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Switching Control for Parameter Identifiability

Giorgio Battistelli^{ID} and Pietro Tesi^{ID}

Abstract—We consider the problem of ensuring the identifiability of the parameters of a multivariable linear system by means of feedback control. It is shown that parameter identifiability can be generically ensured using time-varying (switching) controllers. Moreover, this property holds true for any, arbitrarily small, data observation window. Some practical aspects of the problem are also dealt with, including controller design and multimodel least-squares estimation.

Index Terms—Parameter identification, switching control, systems identification, uncertain systems.

I. INTRODUCTION

Identifying the parameters of a system from the input–output data is a problem of major interest in control engineering. This problem is typically referred to as the *parameter identifiability* problem [1], [2]. This note considers the problem of parameter identifiability for linear systems operating in the *closed loop*. More precisely, we address the problem of designing feedback controllers that make it possible to identify the parameters of a multivariable linear system:

- 1) switching between linear controllers;
- 2) even when the operating conditions are not stationary; and
- 3) in an arbitrary finite time.

The literature on parameter identification is immense, and the goal of this note is not to re-examine well-established research (connections, whenever present, will be discussed throughout the paper). The goal is instead to study the problem in a different context. In a nutshell, mainstream research on parameter identification assumes *stationary operations* and the emphasis is on data generation mechanisms that define *asymptotic* properties (e.g., statistical efficiency) [3], [4]. However, there are important applications where stationarity and infinite-length experiments cannot be assumed. This is the case when controlling systems whose operating mode can vary with time, either genuinely or because of faults, and parameter identification has to be performed in nonstationary conditions, possibly in the presence of closed-loop divergence trends. In fact, this is the scenario that one typically encounters in *fault tolerant* [5], *adaptive* [6], and *supervisory control* [7]. In this context, it is important to establish properties on parameter identification of a *worst-case* type, i.e., properties that hold for any possible closed-loop trajectory.

In this note, the question of parameter identifiability is posed in a purely deterministic framework, and the problem is approached by

searching for feedback control laws, called *discerning*, which guarantee that closed-loop trajectories obtained with different system parameters are never equal to one another. Parameter identifiability is then defined precisely in these terms, and the properties i)–iii) are a natural consequence of such a closed-loop shaping. We show that discerning control laws can be obtained through discontinuous (*switching*) control. The reason for resorting to switching control is twofold. The first reason is that it is in general impossible to guarantee worst-case properties with linear time-invariant (LTI) control laws. In other terms, to guarantee parameter identification for any possible closed-loop trajectory one needs control structures more complex than LTI controllers. Switching control suffices for this goal. The second reason is that in fault tolerant, adaptive and supervisory control, parameter identification is only a preliminary step, which must be followed by a control reconfiguration step. Thus, it is natural to consider schemes that embed reconfiguration capabilities. Switching control accounts for this situation.

The term “discerning control” was first adopted in the literature on *switching/multimode systems* [8], [9], and refers to the problem of finding control inputs under which it is possible to reconstruct the active mode of a switching system from the input–output data. Our approach originates from that context. In fact, multimode systems can be described as parameterized systems in which each mode is associated to a different parameter vector. For multimode systems with a *finite* number of modes, a systematic treatment of discerning control laws has been addressed in [10] and [11]. There, we proved that under mild assumptions almost all LTI controllers are discerning. The solution is obtained by showing that under mild assumptions it is always possible to synthesize an LTI controller (of arbitrary fixed order) in such a way that the closed-loop characteristic polynomials related to different system modes do never share any common root (thus implying that different system modes can never give rise to the same closed-loop trajectories). As mentioned before, in general, an analogous result cannot be established when the number of modes is *not finite*, which is the case of continuously parameterized systems. Inspired by [12], in this note we consider switching controllers (of arbitrary fixed order) and show that this controller class is *generically* discerning if the number of controller modes is sufficiently large. An explicit upper bound on the number of controller modes needed for parameter identifiability is given in terms of the dimension of the parameter set.

While our investigation is mainly theoretical (see Theorem 1), we also discuss some aspects of practical relevance. First, while property ii) is very useful in the context of fault tolerant, adaptive, and supervisory control where parameter identification must be performed possibly in the presence of closed-loop divergence trends, a discerning controller should also be capable of preserving the closed-loop stability during normal operations. We provide sufficient conditions under which identification and closed-loop stability can be simultaneously achieved (see Theorem 2). Further, we analyze the properties of least-squares parameter estimation in connection with the use of discerning controllers, providing bounds on the worst-case estimation error in the presence of finite covering of the parameter set, bounded disturbances affecting the process dynamics and bounded measurement

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noises (see Theorem 3). This latter analysis is also motivated by the widespread use of multimodel parameter estimation techniques in the context of fault-tolerant control and real-time control reconfiguration [13]–[19]. Finally, we discuss a simple procedure to design discerning controllers in practice.

Properties i)–iii) are of major interest for supervisory control schemes based on *certainty equivalence* [10], [13]–[20]. In [10] and [20], it was shown that, when the parameter set is finite (the process under control is a classic switching system), discernibility induces *input-to-state stability* properties to the closed-loop system. The present results indicate that similar properties can be achieved also for continuously parameterized systems. Due to the limited space, such a topic will be the subject of future investigations.

I. Notation

Given a vector v , $|v|$ denotes its Euclidean norm. Given a symmetric, positive semidefinite matrix M , $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote minimum and maximum eigenvalues of M , respectively. Given a matrix M , $M^\top$ is its transpose and $|M| = \sqrt{\lambda_{\max}(M^\top M)}$ its spectral norm. Given a measurable time function $v : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ and a time interval $\mathcal{I} \subseteq \mathbb{R}_{\geq 0}$, we denote the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms of $v(\cdot)$ on $\mathcal{I}$ as $\|v\|_{2,\mathcal{I}} = \sqrt{\int_{\mathcal{I}} |v(t)|^2 dt}$ and $\|v\|_{\infty,\mathcal{I}} = \text{ess sup}_{t \in \mathcal{I}} |v(t)|$, respectively. Finally, we let $\mathcal{L}_2(\mathcal{I})$ and $\mathcal{L}_\infty(\mathcal{I})$ denote the sets of square integrable and, respectively, (essentially) bounded time functions on $\mathcal{I}$.

II. PARAMETER IDENTIFIABILITY AND DISCERNING CONTROLS

Consider a parametric family $\mathcal{P}(\theta)$ of multivariable linear dynamical systems, described as

$$\mathcal{P}(\theta) : \quad \dot{x} = A(\theta)x + B(\theta)u, \quad y = C(\theta)x \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, and $\theta \in \mathbb{R}^{n_\theta}$ is an unknown parameter vector belonging to some known bounded set $\Theta \subseteq \mathbb{R}^{n_\theta}$. The problem of interest is the following: supposing that the input/output data $z = \text{col}(u, y)$, where “col” stands for column vector, are generated by one of the systems in the family (1) with unknown parameter vector $\theta = \theta^\circ$, we want to find conditions under which θ° can be uniquely reconstructed from observations of z . We shall refer to this problem as the *identifiability* problem [1], [2].

In this note, this problem is approached by searching for feedback controllers, called *discerning controls*, under which closed-loop trajectories obtained with different parameters in Θ can never be equal to one another. In [11], the problem was addressed in the special case where the set Θ is finite and it was shown that, under mild assumptions, parameter identifiability can be ensured simply via LTI controllers. In contrast, it turns out that when the parameter set Θ is continuous an analogous result is not always achievable, see for instance the counterexample in the Appendix, and hence more complex control structures have to be considered. In this note, in order to deal with the case when Θ is not finite, we consider a generalization of the notion of discerning control laws by focusing on hybrid controllers in the form of dynamical systems with switched parameters. Let the controller be described by a switching linear system

$$\dot{\xi} = F_\sigma \xi - G_\sigma y, \quad u = H_\sigma \xi - K_\sigma y \quad (2)$$

where $\xi \in \mathbb{R}^{n_\xi}$ is the state and $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathcal{N} := \{1, 2, \dots, N\}$ is the switching signal, i.e., the signal (right continuous) which identifies the index of the active system at each instant of time. For any $i \in \mathcal{N}$,

F_i, G_i, H_i , and K_i are constant matrices of appropriate dimensions. In the sequel, we shall denote by $\mathcal{C}_i$ the LTI system with state-space representation $\{F_i, G_i, H_i, K_i\}$ and by $\mathcal{C}_\sigma$ the time-varying controller associated with the switching signal σ .

Let us denote by $(\mathcal{P}(\theta)/\mathcal{C}_\sigma)$ the parametric family of closed-loop systems resulting from the interconnection of (1) and (2) when the controller switching signal is σ . Further, let $\chi = \text{col}(x, \xi)$ and $z = \text{col}(u, y)$ be the state and output, respectively, of $(\mathcal{P}(\theta)/\mathcal{C}_\sigma)$. The corresponding dynamics can be expressed as

$$\dot{\chi} = \Psi_\sigma(\theta)\chi, \quad z = \Lambda_\sigma(\theta)\chi \quad (3)$$

where, for any $i \in \mathcal{N}$ and $\theta \in \Theta$

$$\Psi_i(\theta) = \begin{bmatrix} A(\theta) - B(\theta)K_i C(\theta) & B(\theta)H_i \\ -G_i C(\theta) & F_i \end{bmatrix} \quad (4)$$

$$\Lambda_i(\theta) = \begin{bmatrix} -K_i C(\theta) & H_i \\ C(\theta) & 0 \end{bmatrix}. \quad (5)$$

Finally, for any parameter vector $\theta \in \Theta$, let $z(t, t_0, x_0, \xi_0, \theta, \sigma)$ denote the value at time t of z when the plant initial state at time t_0 is x_0 , the controller initial state is ξ_0 , and the controller switching signal is σ .

Definition 1: Consider a system as in (1) and assume that u and y are available for measurements. Let $\theta, \theta' \in \Theta$ be any two distinct parameter vectors. A controller as in (2) is said to be (θ, θ') -discerning if, for any interval $\mathcal{I} := [t_0, t_0 + T)$, with $T > 0$, there exists a switching signal $\sigma : \mathcal{I} \rightarrow \mathcal{N}$ such that

$$\|z(\cdot, t_0, x_0, \xi_0, \theta, \sigma) - z(\cdot, t_0, x'_0, \xi'_0, \theta', \sigma)\|_{2,\mathcal{I}} \neq 0 \quad (6)$$

for all nonzero quadruples of vectors $(x_0, \xi_0, x'_0, \xi'_0)$. The switching controller (2) is said to be *globally discerning* if it satisfies condition (6) for all pairs (θ, θ') of different parameter vectors. ■

When the switching controller (2) is (θ, θ') -discerning and a discerning switching signal σ is adopted, then $\mathcal{P}(\theta)$ and $\mathcal{P}(\theta')$ cannot generate the same closed-loop trajectories whenever (2) is in the feedback loop (unless the initial conditions are null). Thus, global discernibility ensures that if the data are generated by $\theta = \theta^\circ$ then we can uniquely identify θ° even if the operating conditions are not stationary and irrespective of the duration of $\mathcal{I}$.

III. MAIN RESULTS

A. Auxiliary Lemma

A necessary condition for the existence of a discerning controller is that all the pairs $(A(\theta), C(\theta))$ are observable (in the sense of Kalman). In fact, the presence of unobservable dynamics would entail the existence of nonzero trajectories of the closed-loop state χ corresponding to zero trajectories of the closed-loop output z and, hence, for which it would be impossible to infer the plant mode. Thus, the following assumption is considered.

A1: The pair $(A(\theta), C(\theta))$ is observable for all $\theta \in \Theta$.

Let $\varphi_i(s, \theta)$ denote the characteristic polynomial of the closed-loop system $(\mathcal{P}(\theta)/\mathcal{C}_i)$ resulting from the feedback interconnection of the plant $\mathcal{P}(\theta)$ with the i th controller $\mathcal{C}_i$.

Lemma 1: Let assumption A1 hold and suppose that, for any pair of distinct parameter vectors $\theta, \theta' \in \Theta$, there exists at least one $i \in \mathcal{N}$ such that the two closed-loop characteristic polynomials $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ are coprime. Then, the following properties are true:

- 1) the switching controller (2) is globally discerning;
- 2) condition (6) holds for any switching signal σ such that each controller mode $i \in \mathcal{N}$ is active, at least, on an interval $\mathcal{I}_i \subset \mathcal{I}$ of positive measure.

Proof: Consider two distinct parameter vectors $\theta, \theta' \in \Theta$ and consider an index i for which $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ are coprime. Let the controller C_i be active on an interval $\mathcal{I}_i = [\underline{t}, \bar{t}] \subset \mathcal{I}$. Consider now a nonzero quadruple of vectors $(x_0, \xi_0, x'_0, \xi'_0)$ representing possible initial states of the two feedback loops $(\mathcal{P}(\theta)/C_\sigma)$ and $(\mathcal{P}(\theta')/C_\sigma)$ at time t_0 . Let $(\underline{x}, \underline{\xi}, \underline{x}', \underline{\xi}')$ be the corresponding states that are reached at time $\underline{t}$, i.e., at the beginning of the time interval $\mathcal{I}_i$, under the switching law σ . Suppose now the switching signal σ is chosen so as to satisfy a dwell-time condition, i.e., in such a way that there exists a lower bound τ_{dwell} on the time interval between subsequent variations of the controller index. Then, it is immediate to see that, under such a switching law, when $(x_0, \xi_0, x'_0, \xi'_0) \neq 0$ then also $(\underline{x}, \underline{\xi}, \underline{x}', \underline{\xi}') \neq 0$. In fact, such a state is reached after switching a finite number of times between autonomous linear systems, i.e., the feedback loops, and it is known that an autonomous linear system cannot reach the zero state in finite time starting from a nonzero initial state. Notice now that, under assumption A1, coprimeness of the polynomials $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ implies observability of the parallel system

$$\begin{cases} \begin{bmatrix} \dot{\chi} \\ \dot{\chi}' \end{bmatrix} = \begin{bmatrix} \Psi_i(\theta) & 0 \\ 0 & \Psi_i(\theta') \end{bmatrix} \begin{bmatrix} \chi \\ \chi' \end{bmatrix} \\ \tilde{z} = [\Lambda_i(\theta) \quad -\Lambda_i(\theta')] \begin{bmatrix} \chi \\ \chi' \end{bmatrix} \end{cases}$$

(see for instance [10, Proposition 1]). Hence, if we initialize such a system with $(\chi(\underline{t}), \chi'(\underline{t})) = (\underline{x}, \underline{\xi}, \underline{x}', \underline{\xi}') \neq 0$ at time $\underline{t}$, then $\tilde{z}$ is different from 0 a.e. on $\mathcal{I}_i = [\underline{t}, \bar{t}]$, where “a.e.” stands for “almost everywhere,” i.e., everywhere except on a set of zero Lebesgue measure. This, in turn, implies that $z(t, \underline{x}, \underline{\xi}, \theta, i) \neq z(t, \underline{x}', \underline{\xi}', \theta', i)$, or, equivalently, $z(t, t_0, x_0, \xi_0, \theta, \sigma) \neq z(t, t_0, x'_0, \xi'_0, \theta', i)$, a.e. on $\mathcal{I}_i = [\underline{t}, \bar{t}]$. Then, by choosing a switching signal σ which satisfies a dwell-time condition and is such that each controller mode i is active, at least, on an interval $\mathcal{I}_i \subset \mathcal{I}$ of positive measure, the same line of reasoning can be repeated for any pair $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$. ■

Let $\bar{n}_\xi$ denote the total number of elements of the controller matrices (F_i, G_i, H_i, K_i) when the controller order is n_ξ and let $\rho_i \in \mathbb{R}^{\bar{n}_\xi}$ be the $\bar{n}_\xi$ -dimensional vector containing all the elements of the matrices (F_i, G_i, H_i, K_i) according to a given arbitrary order. Notice that this allows us to represent a generic LTI controller of order n_ξ as an element of $\mathbb{R}^{\bar{n}_\xi}$ and, accordingly, a generic switching controller of order n_ξ with N modes as an element of $\mathbb{R}^{N\bar{n}_\xi}$. Observe also that the closed-loop characteristic polynomial is a function of ρ_i so that we can write $\varphi_i(s, \theta) = \varphi(s, \theta, \rho_i)$. We denote by $\mathcal{D}(\theta, \theta') \subseteq \mathbb{R}^{N\bar{n}_\xi}$ the set of vectors ρ_i , i.e., of controller matrices (F_i, G_i, H_i, K_i) , such that the two characteristic polynomials $\varphi(s, \theta, \rho_i)$ and $\varphi(s, \theta', \rho_i)$ are coprime. Further, for a given number N of controller modes, let $\mathcal{D}_N \subseteq \mathbb{R}^{N\bar{n}_\xi}$ be the set of vectors $\rho = (\rho_1, \dots, \rho_N)$, such that the corresponding switching controller satisfies the hypotheses of Lemma 1 (i.e., such that for any pair of distinct parameter vectors $\theta, \theta' \in \Theta$, there exist at least one index $i \in \mathcal{N}$ such that $\varphi(s, \theta, \rho_i)$ and $\varphi(s, \theta', \rho_i)$ are coprime). Since by Lemma 1 all the switching controllers belonging to $\mathcal{D}_N$ are globally discerning, we can focus the attention on the properties of the sets $\mathcal{D}(\theta, \theta')$ and $\mathcal{D}_N$.

B. Existence and Genericity of Discerning Controls

The following notion is instrumental for the next results.

Definition 2: Consider two polynomials $a(s) = a_n s^n + \dots + a_0$ and $b(s) = b_m s^m + \dots + b_0$ of positive degrees n and m , respectively.

We denote by

$$S := \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & 0 & 0 & 0 \\ 0 & a_n & a_{n-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \cdots & a_2 & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & 0 & 0 & 0 \\ 0 & b_m & b_{m-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \cdots & b_2 & b_1 & b_0 \end{bmatrix}$$

the $(n+m) \times (n+m)$ matrix (the Sylvester matrix) of a and b with respect to s . The (Sylvester) resultant of a and b with respect to s , denoted by R is the determinant of S . ■

Let $R_i(\theta, \theta')$ be the resultant of the two closed-loop characteristic polynomials $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$. Recall that coprimeness of $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ is equivalent to the fact that their resultant $R_i(\theta, \theta')$ is different from zero. With this respect, we note that $R_i(\theta, \theta')$ is a function of the elements of the controller matrices (F_i, G_i, H_i, K_i) , or equivalently of the vector ρ_i , i.e., $R_i(\theta, \theta') = R(\theta, \theta', \rho_i)$. Further, the dependence on ρ_i is polynomial. Hence, when only a single pair of distinct parameter vectors $\theta, \theta' \in \Theta$ is taken into account, the set of vectors ρ_i for which $R_i(\theta, \theta') = 0$, which is the complement of $\mathcal{D}(\theta, \theta')$, is an algebraic set. Then, only two situations may occur: either $R_i(\theta, \theta') = 0$ is always satisfied; or $R_i(\theta, \theta') = 0$ is satisfied on a set with zero Lebesgue measure. In this latter case, the set $\mathcal{D}(\theta, \theta')$ is *generic*¹ (since it is the complement of a proper algebraic set). The following lemma, proved in [10, Lemma 3], provides sufficient conditions for such a favorable situation to occur.

Lemma 2 (see [10]): Let assumption A1 hold and consider any two distinct parameter vectors $\theta, \theta' \in \Theta$. Let $A_{\bar{c}}(\theta)$ and $A_{\bar{c}}(\theta')$ be the state-transition matrices that corresponds to the uncontrollable blocks of $A(\theta)$ and, respectively, $A(\theta')$ in a Kalman controllability decomposition. Then, a controller (F_i, G_i, H_i, K_i) ensuring coprimeness of the two polynomials $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ exists if and only if the following two conditions hold:

- 1) the transfer functions of $\mathcal{P}(\theta)$ and $\mathcal{P}(\theta')$ are different;
- 2) the characteristic polynomials of $A_{\bar{c}}(\theta)$ and $A_{\bar{c}}(\theta')$ are coprime.

In addition, when a) and b) hold, for any given controller order n_ξ the set $\mathcal{D}(\theta, \theta')$ is generic and of full measure on $\mathbb{R}^{N\bar{n}_\xi}$. ■

Remark 1: In the classic literature on the topic, the definition of parameter identifiability is sometimes divided into two separated concepts: *structural identifiability* and *informativity of the data* [3], [21]. The former requires that the map from the parameters to the system (model) is injective. It is precisely condition a) of Lemma 2. The problem of characterizing (as well as checking) identifiability properties has been largely investigated in the literature for both linearly and nonlinearly parameterized systems; e.g., see [22] and the references therein. Informativity of the data is instead related to the experimental conditions. It requires that the same data pattern cannot be generated by different parameters. In classic approaches, this leads to the notions of persistence of excitation and signal richness; e.g., see [23] for a rather recent discussion on this topic. For deterministic systems, sufficient

¹A subset $\mathcal{X}$ of a topological space is generic when it is open and dense: for any $x \in \mathcal{X}$, then there exists a neighborhood of x contained in $\mathcal{X}$; for any $x \notin \mathcal{X}$, then every neighborhood of x contains an element of $\mathcal{X}$.

conditions for achieving informativity have been recently discussed in [22] and [24]. As we will see, condition b) ensures that informativity can be achieved purely via feedback, in particular through switching. In fact, in view of Definition 1, a globally discerning switching controller ensures informativity of the data in the sense of [22, Definition 2]. As discussed in Section III-C, this turns out to be very useful when parameter estimation must be performed while preserving other control objectives, e.g., stability. ■

Remark 2: It is worth mentioning that switching has already been investigated as a means to achieve identifiability properties. For instance, in [25] and [26] and more recently [27], switching is used as a means to circumvent the problem of loss of identifiability [6, ch. 2]. These approaches, however, are classic in the sense that they assume the quasi-stationary data and closed-loop stability. In that papers, switching is used to guarantee that distinct system parameters never give rise to identical closed-loop transfer matrices, and parameter identification is achieved via external excitation signals. In contrast, discerning controls, as defined in the present paper, achieve the stronger property that distinct system parameters can never generate the same closed-loop trajectories. ■

Building on the above lemmas, under some regularity assumptions on $A(\theta)$, $B(\theta)$, and $C(\theta)$, it is possible to derive conditions for the existence of a global discerning switching controller on the whole parameter set Θ . In particular, by exploiting the results of [12], the following theorem holds true.

Theorem 1: Let the parameter set Θ be contained in an analytic manifold $\mathcal{M} \subset \mathbb{R}^{n_\theta}$ of dimension M and let the elements of $A(\theta)$, $B(\theta)$, and $C(\theta)$ be analytic functions of θ on $\mathcal{M}$. Furthermore, let assumption A1 and conditions a) and b) of Lemma 2 hold on $\mathcal{M}$. Then, provided that $N \geq 2M + 1$, the set $\mathcal{D}_N$ is generic and of full measure on $\mathbb{R}^{N\bar{n}_\epsilon}$.

Proof: Notice that the resultant $R_i(\theta, \theta')$ of the two polynomials $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ is a polynomial (thus analytic) function of the elements of the matrices $(A(\theta), B(\theta), C(\theta))$, $(A(\theta'), B(\theta'), C(\theta'))$, and (F_i, G_i, H_i, K_i) . This, in turn, implies that $R_i(\theta, \theta')$ is an analytic function of θ and θ' (since the composition of analytic functions is analytic). Notice now that, under the stated hypotheses, Lemma 2 ensures that, for any pair $\theta, \theta' \in \mathcal{M}$ with $\theta \neq \theta'$, it is possible to find at least one set of matrices (F_i, G_i, H_i, K_i) such that $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ are coprime and, hence, $R_i(\theta, \theta') \neq 0$. Finally, recall that the set $\mathcal{D}_N$ corresponds to the set of switching controllers (F_i, G_i, H_i, K_i) , $i \in \mathcal{N}$ for which the vector function $\text{col}(R_i(\theta, \theta'), i \in \mathcal{N})$ is different from zero for any pair $\theta, \theta' \in \mathcal{M}$ with $\theta \neq \theta'$. Then, proceeding as in the proof of [12, Th. 2], we can conclude that $\mathcal{D}_N$ is generic and of full measure on $\mathbb{R}^{N\bar{n}_\epsilon}$ whenever $N \geq 2M + 1$. ■

Theorem 1 provides a bound on the number N of controller modes that ensure identifiability of the unknown parameter θ in Θ . This bound is consistent with the results of [10], [11] where it is shown that when the set Θ is finite, i.e., it is a zero-dimensional manifold, one single LTI controller is generically discerning. As discussed in [12], the bound $N \geq 2M + 1$ for parameter distinguishability is tight for general maps (i.e., when $N < 2M + 1$ one can find counterexamples). However, for specific cases, fewer controller modes can be sufficient. For instance, when θ is a scalar, one can consider a single LTI controller (F_i, G_i, H_i, K_i) and plot the root locus of the closed loop polynomial $\varphi_i(s, \theta)$ as a function of θ . Then, global discernibility is guaranteed provided that such a root locus never crosses itself. Notice finally that the set of analytic functions considered in Theorem 1 is quite general as it captures many important classes of functions (e.g., polynomials, trigonometric functions, exponentials, and also rational functions as long as away from singularities).

C. Accounting for Additional Control Objectives

So far, only the discernibility objective has been taken into account. However, generally speaking, a control law should typically satisfy other control objectives, the most fundamental one being stability. In this respect, suppose that we want the switching controller to ensure closed-loop stability in a given subset $\bar{\Theta}$ of Θ together with global discernibility. For instance, $\bar{\Theta}$ can represent the neighborhood of the nominal operating condition and the switching controller should be designed so as to ensure a satisfactory behavior in nominal conditions *jointly* with the possibility of promptly identifying any departure from the nominal behavior (e.g., for fault detection and isolation or for control reconfiguration purposes). An extreme case is when $\bar{\Theta} = \Theta$ so that we want to design a robust discerning controller ensuring stability for any possible operating condition (this may be possible or not depending on the size of the parameter set Θ).

A sufficient condition to ensure stability under switching is the existence of a common Lyapunov function. Consider a quadratic parameter-dependent Lyapunov function $v(\chi) = \chi^\top \Pi(\theta) \chi$. Then in order for the closed-loop system $(\mathcal{P}(\theta)/\mathcal{C}_\sigma)$ to be stable for any $\theta \in \bar{\Theta}$ it is sufficient that there exists $\Pi(\theta) = \Pi(\theta)^\top$ such that

$$\Pi(\theta) \succ 0 \quad (7)$$

$$\Psi_i(\theta)^\top \Pi(\theta) + \Pi(\theta) \Psi_i(\theta) \prec 0 \quad (8)$$

for any $i \in \mathcal{N}$ and for any $\theta \in \bar{\Theta}$. When the set $\bar{\Theta}$ is compact, by means of continuity arguments, it is possible to show that, for any given smooth $\Pi(\theta)$, the set of controller matrices (F_i, G_i, H_i, K_i) satisfying (8) is an open subset of $\mathbb{R}^{\bar{n}_\epsilon}$. Then, recalling that, by definition, any nonempty open set contains a closed ball of positive radius, the following result on the existence of global discerning controllers ensuring also stability can be readily stated.

Theorem 2: Let the same hypotheses of Theorem 1 hold. Further, let the set $\bar{\Theta}$ be compact and suppose that there exists at least one controller (F_i, G_i, H_i, K_i) , $i \in \mathcal{N}$, for which conditions (7) and (8) are satisfied with the Lyapunov matrix $\Pi(\theta)$ depending continuously on θ . Then, whenever $N \geq 2M + 1$, the set of switching controllers (F_i, G_i, H_i, K_i) , $i \in \mathcal{N}$, that jointly satisfy (7) and (8) and ensure global discernibility is nonnegligible, in the sense that it contains a ball of positive radius in $\mathbb{R}^{N\bar{n}_\epsilon}$.

Proof: Let (F_i, G_i, H_i, K_i) be a controller for which (7) and (8) are satisfied with $\Pi(\theta)$ continuous in θ . Further, consider a closed ball $\mathcal{B}(\varepsilon)$ in the controller parameter space $\mathbb{R}^{\bar{n}_\epsilon}$ centered in (F_i, G_i, H_i, K_i) and with radius ε and let

$$\begin{aligned} \beta(\varepsilon) &= \max_{(F, G, H, K) \in \mathcal{B}(\varepsilon)} \max_{\theta \in \bar{\Theta}} \lambda_{\max}(\Psi(\theta)^\top \Pi(\theta) + \Pi(\theta) \Psi(\theta)) \end{aligned}$$

where $\Psi(\theta)$ is defined as in (4) with respect to (F, G, H, K) with $\Psi(\theta) = \Psi_i(\theta)$ when $\varepsilon = 0$. Note that, in view of the compactness of $\bar{\Theta}$ and because of the continuity of $\Psi(\theta)$ and $\Pi(\theta)$, we have that $\beta(0) = \max_{\theta \in \bar{\Theta}} \lambda_{\max}(\Psi_i(\theta)^\top \Pi(\theta) + \Pi(\theta) \Psi_i(\theta)) < 0$. Moreover, under the considered hypotheses, it is easy to show that $\beta(\varepsilon)$ depends continuously on ε (in this respect, notice that $\Psi(\theta)$ is an affine function of (F, G, H, K)). Hence, this implies the existence of $\varepsilon > 0$ such that $\beta(\varepsilon) < 0$, i.e., such that all the controllers in $\mathcal{B}(\varepsilon)$ satisfies (8) with the same Lyapunov matrix $\Pi(\theta)$. Thus, the set $\mathcal{G} \subset \mathbb{R}^{\bar{n}_\epsilon}$ of all controllers satisfying (8) with the Lyapunov matrix $\Pi(\theta)$ turns out to be open, and $\mathcal{G}^N$ will be open as well. Finally, when $N \geq 2M + 1$, the set $\mathcal{D}_N$ is generic and of full measure on $\mathbb{R}^{N\bar{n}_\epsilon}$ and, hence, $\mathcal{G}^N \cap \mathcal{D}_N$ is nonnegligible. ■

Theorem 2 indicates that the considered approach can guarantee identifiability even when other control objectives must be taken into account. The result hinges upon the existence of a common Lyapunov function. Nonetheless, just like what happens in switched systems, this condition can be relaxed for instance by considering multiple Lyapunov functions with bounded ratio [28].

IV. MULTIMODEL LEAST-SQUARES PARAMETER ESTIMATION

Suppose we collect data z on an interval $\mathcal{I} = [t_0, t_0 + T)$, where z is generated by one of the systems in the family (1) with unknown $\theta = \theta^\circ$. In this section, we show that when z results from application of a discerning switching controller then the resulting estimate enjoys some nice properties even in the presence of unknown disturbances and measurement noises.

To this end, recall that, when the unknown parameter vector takes value θ° , the evolution of z on the interval $\mathcal{I}$ takes the form $z(t, t_0, x_0, \xi_0, \theta^\circ, \sigma)$. Then, the set $\mathcal{S}_\sigma(\theta^\circ)$ of all possible closed-loop data on the interval $\mathcal{I}$ associated with θ° and with the switching signal σ can be written as

$$\mathcal{S}_\sigma(\theta^\circ) = \left\{ \hat{z} \in \mathcal{L}_2(\mathcal{I}) : \hat{z}(\cdot) = z(\cdot, t_0, \hat{x}_0, \hat{\xi}_0, \theta^\circ, \sigma) \text{ on } \mathcal{I} \right. \\ \left. \text{for some } \hat{x}_0 \in \mathbb{R}^{n_x}, \hat{\xi}_0 \in \mathbb{R}^{n_\xi} \right\}.$$

Hence, a natural approach for estimating the unknown system parameters is the *least-squares* approach, which amounts to selecting the parameter vector θ for which the distance between the observed closed-loop data z on the interval $\mathcal{I}$ and the set $\mathcal{S}_\sigma(\theta)$ is minimal. Accordingly, the optimal least-squares estimate $\hat{\theta}^\circ$ can be obtained as

$$\hat{\theta}^\circ \in \arg \min_{\hat{\theta} \in \Theta} \delta_\sigma(z, \hat{\theta}) \quad (9)$$

where

$$\delta_\sigma(z, \hat{\theta}) = \min_{\hat{x}_0 \in \mathbb{R}^{n_x}, \hat{\xi}_0 \in \mathbb{R}^{n_\xi}} \left\| z(\cdot) - z(\cdot, t_0, \hat{x}_0, \hat{\xi}_0, \hat{\theta}, \sigma) \right\|_{2, \mathcal{I}}. \quad (10)$$

Concerning the computation of the distance (10), for any possible feedback loop $(\mathcal{P}(\theta)/\mathcal{C}_\sigma)$, let $\Phi_\sigma(t, t_0, \theta)$ denote its state transition matrix and let $W_\sigma(\theta)$ be its observability Gramian on the interval $\mathcal{I}$, i.e.,

$$W_\sigma(\theta) = \int_{\mathcal{I}} \Phi_\sigma(t, t_0, \theta)^\top \Lambda_{\sigma(t)}(\theta)^\top \Lambda_{\sigma(t)}(\theta) \Phi_\sigma(t, t_0, \theta) dt.$$

Notice that, for any globally discerning switching control law, the observability Gramian $W_\sigma(\theta)$ turns out to be positive definite for any $\theta \in \Theta$ (otherwise there would be zero output trajectories corresponding to nonzero state trajectories and parameter identification would not be possible). Hence, in this case, the minimization in (10) yields

$$\delta_\sigma(z, \hat{\theta}) = \left(\int_{\mathcal{I}} \left\| z(t) - \Lambda_{\sigma(t)}(\hat{\theta}) \Phi_\sigma(t, t_0, \hat{\theta}) \left(W_\sigma(\hat{\theta}) \right)^{-1} \right. \right. \\ \left. \left. \times \int_{\mathcal{I}} \Phi_\sigma(\tau, t_0, \hat{\theta})^\top \Lambda_{\sigma(\tau)}(\hat{\theta})^\top z(\tau) d\tau \right\|^2 dt \right)^{1/2}.$$

For a detailed discussion on the multimodel least-squares parameter estimation, including computational aspects, we refer the interested reader to [20, Appendix A].

From Definition 1, it is immediately clear that, when a globally discerning switching control law is adopted, for any nonnull z the quantity $\delta_\sigma(z, \hat{\theta})$ is zero if and only if $\hat{\theta}$ coincides with θ° , the true parameter vector.

Lemma 3: Consider a globally discerning controller as in (2) and let a discerning switching signal σ be adopted on the observation interval $\mathcal{I}$. Let the observed data z be generated by the closed-loop system (3) from initial condition $\chi_0 = (x_0, \xi_0) \neq 0$ and with the unknown parameter vector taking value θ° . Then, $\hat{\theta}^\circ = \theta^\circ$.

Proof: It is a straightforward consequence of the fact that, when $\hat{\theta} \neq \theta^\circ$, we cannot have $z(t) = z(t, t_0, \hat{x}_0, \hat{\xi}_0, \hat{\theta}, \sigma)$ a.e. on $\mathcal{I}$, since the observed data are generated as $z(t) = z(t, t_0, x_0, \xi_0, \theta^\circ, \sigma)$ with $(x_0, \xi_0) \neq 0$ and the control law $\mathcal{C}_\sigma$ is supposed to be discerning. Hence, $\delta_\sigma(z, \hat{\theta}) > 0$ whenever $\hat{\theta} \neq \theta^\circ$, and $\delta_\sigma(z, \theta^\circ) = 0$ since by hypothesis $z \in \mathcal{S}_\sigma(\theta^\circ)$. ■

While the above lemma illustrates the theoretical effectiveness of the least-squares estimation criterion in ideal conditions, in practice computation of the minimum in (9) can be a quite challenging task when the set Θ is not finite. In this case, a standard approach is the *multimodel* one, which amounts to considering only a finite number, say L , of possible parameter values by constructing the finite set $\Theta_L = \{\theta_\ell, \ell = 1, \dots, L\} \subseteq \Theta$. Typically, Θ_L is obtained by sampling the set Θ with a given guaranteed density ε , so that for any $\theta \in \Theta$ there exists at least one $\theta_\ell \in \Theta_L$ such that $|\theta - \theta_\ell| \leq \varepsilon$. When such a condition is satisfied, we say that Θ_L is ε -dense in Θ . Accordingly, the following *multimodel least-squares criterion* can be used to estimate the unknown parameter vector θ

$$\hat{\theta}_L \in \arg \min_{\hat{\theta} \in \Theta_L} \delta_\sigma(z, \hat{\theta}) \quad (11)$$

as an alternative to (9).

A. Properties of Multimodel Least-Squares Estimation

When analyzing the properties of an estimation criterion, either optimal as in (9) or approximate as in (11), it is important to take into account also the effects of process disturbances and measurement noises. With this respect, in the following analysis, we suppose that the plant state and measurement equations are affected by additive disturbances d and n , so that the signals x and y are generated as follows

$$\dot{x} = A(\theta^\circ)x + B(\theta^\circ)u + d, \quad y = C(\theta^\circ)x + n \quad (12)$$

with $d \in \mathbb{R}^{n_x}$ and $n \in \mathbb{R}^{n_y}$. In this case, the closed-loop system takes the form

$$\dot{\chi} = \Psi_\sigma(\theta^\circ)\chi + \Xi_\sigma(\theta^\circ)v, \quad z = \Lambda_\sigma(\theta^\circ)\chi + \Gamma_\sigma v \quad (13)$$

where $v = \text{col}(d, n)$ and

$$\Xi_i(\theta) = \begin{bmatrix} I & B(\theta)K_i \\ 0 & G_i \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 0 & K_i \\ 0 & I \end{bmatrix}$$

for any $i \in \mathcal{N}$ and $\theta \in \Theta$. Moreover, thanks to the linearity assumption, the closed-loop data z can be decomposed as

$$z = z^{(n)} + z^{(f)} \quad (14)$$

where $z^{(f)}$ is the forced response and $z^{(n)}$ is the natural one which can be written as $z^{(n)}(t) = z(t, t_0, x_0, \xi_0, \theta^\circ, \sigma)$. Now, the triangular inequality yields

$$\delta_\sigma(z^{(n)}, \hat{\theta}) - \|z^{(f)}\|_{2, \mathcal{I}} \leq \delta_\sigma(z, \hat{\theta}) \leq \delta_\sigma(z^{(n)}, \hat{\theta}) + \|z^{(f)}\|_{2, \mathcal{I}} \quad (15)$$

for any $\hat{\theta} \in \Theta$. Hence, the properties of the least-squares estimate $\hat{\theta}_L$ can be investigated by deriving bounds on $z^{(f)}$ and $\delta_\sigma(z^{(n)}, \hat{\theta})$. These bounds are derived in the next two lemmas.

Lemma 4: Let Θ be a compact set and let $A(\theta)$, $B(\theta)$, and $C(\theta)$ depend continuously on θ . Further, let the observed data z be generated by the closed-loop system (13). Then, for any $\mathcal{I} = [t_0, t_0 + T)$,

there exists a positive real γ (independent of θ°) such that $\|z^{(f)}\|_{2,\mathcal{I}} \leq \gamma \|v\|_{\infty,\mathcal{I}}$.

Proof: Note that the forced response of the switching linear system can be written as

$$z^{(f)}(t) = \Lambda_{\sigma(t)}(\theta^\circ) \int_{t_0}^t \Phi_\sigma(\tau, t_0, \theta^\circ) \Xi_{\sigma(\tau)}(\theta^\circ) v(\tau) d\tau + \Gamma_{\sigma(t)} v(t).$$

Hence, we also have

$$\begin{aligned} |z^{(f)}(t)| &\leq |\Lambda_{\sigma(t)}(\theta^\circ)| \int_{t_0}^t |\Phi_\sigma(\tau, t_0, \theta^\circ) \Xi_{\sigma(\tau)}(\theta^\circ)| d\tau \|v\|_{\infty,\mathcal{I}} \\ &\quad + |\Gamma_{\sigma(t)}| |v(t)|. \end{aligned}$$

The result follows since by hypothesis the switching signal σ contains only a finite number of discontinuity points on $\mathcal{I}$. ■

Lemma 5: Consider the same assumptions as in Lemma 3, and let the set Θ be compact and let $A(\theta)$, $B(\theta)$, and $C(\theta)$ depend continuously on θ . Moreover, let the switching controller (2) be globally discerning and a discerning switching signal σ be adopted on the observation interval $\mathcal{I}$. Finally, let the observed data z be generated by the closed-loop system (13). Then, there exist two class $\mathcal{K}$ functions² $\alpha(\cdot)$ and $\beta(\cdot)$ (independent of θ°) such that

$$\alpha(|\theta^\circ - \hat{\theta}|) |\chi_0| \leq \delta_\sigma(z^{(n)}, \hat{\theta}) \leq \beta(|\theta^\circ - \hat{\theta}|) |\chi_0| \quad (16)$$

for any $\hat{\theta} \in \Theta$. ■

Proof: By replacing $z^{(n)}(t)$ with $\Lambda_{\sigma(t)}(\theta^\circ) \Phi_\sigma(t, t_0, \theta^\circ) \chi_0$ in the expression for $\delta_\sigma(z^{(n)}, \hat{\theta})$, simple calculations yield

$$\begin{aligned} [\delta_\sigma(z^{(n)}, \hat{\theta})]^2 &= \chi_0^\top \underbrace{\begin{bmatrix} I \\ -V_\sigma(\theta^\circ, \hat{\theta}) \end{bmatrix}^\top W_\sigma(\theta^\circ, \hat{\theta}) \begin{bmatrix} I \\ -V_\sigma(\theta^\circ, \hat{\theta}) \end{bmatrix}}_{X_\sigma(\theta^\circ, \hat{\theta})} \chi_0 \quad (17) \end{aligned}$$

for any $\hat{\theta} \in \Theta$, where

$$\begin{aligned} U_\sigma(\theta^\circ, \hat{\theta}) &= \int_{\mathcal{I}} \Phi_\sigma(\tau, t_0, \hat{\theta})^\top \Lambda_{\sigma(\tau)}(\hat{\theta})^\top \\ &\quad \times \Lambda_{\sigma(\tau)}(\theta^\circ) \Phi_\sigma(\tau, t_0, \theta^\circ) d\tau \quad (18) \end{aligned}$$

$$V_\sigma(\theta^\circ, \hat{\theta}) = \left(W_\sigma(\hat{\theta}) \right)^{-1} U_\sigma(\theta^\circ, \hat{\theta}) \quad (19)$$

$$W_\sigma(\theta^\circ, \hat{\theta}) = \begin{bmatrix} W_\sigma(\theta^\circ) & U_\sigma(\theta^\circ, \hat{\theta})^\top \\ U_\sigma(\theta^\circ, \hat{\theta}) & W_\sigma(\hat{\theta}) \end{bmatrix}. \quad (20)$$

Then, it follows from (17) that $\delta_\sigma(z^{(n)}, \hat{\theta}) \leq \beta_1(\theta^\circ, \hat{\theta}) |\chi_0|$ with $\beta_1^2(\theta^\circ, \hat{\theta}) = \lambda_{\max}(X_\sigma(\theta^\circ, \hat{\theta}))$. Note that $\beta_1(\theta^\circ, \hat{\theta})$ depends continuously on θ° and $\hat{\theta}$ and, in addition, $\beta_1(\theta^\circ, \hat{\theta}) = 0$ if and only if $\theta^\circ = \hat{\theta}$ since the switching law is discerning by hypothesis. Then, $\beta(\rho)$ can be taken equal to $\max_{\theta^\circ, \hat{\theta} \in \Theta, |\theta^\circ - \hat{\theta}| \leq \rho} \beta_1(\theta^\circ, \hat{\theta})$.

As for the lower bound, notice that the *joint observability Gramian* $W_\sigma(\theta^\circ, \hat{\theta})$ gives information about the degree of distinguishability between the two systems $(\mathcal{P}(\theta^\circ)/\mathcal{C}_\sigma)$ and $(\mathcal{P}(\hat{\theta}), \mathcal{C}_\sigma)$. In fact, whenever the switching control law $\mathcal{C}_\sigma$ is globally discerning, $W_\sigma(\theta^\circ, \hat{\theta})$ is singular if and only if $\theta^\circ = \hat{\theta}$ (this is a direct consequence of Lemma 3). Then

$$\begin{aligned} [\delta_\sigma(z^{(n)}, \hat{\theta})]^2 &\geq \lambda_{\min}(W_\sigma(\theta^\circ, \hat{\theta})) (|\chi_0|^2 + |V_\sigma(\theta^\circ, \hat{\theta}) \chi_0|^2) \\ &\geq \lambda_{\min}(W_\sigma(\theta^\circ, \hat{\theta})) |\chi_0|^2. \end{aligned}$$

²Recall that a function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing, and $\varphi(0) = 0$.

Since $\lambda_{\min}(W_\sigma(\theta^\circ, \hat{\theta}))$ depends continuously on $\theta^\circ, \hat{\theta}$ and is equal to 0 if and only if $\theta^\circ = \hat{\theta}$ (again thanks to the discernibility of the switching law), then a class $\mathcal{K}$ function $\alpha(|\theta^\circ - \hat{\theta}|)$ can be found that satisfies the inequality $\lambda_{\min}(W_\sigma(\theta^\circ, \hat{\theta})) \geq \alpha^2(|\theta^\circ - \hat{\theta}|)$ for any $\theta^\circ, \hat{\theta} \in \Theta$. In particular, $\alpha(|\theta^\circ - \hat{\theta}|)$ can be constructed as in the proof of [29, Th. 2]. ■

The importance of Lemma 5 is that it allows to bound the distance $\delta_\sigma(z^{(n)}, \hat{\theta})$, pertaining to the noise-free dynamics, in terms of the initial state χ_0 and of the distance between the true parameter vector θ° and the candidate estimate $\hat{\theta}$. In particular, the left-most inequality in (16) ensures that $\delta_\sigma(z^{(n)}, \hat{\theta})$ cannot be small when the discrepancy $\theta^\circ - \hat{\theta}$ is large, whereas the right-most inequality ensures that $\delta_\sigma(z^{(n)}, \hat{\theta})$ nicely degrades to 0 as the estimate $\hat{\theta}$ approaches the true value θ° . With respect to the latter observation, from (16) it follows that when Θ_L is ϵ -dense in Θ there always exists a parameter vector $\theta_\ell \in \Theta_L$ such that $\delta_\sigma(z^{(n)}, \theta_\ell) \leq \beta(\epsilon) |\chi_0|$. By exploiting inequalities (15) and (16), the main result of this section can finally be stated.

Theorem 3: Let the same assumptions as in Lemma 5 hold. Further, let Θ_L be ϵ -dense in Θ . Then, the estimate $\hat{\theta}_L$ obtained as in (11) is such that

$$|\theta^\circ - \hat{\theta}_L| \leq \alpha^{-1} \left(\beta(\epsilon) + \frac{2\gamma \|v\|_{\infty,\mathcal{I}}}{|\chi_0|} \right) \quad (21)$$

where $\alpha^{-1}(\cdot)$ is the inverse of $\alpha(\cdot)$.

Proof: Since Θ_L is ϵ -dense in Θ , there exists at least one $\hat{\theta}^* \in \Theta_L$ such that $|\theta^\circ - \hat{\theta}^*| \leq \epsilon$. For such a $\hat{\theta}^*$, one has

$$\begin{aligned} \delta_\sigma(z, \hat{\theta}^*) &\leq \delta_\sigma(z^{(n)}, \hat{\theta}^*) + \|z^{(f)}\|_{2,\mathcal{I}} \\ &\leq \beta(|\theta^\circ - \hat{\theta}^*|) |\chi_0| + \|z^{(f)}\|_{2,\mathcal{I}} \\ &\leq \beta(\epsilon) |\chi_0| + \|z^{(f)}\|_{2,\mathcal{I}}. \end{aligned}$$

Since the estimate $\hat{\theta}_L$ is optimal in Θ_L , one has also

$$\delta_\sigma(z, \hat{\theta}_L) \leq \delta_\sigma(z, \hat{\theta}^*) \leq \beta(\epsilon) |\chi_0| + \|z^{(f)}\|_{2,\mathcal{I}}.$$

Further, by exploiting the lower bound in Lemma 5, we can write

$$\begin{aligned} \delta_\sigma(z, \hat{\theta}_L) &\geq \delta_\sigma(z^{(n)}, \hat{\theta}_L) - \|z^{(f)}\|_{2,\mathcal{I}} \\ &\geq \alpha(|\theta^\circ - \hat{\theta}_L|) |\chi_0| - \|z^{(f)}\|_{2,\mathcal{I}}. \end{aligned}$$

Hence

$$\alpha(|\theta^\circ - \hat{\theta}_L|) |\chi_0| \leq \beta(\epsilon) |\chi_0| + 2 \|z^{(f)}\|_{2,\mathcal{I}}$$

which can be written as (21) using Lemma 4 and the fact that any class $\mathcal{K}$ function is invertible. ■

Remark 3: Concerning the upper bound on the estimation error provided by inequality (21), it can be seen that the term $\beta(\epsilon)$ decreases as the number of models increases, whereas the term $2\gamma \|v\|_{\infty,\mathcal{I}}/|\chi_0|$ can be regarded as a sort of noise-to-signal ratio, and decreases as χ_0 increases. This latter property is very relevant in the context of adaptive and fault-tolerant control. In fact, in the presence of a plant variation that renders the closed-loop system unstable, the term $2\gamma \|v\|_{\infty,\mathcal{I}}/|\chi(t)|$ becomes smaller and smaller, thus increasing the reliability of the estimate. This idea has been already exploited, in the case of a finite parameter set, in order to derive a control reconfiguration scheme able to ensure input-to-state stability (ISS); see for instance [10], [20]. The fact that a bound like (21) can be derived also when the parameter set is not finite opens the way to the development of novel control reconfiguration schemes capable of achieving ISS for time-varying systems even when the parameters can take on a continuum of values. ■

V. DESIGN EXAMPLE AND DISCUSSION

In this section, some considerations on how to design discerning switching controllers are provided.

A. Example

Consider an LTI plant with system matrices

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad C(\theta) = [1 \quad 0]$$

with $\theta = (a, b)$. Notice that assumption A1 holds for any $\theta \in \Theta$ since the plant is in observer form. Let the switching controller be a purely proportional one

$$u = -K_\sigma y. \quad (22)$$

1) One Unknown Parameter: Suppose that only b is unknown, while a is perfectly known, that is $\Theta = \{(a, b) : a = a_0, b \in [b_1, b_2]\}$. In view of Lemma 2, consider the two characteristic polynomials $\varphi_i(s, \theta) = s^2 + a_0 s + bK_i$ and $\varphi_i(s, \theta') = s^2 + a_0 s + b'K_i$. The Sylvester matrix of the two polynomials is

$$S_i(\theta, \theta') = \begin{bmatrix} 1 & a_0 & bK_i & 0 \\ 0 & 1 & a_0 & bK_i \\ 1 & a_0 & b'K_i & 0 \\ 0 & 1 & a_0 & b'K_i \end{bmatrix}.$$

Accordingly, the Sylvester resultant is

$$R_i(\theta, \theta') = K_i^2 (b - b')^2.$$

It can be seen that if $K_i \neq 0$ then the resultant is different from zero whenever b and b' are different. Hence, in this case, a single proportional controller with nonnull gain is globally discerning and there is no need for considering a switching controller in that $\mathcal{D}_1 = \{K_1 \neq 0\}$. Similar considerations hold when a is unknown while b is known.

2) Two Unknown Parameters: Suppose now that both a and b are unknown, i.e., $\Theta = \{(a, b) : a \in [a_1, a_2], b \in [b_1, b_2]\}$. Accordingly, we consider the two polynomials $\varphi_i(s, \theta) = s^2 + as + bK_i$ and $\varphi_i(s, \theta') = s^2 + a's + b'K_i$. The Sylvester matrix of the two polynomials is

$$S_i(\theta, \theta') = \begin{bmatrix} 1 & a & bK_i & 0 \\ 0 & 1 & a & bK_i \\ 1 & a' & b'K_i & 0 \\ 0 & 1 & a' & b'K_i \end{bmatrix}.$$

The Sylvester resultant is

$$R_i(\theta, \theta') = K_i^2 (b - b')^2 - K_i (ba' - ab')(a - a').$$

Thus, a single controller is not sufficient for global discernibility since by choosing $(b - b')^2 = \varepsilon$ and $(ba' - ab')(a - a') = K_i \varepsilon$ one has $R_i(\theta, \theta') = 0$. In fact, since ε can be arbitrarily small, one can always find a, b, a', b' so as to satisfy these two equations regardless of the amplitude of the parameter set Θ . Then, $\mathcal{D}_1 = \emptyset$.

On the contrary, a switching controller with two modes, $N = 2$, is generically globally discerning. To see this, notice that

$$\begin{bmatrix} R_1(\theta, \theta') \\ R_2(\theta, \theta') \end{bmatrix} = \underbrace{\begin{bmatrix} K_1^2 - K_1 \\ K_2^2 - K_2 \end{bmatrix}}_Q \begin{bmatrix} (b - b')^2 \\ (ba' - ab')(a - a') \end{bmatrix}$$

and $\det(Q) = K_1 K_2 (K_2 - K_1)$. If we choose K_1 and K_2 such that $K_1 \neq K_2$, $K_1 \neq 0$, and $K_2 \neq 0$, then $\det(Q) \neq 0$. Accordingly,

the two resultants $R_1(\theta, \theta')$ and $R_2(\theta, \theta')$ can simultaneously vanish if and only if $(b - b')^2 = 0$ and $(ba' - ab')(a - a') = 0$ which is equivalent to $a = a'$ and $b = b'$. Under such conditions, we have $\mathcal{D}_2 = \{(K_1, K_2) : K_1 \neq K_2, K_1 \neq 0, K_2 \neq 0\}$, which is generic and of full measure in $\mathbb{R}^2$.

B. Design Considerations

The foregoing example can be used as a starting point to derive design conditions for discerning switching controllers in case of a generic SISO plant. To this end, consider the polynomials

$$a(s, \theta) = \det(sI - A(\theta)), \quad b(s, \theta) = C(\theta) \text{adj}(sI - A(\theta))^T B(\theta)$$

and, for the time being, let the switching controller be again a purely proportional one (22). Then, for any $i \in \mathcal{N}$ and any $\theta \in \Theta$, the closed-loop feedback polynomial is

$$\varphi_i(s, \theta) = a(s, \theta) + K_i b(s, \theta).$$

Since the coefficients of the two polynomials $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ depend affinely on the controller gain K_i , their Sylvester Resultant $R_i(\theta, \theta')$ is a polynomial function of K_i , i.e.,

$$R_i(\theta, \theta') = \sum_{j=0}^M K_i^j c_j(\theta, \theta')$$

for some integer M and scalar functions $c_j(\theta, \theta')$.

Notice now that, under assumption A1, the greatest common divisor of the two polynomials $a(s, \theta)$ and $b(s, \theta)$, denoted hereafter as $\gcd\{a(s, \theta), b(s, \theta)\}$, corresponds to the characteristic polynomial of the uncontrollable part of $\mathcal{P}(\theta)$ (with $\gcd\{a(s, \theta), b(s, \theta)\} = 1$ when $\mathcal{P}(\theta)$ is controllable). Then, when condition b) of Lemma 2 holds, we have $\gcd\{a(s, \theta), b(s, \theta), a(s, \theta'), b(s, \theta')\} = 1$ for any pair (θ, θ') of different parameter vectors. Moreover, condition a) of Lemma 2 implies that $a(s, \theta) b(s, \theta') - a(s, \theta') b(s, \theta) \neq 0$ when $\theta \neq \theta'$. As a consequence, under that assumptions of Theorem 1, we can apply [30, Lemma 3] and conclude that for any $\theta \neq \theta'$ there exists at least one K_i (actually infinitely many) such that the two characteristic polynomials $\varphi_i(s, \theta)$ and $\varphi_i(s, \theta')$ are coprime, i.e., such that their resultant $R_i(\theta, \theta')$ is different from zero. This, in turn, implies that, for any $\theta \neq \theta'$, the vector $c(\theta, \theta') = [c_M(\theta, \theta'), \dots, c_0(\theta, \theta')]^\top$ is different from zero.

Consider now a switching controller with exactly $N = M + 1$ modes. Then, we can write

$$\begin{bmatrix} R_1(\theta, \theta') \\ \vdots \\ R_{M+1}(\theta, \theta') \end{bmatrix} = \underbrace{\begin{bmatrix} K_1^M & \dots & K_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ K_{M+1}^M & \dots & K_{M+1} & 1 \end{bmatrix}}_Q c(\theta, \theta').$$

Since for different parameter vectors θ, θ' the vector $c(\theta, \theta')$ is never zero, we see that like in the previous example, by choosing the $M + 1$ gains $K_1, \dots, K_{M+1}$ such that $\det(Q) \neq 0$, we ensure that the $M + 1$ resultants $R_1(\theta, \theta'), \dots, R_{M+1}(\theta, \theta')$ cannot simultaneously vanish, thus making the switching controller discerning. Finally, one can easily verify that the conditions on the gains ensuring $\det(Q) \neq 0$ (and hence parameter identifiability) are simply $K_i \neq 0$, for any $i \in \mathcal{N}$, and $K_i \neq K_j$, for any $i, j \in \mathcal{N}$ with $i \neq j$.

While the foregoing analysis accounts only for the case of purely proportional switching controllers, it is an easy matter to see that analogous design conditions can be derived whenever the switching controller is obtained from an initial LTI controller by varying one single parameter (gain, location of a pole/zero, etc.) entering affinely into the closed-loop characteristic polynomial. These considerations suggest that a convenient way to design a discerning switching controller amounts to the

following two-step procedure: i) design an LTI controller accounting for all the control specifications; ii) design the switching controller by varying one parameter of the LTI controller designed in step i) in order to ensure discernibility while preserving all the other specifications. Of course, this is just one of the possible ways to tackle the design problem since, in view of Theorem 1, almost all switching controllers are discerning.

VI. CONCLUSION

We have addressed the problem of identifying the parameters of a multivariable linear system by means of switching control. We have shown that even when the parameter set is not finite, parameter identifiability can be generically ensured by switching among a finite number of LTI controllers. In particular, an upper bound on the number of controller modes needed for parameter identifiability can be given in terms of the dimension of the parameter set. The results also indicate that the goals of parameter identifiability as well as stabilization can be simultaneously accomplished by means of switching control. Other aspects of practical relevance, i.e., noisy environments and finite sampling of the parameter set, have also been discussed. As a future step, it is worth investigating how the results can be extended to secure identifiability under sampled-data observations, which is a nontrivial issue when the uncertainty set is described by a continuum, as in the present case.

APPENDIX

Hereafter, we show by means of a simple example that a single LTI controller need not be sufficient to ensure discernibility when the parameter set Θ is not finite. Consider an LTI plant with system matrices

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix}, B(\theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C(\theta) = [1 \quad 0 \quad 0]$$

with $\theta = (a, b, c)$ and $\Theta = [-1, 1]^3$. Since the plant is single-input single-output (SISO), a necessary condition for an LTI controller with parameter vector ρ to be discerning is that for any pair $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$ the two closed-loop characteristic polynomials $\varphi(s, \theta, \rho)$ and $\varphi(s, \theta', \rho)$ are coprime (see [10, Proposition 2]). We show below that such a condition cannot be achieved in this example in that, for any controller order and any choice of ρ , there always exist θ and θ' such that $\varphi(s, \theta, \rho)$ and $\varphi(s, \theta', \rho)$ have a common root. To see this, consider the closed-loop characteristic polynomial $\varphi(s, \theta, \rho) = D_C(s, \rho)(s^3 + as^2 + bs + c) + N_C(s, \rho)$, where $D_C(s, \rho)$ and $N_C(s, \rho)$ are arbitrary polynomials representing denominator and, respectively, numerator of the controller transfer function. Let s_0 be any of the root of $\varphi(s, \theta, \rho)$ for $\theta = 0$. Then, we have $D_C(s_0, \rho)s_0^3 + N_C(s_0, \rho) = 0$ and, hence, $\varphi(s_0, \theta', \rho) = D_C(s_0, \rho)(a's_0^2 + b's_0 + c')$. Moreover, it can be easily checked that, for any triple $\theta' = (a', b', c')$ with $b' = -2a'\text{Re}\{s_0\}$ and $c' = a'(\text{Re}\{s_0\}^2 + \text{Im}\{s_0\}^2)$ it turns out that $a's_0^2 + b's_0 + c' = 0$ and, hence, $\varphi(s_0, \theta', \rho) = 0$ for any ρ . Hence, there exist infinitely many parameter vectors in Θ such that s_0 is a root of the closed-loop characteristic polynomial $\varphi(s, \theta', \rho)$ for any ρ , thus making it impossible to achieve discernibility with a single LTI controller.

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