, now relying on our more general Lemma 1, can be found in Kosinka and Bartoň (2015). In the uniform case when the points t_i are uniformly spaced, $k_i = h$ and $h_i - h_{i-1} = 0$, Lemma 1 implies that

$$w_i(\mathbf{x}) = h w(\mathbf{x}, t_i) + \mathcal{O}(h^3).$$

Combined with Lemma 2, we have

$$\lambda_i(\mathbf{x}) = h \lambda(\mathbf{x}, t_i) + \mathcal{O}(h^3).$$

It follows from the results in Belyaev (2006), Schaefer et al. (2007) that the limit kernels of barycentric coordinates are always barycentric. Nevertheless, we now present an alternative insight into this fact based on our convergence results.

Consider the limit kernel $\lambda(\mathbf{x}, t)$ of any set of barycentric coordinates $\lambda_i(\mathbf{x})$. Property (Q2) is satisfied simply by normalising $w(\mathbf{x}, t)$ using (13). It thus remains to show that $\lambda(\mathbf{x}, t)$ meets (Q3). From the partition of unity (P2) and the barycentric property (P3) of $\lambda_i(\mathbf{x})$ it follows that

$$\sum_{i=1}^n \lambda_i(\mathbf{x})(\mathbf{p}_i - \mathbf{x}) \equiv 0 \quad \text{and thus} \quad \sum_{i=1}^n w_i(\mathbf{x})(\mathbf{p}_i - \mathbf{x}) \equiv 0.$$

Using Lemma 2 again, this time with $u(t) = \mathbf{p}(t)$ (understood coordinate-wise), we obtain that

$$\int_a^b w(\mathbf{x}, t)(\mathbf{p}(t) - \mathbf{x}) dt \equiv 0$$

in the limit as $h \rightarrow 0$. Consequently, the (limit) kernel $\lambda(\mathbf{x}, t)$ is barycentric.

4. Barycentric interpolants and mappings

Key applications of barycentric coordinates and kernels are based on the interpolants and mappings they induce. Barycentric interpolation (Floater, 2015) over P is governed by

$$\hat{g}(\mathbf{x}) = \sum_{i=1}^n \lambda_i(\mathbf{x}) f(\mathbf{p}_i), \quad (30)$$

where $\lambda_i(\mathbf{x})$ are barycentric coordinates on P and $f(\mathbf{p}_i)$ are given function values on the vertices of P that should be interpolated. This interpolation property of continuous barycentric coordinates follows from the Lagrange property (Floater et al., 2006). The interpolant $\hat{g}$ restricted to ∂P is the piece-wise linear curve interpolating $f(\mathbf{p}_i)$.

The transfinite case (Floater and Kosinka, 2010a) over Ω is described by

$$g(\mathbf{x}) = \int_a^b \lambda(\mathbf{x}, t) f(\mathbf{p}(t)) dt, \quad (31)$$

where $\lambda(\mathbf{x}, t)$ is a barycentric kernel on Ω and $f(\mathbf{p}(t))$ is a function that should be interpolated along the boundary curve $\mathbf{p}(t)$ of Ω . This interpolation property was, for smooth functions f defined on $\partial\Omega$, established in Floater and Kosinka (2010a).

With the same assumptions as in Section 3, i.e. $P_h \rightarrow \Omega$ as $h \rightarrow 0$, we now generalise Theorem 1 of Kosinka and Bartoň (2015) from the Wachspress case to any barycentric interpolant.

Theorem 1. Let $f(t)$ be a C^2 -continuous function defined on $\mathbf{p}(t)$, $g(\mathbf{x})$ the barycentric interpolant (31) of $f(t)$, and $\hat{g}_h(\mathbf{x})$ the barycentric interpolant (30) of the discrete values of $f(t)$ at $\mathbf{p}_i$ of P_h . For $\mathbf{x} \in \Omega$ it holds $g(\mathbf{x}) = \hat{g}_h(\mathbf{x}) + \mathcal{O}(h^2)$ as $h \rightarrow 0$.

Proof. As in Kosinka and Bartoň (2015), this is a direct consequence of Lemma 2. $\square$

This shows that the sequence of interpolants $\hat{g}_h(\mathbf{x})$ converges quadratically to the limit kernel interpolant $g(\mathbf{x})$ as P_h tends to Ω for any set of barycentric coordinates.

We now turn to barycentric mappings. With a second polygon $Q \subset \mathbb{R}^2$ with vertices $\mathbf{q}_1, \dots, \mathbf{q}_n$, a barycentric mapping $\hat{\mathbf{f}}$ from $\bar{P}$ to $\bar{Q}$ is defined by

$$\hat{\mathbf{f}}(\mathbf{x}) = \sum_{i=1}^n \lambda_i(\mathbf{x}) \mathbf{q}_i. \quad (32)$$

The edges of P are mapped to those of Q piece-wise linearly by $\hat{\mathbf{f}}$. Properties of barycentric mappings were studied in Floater and Kosinka (2010b), Schneider et al. (2013).

The limit counterparts of barycentric mappings are defined with the help of a second domain $\Psi \subset \mathbb{R}^2$ given by its boundary curve $\mathbf{q}: [a, b] \rightarrow \mathbb{R}^2$ with an injective parametrisation in $[a, b]$, and $\mathbf{q}(a) = \mathbf{q}(b)$. A barycentric mapping $\mathbf{f}$ from Ω to Ψ is defined by

$$\mathbf{f}(\mathbf{x}) = \int_a^b \lambda(\mathbf{x}, t) \mathbf{q}(t) dt. \quad (33)$$

The interpolation property of $\lambda(\mathbf{x}, t)$ ensures that this mapping extends continuously to $\partial\Omega$.

Similarly to the case of barycentric interpolants, we now show that an analogous result applies also to barycentric mappings, again generalising previous results of Kosinka and Bartoň (2015) from the Wachspress case to any barycentric mapping.

Theorem 2. Let Q_h be a sequence of inscribed polygonal approximations of Ψ in correspondence to P_h , $\mathbf{f}(\mathbf{x}) : \Omega \rightarrow \Psi$ the mapping (33) and $\hat{\mathbf{f}}_h(\mathbf{x}) : P_h \rightarrow Q_h$ the mapping (32). For $\mathbf{x} \in \Omega$ it holds $\mathbf{f}(\mathbf{x}) = \hat{\mathbf{f}}_h(\mathbf{x}) + \mathcal{O}(h^2)$ as $h \rightarrow 0$.

Proof. Again, this result is a direct consequence of Lemma 2. $\square$

In summary, we have shown that our convergence results for barycentric coordinates and kernels from Section 3 carry over to barycentric interpolants and mappings.

5. Examples and numerical results

We now present examples of transfinite interpolation and barycentric mappings that numerically validate our theoretical results.

Fig. 2 shows sequences of interpolants for various barycentric coordinates as finer and finer polygons are being inscribed into a smooth convex curve $\mathbf{p}(t)$. Function values $f(\mathbf{p}_1), \dots, f(\mathbf{p}_n)$ associated to the vertices of a polygon P , see Fig. 2 top right, are obtained from $f(\mathbf{p}(t))$ that is known along the smooth boundary $\mathbf{p}(t)$ of a smooth convex domain, see Fig. 2 bottom right. Note how the shape of the barycentric interpolant, both in the polygonal and transfinite case, depends on the function $c_i(\mathbf{x})$. This is a consequence of the heterogeneity of the coordinate functions λ_i , considered again as functions of $c_i(\mathbf{x})$. The coordinate functions λ_i are shown in Fig. 3. Note that these functions are not, in general, guaranteed to be non-negative, see Fig. 3 left. This fact makes them less favourable, for instance from the point of view of numerical stability.

Barycentric coordinates and kernels can also be used in applications that demand shape or image deformation (Floater and Kosinka, 2010b; Floater and Kosinka, 2010a; Schneider et al., 2013; Floater, 2015). The approach is based on mappings

Fig. 2. Convergence of polygonal barycentric interpolants to their limits, the transfinite interpolants governed by the corresponding barycentric kernels. From left to right: Wachspress coordinates, mean value coordinates, and three sets of barycentric coordinates that do not belong to the family of three-point coordinates. These are defined via bivariate functions $c_i(\mathbf{x})$, see Eq. (6). The functions included in the figure are, from left to right: $c_i(\mathbf{x}) = 1$, $c_i(\mathbf{x}) = r_i$, $c_i(\mathbf{x}) = \log(1 + r_i(\mathbf{x}))$, $c_i(\mathbf{x}) = \frac{r_i(\mathbf{x})}{(1+r_i(\mathbf{x}))}$, and $c_i(\mathbf{x}) = \frac{r_i^2(\mathbf{x})}{(1+r_i^2(\mathbf{x}))}$. The initial interpolants for $n = 6$ (top), after one, two, and three subdivisions (middle), and the limits (bottom) are shown. The colour-coding reflects the Gaussian curvature of the limit interpolants. Numerical data indicating uniform quadratic convergence are displayed below in Fig. 6. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

between polygons or smooth domains. An example of a mapping using the kernel of Warren et al. (2007) is shown in Fig. 4. A sequence of examples that use various barycentric coordinates is shown in Fig. 5. Note that as in the case of interpolants, the result of the deformation depends significantly on the particular barycentric coordinates. But more importantly, owing to the quadratic convergence of any set of barycentric coordinates, observe that only the first two iterations (rows) in Fig. 5 are visually different from the corresponding limits (bottom row). Already the third iteration (row) captures all the visual features of the limit mapping across all displayed coordinate types, and is obtained at a significantly lower computational cost than the limit.

It was conjectured in Kosinka and Bartoň (2015) that the quadratic convergence of barycentric interpolants and mappings (see Section 3) based on Wachspress and mean value coordinates is uniform. We investigated this numerically for various barycentric coordinates.

As in Fig. 2, we denote by $\hat{g}_j(\mathbf{x})$ the interpolant at subdivision level $j = 0, \dots, 8$ corresponding to the underlying polygon with $6 \cdot 2^j$ vertices. The limit interpolant, guaranteed to exist by Theorem 1, is denoted by $g(\mathbf{x})$. For each considered barycentric coordinate type, we define

$$m_j = \max_{\mathbf{x} \in \Omega} |\hat{g}_j(\mathbf{x}) - \hat{g}_{j-1}(\mathbf{x})|,$$

i.e., the maximum absolute difference between two consecutive barycentric interpolants at levels j and $j - 1$.

Fig. 3. The barycentric coordinate functions λ_i are shown. We show three examples from the family of the three-point coordinates for various values of p and one example that lies outside this family. The rows reflect subdivision levels. The functions λ_i are piece-wise linear on the boundary (orange polygon) and the colour coding visualises their height field. The level sets for $z = \frac{k}{7}$, $k = 1, \dots, 6$ are shown in white. Recall that the three-point coordinates are guaranteed to be non-negative only for $p = 0$ and $p = 1$. In this particular example, non-negativity is violated for $p = -1$ (bottom left); the corresponding level set for $z = 0$ is highlighted in yellow. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 4. A deformations based on a barycentric mapping. The boundary curves $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are uniform B-spline curves of degree three and four, respectively. Their control polygons are shown in red, with control points in cyan. The barycentric mapping (33) maps $\mathbf{p}(t)$ to $\mathbf{q}(t)$ for all $t \in [a, b]$; $\mathbf{p}(a)$ and $\mathbf{q}(a)$ are shown in yellow. The deformed text is colour-coded by the Jacobian of the mapping, $J(\mathbf{f}(\mathbf{x}))$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Let us assume, as our numerical tests reported in Fig. 6 suggest, that there is a constant K such that $m_j \leq K4^{-j}$. This then yields

$$\begin{aligned} \max_{\mathbf{x} \in \Omega} |\hat{g}_j(\mathbf{x}) - g(\mathbf{x})| &= \max_{\mathbf{x} \in \Omega} \left| \sum_{i=0}^{\infty} (\hat{g}_{j+i}(\mathbf{x}) - \hat{g}_{j+i+1}(\mathbf{x})) \right| \leq \max_{\mathbf{x} \in \Omega} \sum_{i=0}^{\infty} |\hat{g}_{j+i}(\mathbf{x}) - \hat{g}_{j+i+1}(\mathbf{x})| \\ &\leq \sum_{i=0}^{\infty} \max_{\mathbf{x} \in \Omega} |\hat{g}_{j+i}(\mathbf{x}) - \hat{g}_{j+i+1}(\mathbf{x})| = \sum_{i=0}^{\infty} m_{j+i} \leq K4^{-j} \sum_{i=0}^{\infty} 4^{-i} = \frac{4K}{3} 4^{-j}. \end{aligned}$$

Or in other words, $m_j \leq K4^{-j}$ implies that the convergence of $\hat{g}_j(\mathbf{x})$ is quadratic and uniform.

This observation, combined with our numerical results shown in Fig. 6 and similar results obtained for barycentric mappings, encourages us to extend the above mentioned conjecture to barycentric interpolants and mappings governed by any set of barycentric coordinates satisfying the assumptions of Lemma 1, and Theorems 1 and 2, respectively. On the other hand, we emphasise that the convergence of the coordinates themselves is not uniform over $\bar{P}$ due to the Dirac delta function property.

Fig. 5. The mappings for various barycentric coordinates are shown. The mappings correspond to the input from Fig. 4 when the polygon $\mathbf{p}_i$, $i = 1, \dots, n$ is sampled from the source parametric curve $\mathbf{p}(t)$. The image control pentagons (top) are displayed in red. Three levels of refinement of the initial polygons are displayed (middle), together with the limit case when the mapping is governed by the corresponding barycentric kernels (bottom). The image is colour-coded by the Jacobian of the mapping, showing large local distortion in red while locally area-preserving regions are displayed in blue. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 6. A $\log_4$ plot indicating uniform quadratic convergence of the sequence of the barycentric interpolants shown in Fig. 2. The number of samples, uniformly distributed over $\bar{\Omega}$, used to estimate m_j , $j = 1, \dots, 8$ was set to 15,000 in all cases.

6. Conclusion

We have shown that barycentric coordinates on polygons and barycentric kernels on smooth domains are closely linked by the limit process that involves denser and denser polygons converging to a domain with smooth boundary. The convergence rate of this limit process is quadratic and also applies to barycentric interpolants and barycentric mappings. Therefore, barycentric coordinates defined over polygons with only dozens of vertices could be used instead of computationally more demanding barycentric kernels. A detailed (non-asymptotic) analysis that would give bounds on the error between the smooth and discrete interpolants/mappings goes beyond the scope of the current paper. We plan to investigate it for certain classes of functions c_i in (6) in the future.

Our theoretical results were confirmed by several numerical experiments on barycentric coordinates and kernels as well as the interpolants and mappings that they induce. Since generalisations of barycentric coordinates (Floater et al., 2006) from 2D to 3D are readily available (Ju et al., 2007), we also plan to look in this direction.

Acknowledgement

We thank the anonymous reviewers of GMP 2016 for helping us to improve this paper.

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