A power consensus algorithm for DC microgrids
De Persis, Claudio; Weitenberg, Erik R. A.; Dorfler, Florian

Published in:
Automatica

DOI:
10.1016/j.automatica.2017.12.026

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2018

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
A power consensus algorithm for DC microgrids

Claudio De Persis a,*, Erik R.A. Weitenberg a, Florian Dörfler b

a Engineering and Technology Institute Groningen, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands
b Automatic Control Laboratory, Swiss Federal Institute of Technology, 8092 Zürich, Switzerland

ARTICLE INFO

Article history:
Received 16 February 2017
Received in revised form 18 August 2017
Accepted 15 November 2017
Available online 11 January 2018

Keywords:
DC microgrids
Power sharing
Distributed control
Nonlinear consensus
Lyapunov stability analysis

ABSTRACT

A novel power consensus algorithm for DC microgrids is proposed and analyzed. DC microgrids are networks composed of DC sources, loads, and interconnecting lines. They are represented by differential–algebraic equations connected over an undirected weighted graph that models the electrical circuit. The proposed algorithm features a second graph, which represents the communication network over which the source nodes exchange information about the instantaneous powers, and which is used to adjust the injected current accordingly. This gives rise to a nonlinear consensus-like system of differential–algebraic equations that is analyzed via Lyapunov functions inspired by the physics of the system. We establish convergence to the set of equilibria, where weighted power consensus is achieved, as well as preservation of the weighted geometric mean of the source voltages. The results apply to networks with constant impedance, constant current and constant power loads.

© 2018 Elsevier Ltd. All rights reserved.

1. Introduction

The proliferation of renewable energy sources and storage devices that are intrinsically operating using the DC regime is stimulating interest in the design and operation of DC microgrids, which have the additional desirable feature of preventing the use of inefficient power conversions at different stages. These DC microgrids might have to be deployed in areas where an AC microgrid is already in place, creating what is called a hybrid microgrid (Loh, Li, Chai, & Blaabjerg, 2013), for which rigorous analytical studies are still in their infancy. Furthermore, the envisioned future in which power generation is far away from the major consumption sites raises the problem of how to transmit power with low losses, a problem for which High Voltage Direct Current (HVDC) networks perform comparatively better than AC networks. Finally, also mobile grids on ships, aircrafts, and trains are based on a DC architecture.

With DC and hybrid microgrids, as well as HVDC networks, on the rise, we need to develop a deeper system-theoretic understanding of this interesting class of dynamical networks. In this paper we propose and analyze a control algorithm for a DC microgrid that enforces power sharing among the different power sources.

1.1. Literature review

The literature on DC microgrids is rapidly growing. We summarize below the contributions that share a systems and control-theoretic point of view on these networks. The work (Nasirian, Moayedi, Davoudi, & Lewis, 2015) relies on a cooperative control paradigm for DC microgrids to replace the conventional secondary control by a voltage and a current regulator. In Zhao and Dörfler (2015) a voltage droop controller for DC microgrids inspired by frequency droop in AC powernetworks is analyzed, and a secondary consensus control strategy is added to prevent voltage drift and achieve optimal current injection. The paper (Belk, Inam, Perreault & Turitsyn, 2016) models the DC microgrid via the Brayton–Moser equations and uses this formalism to show that with the addition of a decentralized integral controller voltage regulation to a desired reference value is achieved. Other schemes achieving desirable power sharing properties are proposed but no formal analysis is provided. In Tucci, Meng, Guerrero, and Ferrari-Trecate (2016), a secondary consensus-based control scheme for current sharing and voltage balancing in DC microgrids is designed in a Plug-and-Play fashion to allow for the addition or removal of generation units. A distributed control method to enforce power sharing among a cluster of DC microgrids is proposed in Moayedi and Davoudi (2016). Other work has focused on the challenges in the stability analysis of DC microgrids using consensus-like algorithms due to the interaction between the communication
network and the physical one (Meng, Dragicevic, Roldán-Pérez, Vasquez, & Guerrero, 2016). Finally, feasibility of the nonlinear algebraic equations in DC power circuits is studied by Barabanov, Ortega, Grino, and Polyak (2016), Lavei, Rantzer, and Low (2011), and Simpson-Porco, Dörfler, and Bullo (2015).

A closely related research area is that of multi-terminal HVDC transmission systems. The paper (Sarlette, Dai, Phulpin, & Ernst, 2012) focuses on cooperative frequency control for these networks. In Andréasson et al. (2014) distributed controllers that keep the voltages close to a nominal value and guarantee a fair power sharing are considered, whereas passivity-based decentralized PI control for the global asymptotic stabilization of multi-terminal high-voltage is studied in Zonetti, Ortega, and Benchab (2015). The paper (Zonetti, Ortega, & Schiffer, 2016) studies feasibility and power sharing under decentralized droop control. We refer to Zonetti (2016, Chapter 4) for an annotated bibliography of HVDC transmission systems.

1.2. Main contribution

The objective of the paper is to propose a novel control algorithm that exhibits three main features: (i) it makes sources provide power in prescribed ratios for a wide range of load magnitudes; (ii) it simultaneously guarantees that all voltages stay within a compact set around an operating point, and that the geometric voltage of the source voltages is maintained constant for all time; (iii) it tackles so-called “ZIP” (constant impedance, constant current and constant power) loads, which are known to substantially affect the stability of the system.

Power sharing is essential in microgrid operations because changing load conditions may lead to an imbalance situation in which few sources, if not a single one, provide the majority of the power demand. This might result in cases in which the overloaded sources exceed their capacity limit driving the microgrid to instability.

While droop controllers are usually employed to achieve power distribution in DC microgrids, they cannot achieve such a task exactly because they can only strike a tradeoff between power sharing and voltage control. As such, there are no generally acknowledged criteria for the selection of droop gains that provide guaranteed power sharing properties in the presence of unknown or uncertain variable load conditions. Our controller overcomes such limitations, providing a substantial improvement with respect to existing controllers.

The proposed controller is enabled by communicating the instantaneous source power measurements among neighboring source nodes, averaging these measurements and setting the voltage at the source terminals accordingly. An additional feature of the algorithm is that a weighted geometric average of the source voltages is preserved. In absence of a communication environment, our distributed consensus-based algorithm can also be implemented by power talk communication via the DC microgrid (Angelichinoski et al., 2015).

The system dynamics present interesting features. By averaging the power measurements that the sources communicate amongst each other, the system dynamics becomes an intriguing combination of the physical network (the weighted Laplacian of the electrical circuit appearing in the power measurements) and the communication network (over which the information about the power measurements is exchanged): ZIP loads introduce algebraic equations in the system’s dynamics, adding additional complexity and non-linearities.

To analyze this system of nonlinear differential–algebraic equations without going through a linearization of the dynamics, Lyapunov-based arguments become very convenient. The Lyapunov functions in this case are constructed starting from the power dissipated in the network that is further shaped to take into account the specifics of the dynamics. In fact, it is shown that the closed-loop system can be written as a weighted gradient of the Lyapunov function (Lemma 2), a form that is crucial to carry out the stability analysis. The presence of the loads, which shift the equilibrium of interest, is taken into account by the so-called Bregman function (De Persis & Monshizadeh, 2018). The level sets of the Lyapunov functions are used to estimate the excursion of the state response of these systems and therefore, combined with the preservation of the geometric average of the source voltages, can be used to obtain an estimate of the voltage at steady state.

Reactive power sharing algorithms have been first suggested by Schiffer, Seel, Raisch, and Sezi (2016) for network-reduced AC microgrids whose voltage dynamics show similar features as in DC grids. In this paper we show that a related idea can be adopted also for network preserved DC microgrids. The novelties of this contribution with respect to Schiffer et al. (2016) are the different dynamics of the system under study, the explicit consideration of algebraic equations in the model and the use of Lyapunov arguments to prove the main results.

1.3. Paper organization

The model of the DC microgrid is introduced in Section 2. The power consensus algorithm is introduced in Section 3. The analysis of the closed-loop system is carried out in Section 4 for the general case of ZIP loads, and then specialized to the case of ZI loads, since the latter permits to obtain stronger results under weaker conditions. The simulations of the algorithm are provided in Section 5. Conclusions are drawn in Section 6.

1.4. Notation

Given a vector $v$, the symbol $[v]$ represents the diagonal matrix whose diagonal entries are the components of $v$. The notation $\col(v_1, v_2, \ldots, v_n)$ with $v_i$ scalars, represents the vector $[v_1 \ v_2 \ \ldots \ v_n]^T$. If $v_i$ are matrices having the same number of columns, then $\col(v_1, v_2, \ldots, v_n)$ denotes the matrix $[v_1 \ | \ v_2 \ | \ \ldots \ | \ v_n]^T$. The symbol $s_m$ represents the $m$-dimensional vector of all 1’s, whereas $\delta_m$ is the $m \times n$ matrix of all zeros. When the size of the matrix is clear from the context the index is omitted. The $n \times n$ identity matrix is represented as $I_n$. Given a vector $v \in \mathbb{R}^n$, the symbol $\log(v)$ denotes the element-wise logarithm, i.e., the vector $[\log(v_1) \ | \ \ldots \ | \ \log(v_n)]$. Given a set $S$, the symbol $|S|$ indicates the cardinality of the set.

2. DC resistive microgrid

The DC microgrid is modeled as a undirected connected graph $G = (V, E)$, with $V = \{1, 2, \ldots, n\}$ the set of nodes (or buses) and $E \subseteq V \times V$ the set of edges. The edges represent the interconnecting lines of the microgrid, which we assume here to be resistive. Associated to each edge is a weight modeling the conductance (or reciprocal resistance) $1/\kappa_k > 0$, with $k \in E$. The set of nodes is partitioned into the two subsets of $n_l$ DC sources $V_l$ and $n_i$ loads $V_i$, with $n_l + n_i = n$.

Let $I \in \mathbb{R}^n, V \in \mathbb{R}^n_0$ denote the vectors of currents and potentials respectively at the nodes of $G$. The current–potential relation in a resistive network is given by the identity $I = B^{-1}V$, with $B \in \mathbb{R}^{n \times |E|}$ being the incidence matrix of $G$ and $\Gamma = \text{diag}(r_{1}, \ldots, r_{|E|})$ the diagonal matrix of conductances. Considering the partition of the nodes in sources and loads, we let $I = \col(I_l, I_l)$ and $V = \col(V_s, V_l)$ without loss of generality, where $I_l = \col(I_{1_l}, \ldots, I_{n_l}), I_l = \col(I_{n_l+1}, \ldots, I_{n}), V_s = \col(V_s, \ldots, V_s), V_l = \col(V_{n_l+1}, \ldots, V_n)$ and we correspondingly partition the...
Then, the current-potential relation can be rewritten as
\[ \begin{bmatrix} I_s \\ \hat{I}_l \end{bmatrix} = \begin{bmatrix} B_l & B_s G_l \\ B_l G_s & B_s G_s \end{bmatrix} \begin{bmatrix} V_s \\ \hat{V}_s \end{bmatrix} = \begin{bmatrix} Y_{ss} & Y_{sl} \\ Y_{ls} & Y_{ll} \end{bmatrix} \begin{bmatrix} V_s \\ \hat{V}_s \end{bmatrix}. \]

Observe that both \( Y_{ss} \) and \( Y_{ll} \) are positive definite since they are principal submatrices of a Laplacian of a connected undirected graph. This allows us to eliminate the load voltages as \( V_l = Y_{ll}^{-1} \hat{I}_l \) and \( Y_{ls}^{-1} I_s \) and reduce the network to the source nodes \( V_s \) with balance equations
\[ I_s = Y_{ll}^{-1} \hat{I}_l = Y_{red} V_s, \]
where \( Y_{red} = Y_{ss} - Y_{sl} Y_{ll}^{-1} Y_{ls} \) is known as the Kron-reduced conductance matrix (Döffler & Bullo, 2013) and \(-Y_{ss}^{-1} I_s\) is the mapping of the load current injections to the sources.

### 3. Power consensus controllers

We propose controllers that force the different sources to share the total power injection in prescribed ratios (Schiffer et al., 2016). For this purpose, a communication network is deployed to connect the source nodes, through which the controllers exchange information about the instantaneous injected powers. This communication network is modeled as an undirected unweighted graph \( (\mathcal{V}_c, \mathcal{E}_c) \), where \( \mathcal{V}_c = \mathcal{V}_s \). Associated with the communication graph is the \( n_c \times n_c \) Laplacian matrix \( L_c = D_c - A_c \), where \( D_c = \text{diag}(D_{c1}, \ldots, D_{cn}) \) is the degree matrix, \( D_{ci} \) is the degree of node \( i \), and \( A_c \) is the adjacency matrix of the communication graph. Note that the nodes of the communication network (but not necessarily the edges) coincide with the source nodes of the microgrid. For each node \( i \in \mathcal{V}_c \), the set \( \mathcal{N}_{ci} = \{ j \in \mathcal{V}_c : (i, j) \in \mathcal{E}_c \} \) represents the neighbors connected to node \( i \) via the communication graph.

**Controllers.** We assume that all sources \( V_s \) are controllable voltage sources (e.g., realized by boost converters), which are controlled as a function of the measured local current and power injections \( I_s \) and \( P_i \) as well as the injected power \( P_j \) at neighboring sources that need to be communicated. In the following, we will design powers consensus controllers in such a way that the algorithm achieves weighted proportional power sharing according to ratios \( K_{ij} > 0 \) chosen by the operator, that is, given \( P_i := I_i V_i \) and \( i \in \mathcal{V}_c \), the control objective is to guarantee that at steady state the following identities hold:
\[ \frac{P_j}{K_{ij}} = \frac{P_i}{K_{ii}}, \quad \forall i, j \in \mathcal{V}_c. \]

Keeping this in mind, the proposed controllers are of the form
\[ C_i(V_s) \hat{V}_i = -I_i + u_i, \quad i \in \mathcal{V}_c, \] (4)
where
\[ C_i(V_s) = V_i^{-1} D_{ci} K_{ii}^{-1} \]
\[ = V_i^{-1} (\text{const.}), \quad i \in \mathcal{V}_c \] (5)
can be interpreted as a nonlinear capacitance, \( K_{ii} > 0 \), the power sharing coefficient, is of suitable units such that \( C_i(V_s) \) actually has the units of a capacitance, \( I_i \) is the injected current at node \( i \in \mathcal{V}_c \) as defined in (1), and the term
\[ u_i = V_i^{-1} D_{ci} K_{ii}^{-1} \sum_{j \in \mathcal{N}_{ci}} K_{ij}^{-1} P_j, \quad i \in \mathcal{V}_c \] (6)
represents an ideal current source that is controlled as a function of the local voltage \( V_i \) and the injected power \( P_j = V_j I_j \) at the neighboring node sources \( j \in \mathcal{N}_{ci} \). The current sources \( u_i \), \( i \in \mathcal{V}_c \), are designed to make the right-hand side of (4) equal to \( V_i^{-1} D_{ci}^{-1} K_{ii} \sum_{j \in \mathcal{N}_{ci}} (K_{ij}^{-1} P_j - K_{ii}^{-1} P_i) \), which is a weighted average of the power variables, multiplied by the factor \( V_i^{-1} D_{ci}^{-1} \).

### Remark 1

*(Digital Implementation and Circuit Realization.)* The control algorithm (4), (5), (6) can be implemented at each controllable voltage source as follows. The local current and power injections \( I_s \) and \( P_i \) are measured, and the power injections are broadcast through a communication network. The local current measurements \( I_s \) and the power injections \( P_i \) at neighboring sources \( j \in \mathcal{N}_{ci} \) are processed along with the current measurement \( I_s \) to compute the source voltage value \( V_i \) applied at the source terminals as in (4), (5), (6).

Since the signal \( u_i \) has the dimension of amps and appears as a current signal in (4), we have drawn an equivalent circuit realization of Eq. (4) in Fig. 1. Comparing with Belk et al. (2016), the equivalent current source \( u_i \) can be generated also by a voltage source with value \( v_i \) in series with a resistance \( r_i \) provided that \( v_i = r_i u_i + V_i \). Finally, the dynamic droop controller in Zhao and Döffler (2015) corresponds in our notation to a constant capacitance \( c_i \) and current source \( u_i \). We remark again that this is merely an equivalent circuit that helps interpreting the controllers [4].

In the end, the most convenient realization of (4), (5), (6) is by means of a converter controlled as a voltage source.

Multiplying both sides of (4) by \( V_i^2 D_{ci} K_{ii}^{-1} \), one arrives at the closed-loop system
\[ K_i \hat{V}_i = -V_i D_{ci} K_{ii}^{-1} P_i + V_i \sum_{j \in \mathcal{N}_{ci}} K_{ij}^{-1} P_j \]
\[ = V_i \sum_{j \in \mathcal{N}_{ci}} (K_{ij}^{-1} P_j - K_{ii}^{-1} P_i), \quad i \in \mathcal{V}_s, \] (7)
that is, the voltage at the source terminal is updated according to a weighted power consensus algorithm scaled by the voltage. Provided that \( V_i \neq 0 \) (a property that will be established in the next sections), Eq. (7) shows that at steady state the proposed algorithm achieves proportional power sharing as in (3).

A detailed characterization of the steady-state power signals is given in the next section *(Lemma 1).*

For interpretation purposes, we write (7) as
\[ \frac{d}{dt} K_i \ln(V_i) = \sum_{j \in \mathcal{N}_{ci}} (K_{ij}^{-1} P_j - K_{ii}^{-1} P_i), \quad i \in \mathcal{V}_s, \]

In a classic power system analysis (Chiang, 2011), the term \( K_i \ln(V_i) \) is the natural energy representation of a source of constant value \( K_i \). The interpretation of the closed loop (7) is then that the voltage at this constant power source is adapted according to a power consensus algorithm.
Remark 2 (Alternative Power Sharing Control). A possibly more simplistic and obvious power sharing controller inspired by the current-sharing controller in Zhao and Dörfler (2015) is based on a distributed averaging integral control given by

\[ C_i \dot{V}_i = -i_l + p_i, \]
\[ D_i p_i = l - p_i + \sum_{j \in N_{i,j}} (K_j^{-1}V_j p_j - K_i^{-1}V_i p_i), \quad i \in V_s \]

(8)

where \( p_i \) is a control variable in units of currents, \( C_i > 0 \) a gain with capacitance units and \( D_i > 0 \) a time constant. Note that the power sharing coefficients \( K_i \) in (8) have the units of voltages. Any steady state of this controller would guarantee for all \( i \in V_s \) that \( V_i = 0 \), and \( p_i = i_l \) is the steady-state current injection, and the vector of power injections \( K_i^{-1}[V_i, p] \) has all identical entries (power sharing). Numerical results (see Section 5) show that (7) and (8) perform similarly. Indeed, in the limit \( D_i = 0 \), near steady-state, and for nearly unit voltages (in per unit system), the closed-loops (8) and (7) have similar dynamics. In the rest of the paper we focus on the analysis of (7), since no analytical guarantee on the stability of system (8) is available at this moment.

Loads. Depending on the particular load models, the term \( i_l \) in (1) takes different expression and will henceforth be denoted as \( i_l(V_i) \) to stress the functional dependence on the load voltages. Prototypical load models that are of interest include the following:

(i) constant current loads: \( i_l(V_i) = I_l^* \in \mathbb{R}_{>0}^{n_0} \),
(ii) constant impedance: \( i_l(V_i) = -Y_l^* V_i \) with \( Y_l^* > 0 \) a diagonal matrix of load conductances, and \( V_l = \text{col}(V_{n_1+1}, \ldots, V_{n_{n_0}}) \), and
(iii) constant power: \( i_l(V_i) = [V_i]^{-1}P_l^* \), with \( P_l^* \in \mathbb{R}_{>0}^{n_0} \).

To refer to the three load cases above, we will use the indices “I”, “Z” and “P” respectively. The analysis of this paper will focus on the more general case of a parallel combination of the three loads, thus on the case of “ZIP” loads, for which

\[ i_l(V_i) = I_l^* - Y_l^* V_i + [V_i]^{-1}P_l^*. \]

(9)

Moreover, additional and stronger results on the “ZIP” case will be reported. The following analysis also applies to any load scenario where components of \( I_l^* \), \( Y_l^* \) and \( P_l^* \) are possibly zero.

Bearing in mind (1), (7), and vectorizing the expressions to avoid cluttered formulas, the closed-loop system is

\[
\begin{bmatrix}
K_{l} \dot{V}_s \\
-l_l(V_i)
\end{bmatrix} = \begin{bmatrix}
[V_i]_l K_l^{-1} P_l \\
B_l \Gamma B_l^T V
\end{bmatrix},
\]

(10)

where \( V = \text{col}(V_s, V_i) \), \( K_l = \text{diag}(K_1, \ldots, K_{n_l}) \), \( P_l = \text{col}(P_1, \ldots, P_{n_l}) \) given by

\[ P_l = [V_i]_l K_l^{-1} [V_i]_{n_0} V_s + Y_a V_l \]

(11)

is the vector of source power injections and \( i_l(V_i) \) as defined in (9) are the load currents. The interconnected closed-loop DC microgrid is then entirely described by Eqs. (10), (11) and (9). An example of a simple closed-loop DC microgrid with two sources and one constant impedance load is given in Fig. 2.

Remark 3 (Nonlinear Consensus Algorithms). To compare the control algorithm (7) with related nonlinear consensus algorithms proposed in the literature (Bauso, Giare, & Pesenti, 2006; Cortes, 2008), we neglect the algebraic constraints and the differentiation between sources and loads. This allows us to rewrite (7) as

\[ KV = -[V_i]_l K_l^{-1} [V_i] B_l \Gamma B_l^T V. \]

The weighted power mean consensus algorithms of Bauso et al. (2006) and Cortes (2008), on the other hand, can be written as

\[ [W] \dot{V} = [V]^{1-r} B_l \Gamma B_l^T V, \]

where \( W \) is a vector of weights satisfying \( 1^T W = 0 \) and \( r \in \mathbb{R} \). In the special case \( r = 0 \), we get

\[ [W] \dot{V} = [V] B_l \Gamma B_l^T V, \]

which is known to converge to the consensus value \( V_1^{w_1} \ldots V_n^{w_n} \). The analysis is based on the Lyapunov function \( \sum_{i=1}^n W_i V_i - \prod_{i=1}^n V_i^{w_i} \).

The nonlinear power consensus algorithm presented in this paper is different in that it uses another layer of averaging in addition to the averaging induced by the physical network. This, and the algebraic constraints, requires a different analysis based on physics-inspired Lyapunov functions.

4. Power consensus algorithm with ZIP loads

In this section we analyze the closed-loop system (10), (11), (9). We start by studying its equilibria, namely the set of points \( V \in \mathbb{R}_{>0}^{n_0} \) that satisfy (11) and (9), and

\[
\begin{bmatrix}
0 \\
-l_l(V_i)
\end{bmatrix} = \begin{bmatrix}
[V_i]_l K_l^{-1} P_l \\
B_l \Gamma B_l^T V
\end{bmatrix},
\]

(12)

4.1. Steady-state characterization

In the following, we show that the equilibria are fully characterized by power balance equations at the sources and current balance equations at the loads, respectively.

Lemma 1 (System Equilibria). The equilibria of the system (10), (11), (9) are equivalently characterized by

\[ \varepsilon_{ZIP} = \{ V \in \mathbb{R}_{>0}^{n_0} : \mathcal{L}_{ZIP}(V) = 0, \mathcal{P}_{ZIP}(V) = 0 \}. \]

where \( \mathcal{L}_{ZIP}(V) = 0 \) is the current balance at the loads

\[ \mathcal{L}_{ZIP}(V) = l_l(V) - Y_s V_l - Y_a V_s, \]

\[ \mathcal{P}_{ZIP}(V) = 0 \]

depicts the power balance at the sources

\[ \mathcal{P}_{ZIP}(V) = \sum_{i=1}^n W_i Y_{l,l}(V_i) + \sum_{i=1}^n Y_{l,l}(V_i) l_l(V_i) - \sum_{i=1}^n P_i, \]

where \( Y_{l,l}(V_i) \) is the Kron-reduced conductance matrix, \( Y_{l,l}^{-1} Y_{l,l}(V_i) \) is the mapping of the ZIP loads \( l_l(V_i) \) to the source bases in the Kron-reduced network as in (2), and \( P_i \) is the vector of power injections by the sources written for \( V \in \mathbb{R}_{>0}^{n_0} \) as

\[ P_i = -K_l P_l^*, \quad P_l^* = \frac{1}{x^T} l_l(V_i). \]

(13)

Observe that the steady-state injections (13) achieve indeed power sharing, and the asymptotic power value \( p_l^* \) to which the source power injections converge (in a proportional fashion according to the coefficients \( K_i, i \in V_s \)) is the total current demand divided by the weighted sum of the steady-state source voltages. The latter values and those of the load voltages are entangled by...
the power balance at the sources $\mathcal{P}_{ZIP}(V) = 0$ and the current balance
equations at the loads $\mathcal{I}_{ZIP}(V) = 0$, similar to the related
studies (Zonetti et al., 2015, Proposition 3.3), (Sanchez, Ortega,
Bergna, Molinas, & Griño, 2013, Lemma 2).

**Proof.** Let $V$ be an equilibrium of (10), (11), (9), that is let $V \in \mathbb{R}^n_{+}$ satisfy (12). From the first equation, $0 = [V]l_nK_s^{-1}P_l$, it immediately follows that $P_l = K_s x_n p_s^*$. We rewrite the current balances as

$$\begin{bmatrix} [V]^{-1}K_s x_n p_s^* \end{bmatrix} \begin{bmatrix} B_l \Gamma B_l^T \end{bmatrix} = \begin{bmatrix} B_l \Gamma B_l^T \end{bmatrix} V = l(V).$$

(14)

Next, we left-multiply (14) by $[z_n^T \ 1_n^T]$ to obtain

$$z_n^T [V]^{-1}K_s x_n p_s^* + z_n^T l(V) = 0.$$  

(15)

The latter equation can be solved for $p_s^*$ in (13). From $l(V) = B_l \Gamma B_l^T V$, we obtain (2) $\mathcal{I}_{ZIP}(V) = 0$ or

$$V_l = -Y_s^{-1}Y_s V_l + Y_s^{-1}l(V_l),$$

(16)

which replaced in the first equation of (14) returns

$$Y_s V_l + Y_s^{-1}Y_s V_l = [V]^{-1}K_s x_n p_s^*.$$  

By rearranging the terms, we arrive at

$$V_{ref} + Y_s^{-1}l(V_l) - [V]^{-1}K_s x_n p_s^* = 0,$$

which can be reformulated as $\mathcal{P}_{ZIP}(V) = 0$ after left-multiplying by $[V]$, and bearing in mind (13). The latter and (15) show that $V \in \mathcal{E}_{ZIP}$.

Conversely, let $V \in \mathcal{E}_{ZIP}$. Then the equation $l(V_l) = B_l \Gamma B_l^T V_l$ in (12) is trivially satisfied. From $\mathcal{P}_{ZIP}(V) = 0$, and $l(V_l) = B_l \Gamma B_l^T V_l$ written as (15), and going backwards through the passages above, we arrive at

$$Y_s V_l + Y_s^{-1}l(V_l) - [V]^{-1}K_s x_n p_s^*,$$

or equivalently at $[V]B_l \Gamma B_l^T V = K_s x_n p_s^*$. Hence, the power vector $P_l = [V]B_l \Gamma B_l^T V$ satisfies $L_lK_s^{-1}P_l = 0$, that is, the first equation in (12). Hence, $V \in \mathcal{E}_{ZIP}$ implies that the equilibrium equations (12) are met.

We make the standing assumption that equilibria exist:

**Assumption 1.** $\mathcal{E}_{ZIP} \neq \emptyset$.

**Remark 4 (Existence of the Equilibria $\mathcal{E}_{ZIP}$).** The analytical investigation of the existence of the equilibria $\mathcal{E}_{ZIP}$ is deferred to a future research. This is a topic of interest on its own and similar problems have been dealt with in recent work about the solvability of reactive or DC power flow equations (Barabanova et al., 2016; Bolognani & Zampieri, 2016; Sanchez et al., 2013; Simpson-Porco et al., 2015; Simpson-Porco, Dörrer, & Bullo, 2016). For instance, the problem in Simpson-Porco et al. (2016) boils down to the solution of quadratic algebraic equations of the form $[V]_i [Y]_i [V]_i - [V]_i [Y]_i [V]_i + Q_i = 0$, where $Q_i$ is the vector of constant power load demands and $V^*$ is the so-called vector of open circuit voltages (again constant). Although similarities between these equations and the equations $\mathcal{P}_{ZIP}(V) = 0 = [V]l_n V_l + [V]_l Y_s^{-1}l(V_l) + P_l$, could be used to investigate the nature of the set $\mathcal{E}_{ZIP}$, the non-quadratic nature of $\mathcal{P}_{ZIP}(V) = 0$, as well as the presence of the additional equations $Y_s^{-1}l(V_l) = V_l - Y_s^{-1}Y_s V_l$ pose additional challenges. Extra insights could come from the convex relaxation of the DC power flow equations in the context of optimal DC power flow dispatch (Lavei et al., 2011).

**Remark 5 (Equilibrium Power Balance and Voltage Inequalities).** To gain further insights into the equilibrium set $\mathcal{E}_{ZIP}$, recall that the vector of power injections is $P = \text{col}(P_l, P_r) = [V]B_l \Gamma B_l^T V$, where $P_l = [V]l_l(V_l)$. Thus, we have the inherent power balance

$$V_l^T P_l + r^T P_r = V_l^T B_l \Gamma B_l^T V \geq 0$$

(16)

implying that the amount of supplied power has to make up for load demands and resistive losses. In the special case of constant power loads, $l(V_l) = [V]^{-1} P_l$, we obtain the total (or average) power inequality $V_l^T P_l + r^T P_r \geq 0$. Equivalently, after using (13), we arrive at

$$-1^T K_s^{-1} \frac{1}{V_l^T} [V]^{-1} P_s^* + 1^T P_s^* \geq 0.$$  

This inequality can be reformulated as

$$\sum_{i \in V_l} a_i \frac{1}{V_i} \geq \sum_{i \in V_l} b_i \frac{1}{V_i},$$

(17)

with $a_i = P_{si}^* / \sum_{i \in V_l} P_{si}^*$ and $b_i = K_i / \sum_{i \in V_l} K_i$, which relates a convex combination of the reciprocals of the voltages at the loads, with a convex combination of the reciprocals of the voltages at the sources, and represents another relation between $V_l$, $V_l$ in addition to those in (16). The average voltage inequality (17) implies that the reciprocal of the harmonic average source voltage must be larger than the reciprocal of the harmonic average load voltage so that power can flow from sources to loads.

In a special case reviewed in the example below, an explicit characterization of the equilibria can be given.

**Example 1.** Consider the case of two sources ($n_s = 2$) and one load ($n_l = 1$) as in Fig. 2, in which the constant impedance load is replaced by a ZIP load. The equations $\mathcal{P}_{ZIP}(K_l) = 0$, assuming $K_1 = K_2$, are in this case

$$\gamma_1 V_1 (V_1 - V_2) - \gamma_1 V_1 l_l(V_l) + l_l(V_l) V_1 V_2 \gamma_1 \gamma_2 V_1 V_2 V_1 + V_2 = 0$$

$$\gamma_1 V_2 (V_2 - V_1) - \gamma_2 V_2 l_l(V_l) + l_l(V_l) V_2 V_1 V_2 V_1 + V_2 = 0.$$  

We study solutions to the algebraic equations on the curve $V_1 V_2 = c$. The reason for this choice will become clear in Section 4.3. On such a curve, the equations simplify as

$$V_1^4 - 2r_l h(V_l) V_1^3 + cr_l h(V_l) V_1 - c^2 = 0$$

$$V_2^4 - 2r_l h(V_l) V_2^3 + cr_l h(V_l) V_2 - c^2 = 0,$$

where $r_l = \gamma_1^{-1}$, $i = 1, 2$ (the resistance of the transmission line $i$ connecting the source $i$ to the load).

We want to study the solutions of these equations as functions of $l_l(V_l)$. Then these can be regarded as two independent quartic functions for which an analytic, although involved, expression of the solutions exist according to the Ferrari–Cardano’s formula. These expressions simplify if one takes $r_1 = r_2$. Then there is a unique positive solution given by $V_1 = V_2 = \sqrt{c}$, independent of $l_l(V_l)$. The value of $V_l$ is obtained from the algebraic equation

$$0 = V_l B_l \Gamma B_l^T V_l - V_l l_l(V_l),$$

solving

$$0 = V_l (-\gamma_1 V_1 - \gamma_2 V_2 + (\gamma_1 + \gamma_2) V_l - V_l l_l(V_l)) = 2 \gamma_1 V_l^3 - 2 \gamma_1 \sqrt{c} V_l - V_l^2 \gamma_1 V_l - P_l^*$$

(19)

$$= (2 \gamma_1 + \gamma_2) V_l^3 - (1^T + 2 \gamma_1 \sqrt{c} V_l - P_l^*.$$  

In the absence of loads, we have two real roots: a root at $V_l = 0$ and a root at $V_l = \sqrt{c} = V_1 = V_2$. Since the roots of a polynomial are continuous in the parameters, the two real-valued roots can vanish and turn to a complex-conjugate pair for large loading. A
classical root-locus analysis shows that, maintaining $p_i^*$, $y_i^*$ equal to zero and letting $p_i^+$ decrease to $-\infty$, the two roots meet halfway at $\sqrt{c}/2$ and then diverge to infinity along the vertical axis passing by the point $(\sqrt{c}/2, 0)$ of the complex plane. This is known as “voltage collapse” in power systems.

4.2. A Lyapunov function and hidden gradient form

We pursue a Lyapunov-based analysis of the stability of the closed-loop system (10), (11), (9). Inspired by the Lyapunov analysis of the reactive power consensus algorithm in De Persis and Monshizadeh (2018), we consider the total power dissipated through the network resistors, $\frac{1}{2}V^TB_iB_i^TV$, as the first natural lyapunov candidate for our analysis, to which we add the power dissipated through the impedance loads, to obtain the power losses at passive devices as

$$J(V) = \frac{1}{2}V^T \left( B_i \Gamma_i^2 B_i^T + \begin{bmatrix} 0 & 0 \\ 0 & Y_i^* \end{bmatrix} \right) V. \quad \text{(20)}$$

Let $\overline{V} \in \mathbb{E}_{ZIP}$, and define $\bar{P}_i = [\overline{V}]_L B_i \Gamma_i^2 \overline{V}$ the source power injection corresponding to the equilibrium source voltage $\overline{V}$ (see (13)). To cope with the asymmetry in the dynamics of the sources and loads we add to $J$ the terms

$$H(V) = -\bar{P}_i^T \ln (V_i),$$
and

$$K(V) = -P_i^T \ln (V_i),$$

which is the way classical power systems transient stability analysis absorbs constant power injections (Chiang, 2011) into a so-called energy function defined here as

$$M(V) := J(V) + H(V) + K(V)$$

$$= \frac{1}{2}V^T (B_i \Gamma_i^2 B_i^T + \begin{bmatrix} 0 & 0 \\ 0 & Y_i^* \end{bmatrix}) V - \bar{P}_i^T \ln (V_i) - P_i^T \ln (V_i). \quad \text{(21)}$$

The natural “energy function” (21) has its critical points at voltages for $P_i = [V_i]_L Y_i^* V_i + P_i^*$, thus different from the power loads prescribed by the ZIP loads. To center the function $M$ with respect to a non-trivial equilibrium $\overline{V} \in \mathbb{E}_{ZIP}$, we use the following Bregman function (De Persis & Monshizadeh, 2018)

$$\mathcal{M}(V) := M(V) - M(\overline{V}) - \frac{\partial M}{\partial V}_{V=\overline{V}} (V - \overline{V}). \quad \text{(22)}$$

The next result shows a (perhaps surprising) gradient relation between the dynamics of system (10), (11), (9) and the Bregman function (22) above:

**Lemma 2 (Gradient Dynamics).** The following holds

$$\begin{bmatrix} L_i K_i^{-1} P_i \\ B_i \Gamma_i^2 B_i^T V - I(V_i) \end{bmatrix} \Rightarrow \begin{bmatrix} \partial M(V) \\ \partial V \end{bmatrix}_{V=\overline{V}} \quad \text{for all } V \in \mathbb{E}_{ZIP}. \quad \text{(23)}$$

Hence the system (10), (11), (9) can be rewritten as a weighted gradient flow

$$\begin{bmatrix} \dot{K}_i V_i \\ \dot{V}_i \end{bmatrix} = -\begin{bmatrix} [V_i]_L [K_i]^{-1} \\ 0 \end{bmatrix} \frac{\partial \mathcal{M}(V)}{\partial V}. \quad \text{(24)}$$

**Proof.** The gradient of the function $M(V)$ writes as

$$\frac{\partial M}{\partial V} = B_i \Gamma_i^2 B_i^T V + \begin{bmatrix} 0 \\ Y_i^* V_i \end{bmatrix} - \begin{bmatrix} [V_i]^{-1} \bar{P}_i \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ [V_i]^{-1} P_i^* \end{bmatrix}. \quad \text{(25)}$$

Hence, the Bregman function (22) satisfies

$$\frac{\partial \mathcal{M}}{\partial V} = \frac{\partial M}{\partial V} - \frac{\partial M}{\partial V} \bigg|_{V=\overline{V}} = B_i \Gamma_i^2 B_i^T (V - \overline{V}) + \begin{bmatrix} 0 \\ Y_i^* (V_i - \overline{V}_i) \end{bmatrix} - \begin{bmatrix} [V_i]^{-1} \bar{P}_i \\ 0 \end{bmatrix} - \begin{bmatrix} [V_i]^{-1} P_i^* \end{bmatrix}. \quad \text{(26)}$$

Bearing in mind the equilibrium condition at the loads

$$B_i \Gamma_i^2 B_i^T \overline{V} = I(V_i) = I_i - Y_i^* (V_i + [\overline{V}]_L)^{-1} P_i^*,$$

and replacing it in the second line of the identity above describing $\partial \mathcal{M}/\partial V$, we obtain

$$\frac{\partial \mathcal{M}}{\partial V_i} = B_i \Gamma_i^2 B_i^T V + Y_i^* V_i - [V_i]^{-1} P_i^* - I_i^* = B_i \Gamma_i^2 B_i^T V - [V_i]^{-1} I_i,$$

which equals precisely the second equation in (23).

Analogously, for the first line $\partial \mathcal{M}/\partial V_i$, we write

$$\frac{\partial \mathcal{M}}{\partial V_i} = B_i \Gamma_i^2 B_i^T (V - \overline{V}) - \begin{bmatrix} [V_i]^{-1} - [V_i]^{-1} \overline{V}_i \end{bmatrix} \bar{P}_i$$

$$= [V_i]^{-1} P_i - [V_i]^{-1} \bar{P}_i - \begin{bmatrix} [V_i]^{-1} - [V_i]^{-1} \overline{V}_i \end{bmatrix} \bar{P}_i$$

$$= [V_i]^{-1} (P_i - \bar{P}_i). \quad \text{(27)}$$

where to write the second equality we have used the identities $P_i = [V_i]_L B_i \Gamma_i^2 B_i^T V$ and $\bar{P}_i = [\overline{V}_i]_L B_i \Gamma_i^2 B_i^T \overline{V}$.

Now note that

$$P_i = [V_i]_L \frac{\partial M}{\partial V_i} + \bar{P}_i$$

and, multiplying both sides by $L_i K_i^{-1}$, we obtain

$$L_i K_i^{-1} P_i = L_i K_i^{-1} \left[ V_i \frac{\partial M}{\partial V_i} \right] + L_i K_i^{-1} \bar{P}_i$$

$$= L_i K_i^{-1} \left[ V_i \frac{\partial M}{\partial V_i} \right] + L_i K_i^{-1} \bar{P}_i.$$

having exploited that $\overline{V} \in \mathbb{E}_{ZIP}$ implies $\bar{P}_i = K_i P_i^*$. The identity $L_i K_i^{-1} P_i = L_i K_i^{-1} \left[ V_i \frac{\partial M}{\partial V_i} \right] + L_i K_i^{-1} \bar{P}_i$ is the first equation in (23).

In view of the dynamics (10), (11), (9), one immediately realizes that

$$\begin{bmatrix} L_i K_i^{-1} P_i \\ B_i \Gamma_i^2 B_i^T V - I(V_i) \end{bmatrix} = \begin{bmatrix} -[V_i]^{-1} K_i V_i \\ 0 \end{bmatrix},$$

showing the identity (24) which concludes the proof. ■

**Remark 6 (Logarithmic Terms of the Lyapunov Function).** As evident from the proof, the logarithmic terms $\bar{P}_i^T \ln (V_i)$ and $\bar{P}_i^T \ln (V_i)$ in the Lyapunov function yield that

$$\frac{\partial \mathcal{M}}{\partial V_i} = [V_i]^{-1} (P_i - \bar{P}_i), \frac{\partial \mathcal{M}}{\partial V_i} = h - I(V_i),$$

that is, they make sure that the critical points $V \in \mathbb{R}^n_{>0}$ of the Lyapunov function are those for which the algebraic equations modeling the loads are satisfied and the vector of injected powers at the sources is equal to the desired one $\bar{P}_i$ as characterized in Lemma 1 (see Eq. (13)).

4.3. Convergence of solutions

The particular form of the dynamics (10), (11), (9) elucidated in Lemma 2 permits a straightforward analysis of the convergence properties of the solutions.

**Theorem 1 (Main Result).** Assume that there exists $\overline{V} \in \mathbb{E}_{ZIP}$ such that

$$Y_i + Y_i^* + [\overline{V}]_L^{-2} [P_i]_L - Y_h (Y_h + [\overline{V}]_L^{-2} [\bar{P}_i]_L^{-1} [Y_i]_L > 0, \quad \text{(28)}$$


where $Y_{st}, Y_{st}, Y_{ss}$ are the submatrices of the Laplacian matrix defined in (1), $P^*_s$ is the constant power load, and $P_s$ is the constant source power injection defined in (13). Then the following statements hold:

1. there exists a compact sublevel set $\Lambda_{ZP}$ of the shifted Lyapunov function $M$ in (22) contained in $\mathbb{R}^n_{>0}$ such that any solution to (10), (11), (9) that originates from initial conditions $V(0)$ belonging to $\Lambda_{ZP}$ exists, always remains in $\Lambda_{ZP}$ with strictly positive voltages for all times, and asymptotically converges to the set of equilibria $\mathcal{E}_{ZP} \cap \Lambda_{ZP} \cap V_{\text{mean}}$, where $V_{\text{mean}}$ specifies the preserved weighted geometric mean of the source voltages

$$V_{\text{mean}} := \{ V \in \mathbb{R}^n_{>0} : V_{K_1} \cdots V_{K_n} = V_{K_1}^*(0) \cdots V_{K_n}^*(0) \}.$$ (27)

Remark 7 (Interpretation of the Main Condition). The main condition (26) guarantees regularity of the algebraic equations and stability of the solutions. Its role is revealed when converting the constant power loads and the asymptotically constant power injections at the sources to the equivalent impedances $[\bar{V}]^{-2}[P_s]$ and $[\bar{V}]^{-2}[P_s]$. In this case, the equivalent conductance matrix in the steady-state current-balance equations (1) read as

$$Y_{eq} = \begin{bmatrix} Y_{ss} & Y_{st} \\ Y_{st} & Y_{tt} \end{bmatrix} + \begin{bmatrix} [\bar{V}]^{-2}[P_s] & 0 \\ 0 & [\bar{V}]^{-2}[P_s] + Y_{tt} \end{bmatrix}.$$ (28)

By a Schur complement argument, observe that $Y_{eq}$ is a well-defined (i.e., positive definite) conductance matrix if and only if the main condition (26) holds.

Proof. Existence and boundedness of solutions. Observe first that

$$\frac{\partial^2 M}{\partial V^2} = \frac{\partial M}{\partial V} T \frac{\partial M}{\partial V} = \begin{bmatrix} 0 & 0 \\ 0 & Y_{tt} \end{bmatrix} + \begin{bmatrix} [\bar{V}]^{-2}[P_s] & 0 \\ 0 & [\bar{V}]^{-2}[P_s] + Y_{tt} \end{bmatrix}.$$ (29)

Let $\bar{V} \in \mathbb{R}^n_{>0}$ be an equilibrium of the system, i.e., $\bar{V} \in \mathcal{E}_{ZP}$. Since $P^*_s, P_s \in \mathbb{R}^n_{>0}$, and $\bar{V} \in \mathbb{R}^n_{>0}$, the steady-state power injection at the sources satisfies $P_s \in \mathbb{R}^n_{>0}$ by (13). Hence, $[\bar{V}]^{-2}[P_s] > 0$ is positive definite. Then the Bregman function $M$ has an isolated minimum at the equilibrium $\bar{V}$, in view of (26), (29) and a standard Schur complement argument. Then there exists a compact sublevel set $\Lambda_{ZP}$ of $M$ around the equilibrium $\bar{V}$ contained in the positive orthant. Without loss of generality this compact sublevel set can be taken so that all the solutions to (10), (11), (9) that originate here locally exist.

The algebraic equations (9) written as in Lemma 1 are

$$0 = \frac{\partial M}{\partial V} V_t = I(V_t) - Y_0 V_t - Y_0 V_t.$$ (30)

To study local solvability of these equations, we analyze

$$\frac{\partial^2 M}{\partial V^2} V_t = - (Y_0 + Y_t^* + [V_t^{-1}P_s^*] V_t).$$ (31)

In view of (26), nonsingularity of $\frac{\partial^2 M}{\partial V^2} / \partial V_t$ and therefore regularity of the algebraic condition holds in a neighborhood of $\bar{V} \in \Lambda_{ZP}$ from the implicit function theorem (Abraham, Marsden, & Ratiu, 1988).

The sublevel set $\Lambda_{ZP}$ can be taken sufficiently small such that it is contained in the neighborhood of regularity for the algebraic equations, thus showing the claim that solutions starting from $\Lambda_{ZP}$ locally exist in time, see Hill and Mareels (1990, Theorem 1) and Schiffer and Dörfler (2016, Lemma 2.3).

When computed along these solutions, $\mathcal{L}(V(t))$ satisfies

$$\mathcal{L}(V(t)) = \frac{\partial M}{\partial V_t} |_{V=V(t)} \dot{V}_t(t) + \frac{\partial M}{\partial V} |_{V=V(t)} \dot{V}_t(t).$$ (32)

Notice that, by the algebraic constraint (23),

$$\frac{\partial M}{\partial V_t} |_{V=V(t)} = B_t \Gamma B_t^T V(t) - l_0 V(t)) = 0$$

for all $t$ for which a solution exists. Hence, we arrive at

$$\mathcal{L}(V(t)) = \frac{\partial M}{\partial V_t} |_{V=V(t)} \dot{V}_t(t) = - \frac{\partial M}{\partial V} |_{V=V(t)} K_s^{-1} [V_s] L_s [V_s] K_s^{-1} \frac{\partial M}{\partial V} |_{V=V(t)} \leq 0,$$

where the second equality holds because of (24). The inequality above shows that $\mathcal{L}(V(t))$ is a non-increasing function of time. By the compactness of the sublevel set around $\bar{V}$, the solutions are bounded, exist and belong to $\Lambda_{ZP}$ for all times. Thus, among others the voltages stay positive for all times.

Convergence. Exploiting the regularity of the algebraic equation, the DAE system can be reduced to an ODE system and then the standard LaSalle invariance principle for ODEs can be used to infer convergence, see also Schiffer & Dörfler (2016). We argue as follows. Any solution $(V_t, \dot{V}_t)$ to the DAE system (10), (11), (9) originating in $\Lambda_{ZP}$ is such that its component $V_t$ is a solution to the system of ODE

$$\dot{V}_t = -K_s^{-1} [V_s] L_s [V_s] K_s^{-1} (Y_{ss} V_t + Y_0 \delta(V_t)),$$ (33)

where the map $V_t = \delta(V_t)$ denotes the solution of the algebraic equation $\mathcal{E}_{ZP}(V) = 0$ in $\mathcal{E}_{ZP}$. Define

$$\Lambda(V_t) := \mathcal{L}(V_t, \delta(V_t))$$ (34)

and observe that

$$\mathcal{L}(V_t(t)) = \frac{\partial M}{\partial V_t} |_{V=V(t)} \dot{V}_t(t) + \frac{\partial M}{\partial V} |_{V=V(t)} \dot{V}_t(t) = \delta(V_t(t)),$$

since

$$\frac{\partial M}{\partial V_t} |_{V=V(t)} V_t(t) = Y_0 V_t(t) + Y_0 \delta(V_t(t)) - l_0(\delta(V_t(t)))$$

$$V_t(t) = \delta(V_t(t))$$

where the second equality holds because $V_t(t) = \delta(V_t(t))$ on $\Lambda_{ZP}$ and the third equality because of the algebraic equation in (10), (11), (9). It then follows that

$$\mathcal{L}(V_t(t)) = (P_s - P_s) T [V_t(t)] V_t(t) = \delta(V_t(t))$$

$$\mathcal{L}(V_t(t)) = Y_0 V_t(t) + Y_0 \delta(V_t(t)) - l_0(\delta(V_t(t)))$$

where the first equality descends from (25), the second from (10), and the third from (13).

Since $V_t$ is bounded, then the standard La Salle invariance principle for ODEs yields convergence of $V_t$ to the largest invariant set where $L_s K_s^{-1} P_s = 0$. Moreover, since the solutions evolve in $\Lambda_{ZP}$, since they satisfy the algebraic equations, and since $L_s K_s^{-1} P_s = 0$, we have from Lemma 1 that at steady state $(V_s, V_t) \in \mathcal{E}_{ZP}$. Since $(V_s, V_t)$ is a solution to (10), (11), (9) that remains in $\Lambda_{ZP}$, convergence to the set $\mathcal{E}_{ZP} \cap \Lambda_{ZP}$ is inferred. Moreover, the
quantity $V_1^{K_1} \cdots V_n^{K_n}$ is conserved, namely $V_i(t)^{K_1} \cdots V_n(t)^{K_n} = V_i^{(0)})^{K_1} \cdots V_n^{(0)})^{K_n}$ for all $i$. In fact, by (30),
\[ K_i \frac{d}{dt} \ln V_i = -L_i [V_i] K_i^{-1} (Y_i V_i + Y_i \delta (V_i)), \]
and therefore $\frac{d}{dt} \ln V_i = 0$. The thesis then follows.

**Example 2.** Consider again the case of two sources ($n=2$) and one load ($n_1=1$) connected in a “T” configuration, as in Example 1. If $K_1 = K_2$, the result above shows that on the convergence set $\mathcal{E}_{ZIP} \cap \mathcal{A}_{ZIP} \cap \mathcal{V}_{\text{mean}}, V_2 = V_1(0) \forall t \geq 0$. Hence, as discussed in Example 1, the expression of the (real and positive) solution to Eqs. (18) takes on a particularly simple form, namely $V_1 = V_2 = \sqrt{c} = \sqrt{\mathcal{V}(0)}$, that is on the convergence set each source voltage is the geometric mean of the initial voltage sources. Accordingly, the load voltage $V_2$ must satisfy (19).

**Remark 8 (Capacitors at the Loads).** If loads are interconnected to the network via capacitors, the load equations are modified as
\[ C_i \frac{d}{dt} V_i = -I_i(V_i) + B_i \Gamma B^T V_i. \]
Notice that the equilibrium of the system remain the same. Bearing in mind (23), the load dynamics read as
\[ C_i \frac{d}{dt} V_i = \frac{1}{2} \frac{d}{dV_i} \mathcal{M} \frac{d}{dV_i} V_i. \]
It follows that
\[ \mathcal{M} = -\frac{1}{2} \frac{d}{dV_i} V_i^T [V_i]^{-1} [V_i] K_i^{-1} \frac{d}{dV_i} C_i \frac{d}{dV_i} V_i, \]
and one can infer convergence to the set $\mathcal{E}_{ZIP} \cap \mathcal{A}_{ZIP} \cap \mathcal{V}_{\text{mean}}$ similarly as for the differential–algebraic model.

**Remark 9 (Constant Voltage Buses).** Similarly to Zhao and Dörfler (2015, Remark 3.3), one can also consider voltage-controlled buses. For example, consider the scenario of all load buses having constant (not necessarily identical) voltages $V_l$ (see Zhao and Dörfler, 2015 for a discussion on this load condition). More precisely, a controller adjust the current injection $I_i$ depending on $V_i$ to maintain the value of the voltage at the constant level $V_l$ so that system (10) reads as
\[ K_i V_i = -[V_i]_L [K_i]_L^{-1} [V_i]_L (Y_i V_i + Y_i V_l) \]
\[ -I_i = Y_i V_i + Y_i V_l. \]
The only relevant equations for stability of (33) are the ordinary differential equations (33a) driven by the constant term $V_l$. We study their stability using a similar Lyapunov argument as before. Since $V_l$ is now constant, we consider a simplified version of the function $\mathcal{M}$, namely $\mathcal{M}(V_i) = \tilde{\mathcal{J}}(V_i) + \mathcal{N}(V_i)$, where
\[ \tilde{\mathcal{J}}(V_i) = \frac{1}{2} \left[ \frac{d}{dV_i} V_i^T [V_i]^{-1} \frac{d}{dV_i} V_i \right], \]
are the (shifted) network losses so that $\frac{d}{dV_i} V_i = Y_i (V_i - \tilde{V}_i) = (Y_i V_i + Y_i \tilde{V}_i) - (Y_i V_i + Y_i \tilde{V}_i) = [V_i]^{-1} (Y_i - \tilde{V}_i)$. Together with $\mathcal{N}(V_i) = -P_i \mathcal{L}_i \ln [V_i] + P_i \ln [V_i] + P_i [V_i]^{-1} (V_i - \tilde{V}_i)$, we obtain that $\frac{d}{dV_i} = [V_i]^{-1} (P_i - \tilde{P}_i)$ and thus $K_i V_i = -[V_i]_L [K_i]_L^{-1} [V_i]_L \frac{d}{dV_i}$. The convergence analysis of the solutions of the system (33) is now analogous to the proof of Theorem 1.

### 4.4. The case of ZI loads

In the case of ZI loads the previous results can be strengthened. First, the set of equilibria can be characterized by two systems of equations, one depending on the source voltages only and the other one allowing for a straightforward calculation of the load voltages once the source voltages are determined. Second, the convergence result can be established without any extra condition on the equivalent conductance matrix in (28). Finally, the convergence is to a point rather than to a set. The first result we present concerns the set of equilibria, which follows by adapting the proof of Lemma 1.

**Lemma 3 (Equilibria for ZI Loads).** The set of equilibria of system (10), (11), (9) with $l(V_i) = I_i^* - Y_i^* V_i$ is
\[ \mathcal{E}_{ZI} = \{ V \in \mathbb{R}_0^n : \mathcal{P}_{ZI}(V) = 0, \}
\]
\[ V_i = (Y_i + Y_i^*)^{-1} (I_i^* - Y_i V_i) \}
\]
where $\mathcal{P}_{ZI}(V_i)$ depicts the power balance at the sources
\[ \mathcal{P}_{ZI}(V) = \frac{1}{2} \mathcal{Y}_{\text{red}} V + \frac{1}{2} \left[ (Y_i + Y_i^*)^{-1} I_i^* - P_i \right]. \]

We remark that in the ZI case the equations $\mathcal{P}_{ZI}(V_i) = 0$ depend on the source voltages only, and once a solution to it is determined, the corresponding voltages at the loads are obtained as $V_i = (Y_i + Y_i^*)^{-1} (I_i^* - Y_i V_i)$ thereby explicitly solving previous $\mathcal{E}_{ZI}(V) = 0$. Similarly to the case of ZIP loads, we introduce the following standing assumption:

**Assumption 2.** $\mathcal{E}_{ZI} \neq \emptyset$.

Our second result concerns the convergence of the dynamics. In the case of ZI loads, convergence can be established without the definiteness condition on the equivalent conductance matrix $Y_{eq}$ in (28). Indeed, for $P_i^* = 0$, the condition (26) is automatically satisfied. Before, this condition was needed to certify strict convexity of the shifted Lyapunov function $\mathcal{M}$ (see (29)) as well as the regularity of the algebraic equation $\mathcal{E}_{ZI}(V) = 0$. Additionally, the limit set in case of ZI loads is $\mathcal{E}_{ZI} \cap \mathcal{A}_{ZI} \cap \mathcal{V}_{\text{mean}}$, where the set of equilibria $\mathcal{E}_{ZI}$ is characterized in Lemma 3. $\mathcal{A}_{ZI}$ is a sublevel set associated with the Lyapunov function $\mathcal{M}$ with $P_i^* = 0$, and the set $\mathcal{V}_{\text{mean}}$ is defined as in (27). Finally, a stronger convergence result can be established, namely any trajectory converges to a point depending on the initial condition. This can be formalized as follows:

**Theorem 2 (Point Convergence).** Assume that there exists $\tilde{V} \in \mathcal{E}_{ZI}$.

1. The solutions to (10), (11), (9) with $P_i^* = 0$ that originate from any initial condition $V(0)$ belonging to a sublevel set $\mathcal{A}_{ZI}$ of the shifted Lyapunov function $\mathcal{M}$ in (22) with $P_i^* = 0$ contained in $\mathbb{R}_0^n$ always remain in $\mathcal{A}_{ZI}$, and
2. converge to an asymptotically stable equilibrium belonging to $\mathcal{E}_{ZI} \cap \mathcal{A}_{ZI} \cap \mathcal{V}_{\text{mean}}$.

**Proof.** First of all we observe that the proof of Theorem 1 holds for the case of ZI loads (it suffices to set $P_i^* = 0$ and $l(V_i) = I_i^* - Y_i^* V_i$ throughout the proof). As an additional feature of ZI loads (to be exploited below) we can explicitly construct $\delta(V_i) = (Y_i + Y_i^*)^{-1} (I_i^* - Y_i V_i)$. From the proof of Theorem 1 (specialized to the case of ZI loads), it is known that any solution $V_i$ of the ODE (30) is bounded. By Birkhoff’s Lemma (Khalil, 1996, Lemma 3.1) the positive limit set $\mathcal{Q}(V_i)$ associated with a solution $V_i(t)$ is non-empty, compact,
and invariant. Moreover, it is contained in $\mathcal{E}_2 \cap \mathcal{A}_2 \cap \mathcal{V}_{\text{mean}}$. We would like to prove that $\Omega(V_i)$ is a singleton. To this end, and similarly to De Persis & Monshizadeh (2018) we appeal to Haddad & Chellaboina, (2008, Proposition 4.7), which states that if the positive limit set $\Omega(V_i)$ of a trajectory contains a Lyapunov stable equilibrium $\overline{V}_i$, then $\Omega(V_i) = \{\overline{V}_i\}$. To see this first notice that $\overline{V}_i$ being in $\Omega(V_i)$ and hence in $\mathcal{E}_2 \cap \mathcal{A}_2 \cap \mathcal{V}_{\text{mean}}$, it is indeed an equilibrium of the system. Thus, following (31), one can construct a shifted function $\mathcal{N}(V_i)$ associated to $\overline{V}_i$. The explicit expression of $\mathcal{N}(V_i)$ is given by

$$
\mathcal{N}(V_i) = -\overline{P}_i \ln(V_i) + \overline{P}_i \ln(\overline{V}_i) + \overline{P}_i (\overline{V}_i - \overline{V}_i) - \frac{1}{2} \left( \frac{V_i - \overline{V}_i}{\delta(V_i) - \delta(\overline{V}_i)} \right)^T \left[ \begin{array}{c} Y_{bs} \\ Y_{hil} + Y_i^* \end{array} \right] \left( \frac{V_i - \overline{V}_i}{\delta(V_i) - \delta(\overline{V}_i)} \right).
$$

The gradient of $\mathcal{N}(V_i)$ is given by

$$
\frac{\partial \mathcal{N}}{\partial V_i} = -[\overline{V}_i]^{-1} \overline{P}_i + \left(V_i - \overline{V}_i\right) + \left(\begin{array}{c} Y_{bs} \\ Y_{hil} + Y_i^* \end{array}\right) \frac{\partial \delta}{\partial V_i} (V_i - \overline{V}_i) + \left(\begin{array}{c} Y_{bs} \\ Y_{hil} + Y_i^* \end{array}\right) \frac{\partial \delta}{\partial V_i} (V_i - \overline{V}_i).
$$

Since $\frac{\partial \delta}{\partial V_i} = -(V_i + Y_i^*)^{-1} Y_{bs}$, the last summand above vanishes.

With the shorthand $\hat{V}_{\text{red}} = Y_{bs} - Y_{hil}(Y_{hil} + Y_i^*)^{-1} Y_{bs}$, the gradient simplifies as

$$
\frac{\partial \mathcal{N}}{\partial V_i} = -[\overline{V}_i]^{-1} \overline{P}_i + \left(V_i - \overline{V}_i\right) + \hat{V}_{\text{red}} (V_i - \overline{V}_i).
$$

Note that the gradient $\frac{\partial \mathcal{N}}{\partial V_i}$ vanishes if $V_i = \overline{V}_i$ and $\mathcal{N}$ has a strict local minimum at $\overline{V}_i$, since

$$
\frac{\partial^2 \mathcal{N}}{\partial V_i^2} = \hat{V}_{\text{red}} + [V_i - \overline{V}_i]^{-1} \overline{P}_i.
$$

By (32), $\mathcal{N} \leq 0$, and these two properties (properness and the nonpositive time derivative) show that $\overline{V}_i$ is a Lyapunov stable equilibrium. Therefore, $\Omega(V_i) = \{\overline{V}_i\}$, and the solution $V_i(t)$ converges to an equilibrium point. Because $V_i(t)$ is the use an IEEE 37 bus system adapted from Alwala, Feliachi, & Choudhry (2012), Distribution Test Feeder Working Group (0000) and Kersting (2001) adding lines to form a mesh topology, and upscaling the values of the microgrid parameters. The grid topology is sketched in Fig. 3, and the network and control parameters are given in Tables 1 and 2, and 3. It can be checked that condition (26) is satisfied for this network. There are 26 loads and 7 sources. Among these are eight constant power loads, nine constant impedance loads and nine constant current loads. Fifteen of the loads are initially turned off.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Table 2</th>
<th>Table 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1</td>
<td>Node 2</td>
<td>Node 3</td>
</tr>
</tbody>
</table>

Fig. 3. The network used for the simulations. Sources are depicted as circles, loads as rectangles. Solid lines denote the interconnecting lines, while dashed blue lines represent the communication graph used by the controllers.

and are turned on gradually between 9.5 and 10.5 ms; see Table 2. This constitutes a load increase of approximately 1.06 MW, from approximately 779 kW to 1.84 MW. The remaining eleven loads are active throughout the experiment. The voltage evolution both at the sources and at the loads for the controllers (7) and (8) is depicted in Fig. 4. Fig. 5 represents a comparison between the evolution of the voltages at node 1 for the two controllers. For a more detailed comparison, the steady state percentage voltage deviation from the nominal value for all the nodes are reported in Table 2. The power injected at the source nodes is shown for both control strategies in Fig. 6. As predicted by the analysis, at steady state proportional power sharing is achieved by the power sources in conformity with (3). Observe that the voltage deviations are rather small even though both the proposed controller (7) and the distributed integral controller (8) do not account for voltage regulation. Notice also that the two controllers perform similarly, though the voltages for the integral controller tend to be slightly lower than those for the power consensus controller.

6. Conclusions

We have proposed controllers for DC microgrids that average power measurement at the sources. The results apply to network preserved model (systems of DAЕ) of the microgrid in the presence of ZIP loads. Capacitors at the terminals of the grid that model either $\Pi$-models of lines or power converter components can be included by means of passivity-based analysis.
Many interesting new research directions can be taken. The first one is to consider more complex scenarios such as the inclusion of dynamical (inductive) lines and loads. Another one is the extensions of the controllers to network preserved AC microgrids. Moreover, although the preservation of the geometric mean of the voltages allows for an estimate of the voltage excursion, no active voltage regulation is present in the proposed scheme. An addition of voltage controllers to the power consensus algorithm is an interesting and important open problem. The sensitivity of the power consensus algorithm to delays in the communication network is an important feature to be assessed. The theoretical analysis of the algorithm under delays, however, is far from trivial and is left for future research. The power consensus algorithms lead to a new set of power flow equations, whose solvability still needs to be investigated, e.g., starting from recent advances concerning power flow feasibility and approximations; see Bolognani and Zampieri (2016), Barabanov et al. (2016), Simpson-Porco et al. (2016) and references therein. The distributed averaging integral controller (8) discussed in Remark 2 enjoys the nice feature of not requiring power measurements and could be an enthralling algorithm to investigate further. Finally, the power consensus algorithms preserves the weighted geometric mean of the voltages and is thus a compelling application for nonlinear consensus schemes (Bauso et al., 2005; Cortes, 2008). We believe this connection deserves a deeper investigation.

### Table 1
Simulation parameter values. The power sharing coefficients \( K_i \) used in [8] have the same numerical values as those in the table, but units of volts.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal voltage ( V^0 )</td>
<td>4.8 kV</td>
</tr>
<tr>
<td>Power sharing coefficients ( K_i )</td>
<td>40 ( \sqrt{\text{kgm/s}} )</td>
</tr>
<tr>
<td>Integral controller gains ( C_i )</td>
<td>0.2 F</td>
</tr>
<tr>
<td>Integral controller gains ( D_i )</td>
<td>0.075 s</td>
</tr>
<tr>
<td>Load values ( Y_i )</td>
<td>3.08 mS</td>
</tr>
<tr>
<td>( -k_i^0 )</td>
<td>14.8 A</td>
</tr>
<tr>
<td>( -F_i^0 )</td>
<td>70.8 kW</td>
</tr>
</tbody>
</table>

### Table 2
Node properties and final percentage voltage deviation from the nominal value for both the proposed controllers (7) and the distributed integral controller (8).

<table>
<thead>
<tr>
<th>Node</th>
<th>ZIP</th>
<th>Turned on</th>
<th>Proposed</th>
<th>DAPI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>Always</td>
<td>-0.10%</td>
<td>-0.25%</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>Always</td>
<td>-0.02%</td>
<td>-0.37%</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>Always</td>
<td>-0.34%</td>
<td>-0.62%</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>Always</td>
<td>-0.04%</td>
<td>-0.43%</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>Always</td>
<td>-0.19%</td>
<td>-0.57%</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>Always</td>
<td>-0.25%</td>
<td>-0.56%</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>Always</td>
<td>-0.13%</td>
<td>-0.26%</td>
</tr>
<tr>
<td>8</td>
<td>Z</td>
<td>Gradual</td>
<td>-0.19%</td>
<td>-0.57%</td>
</tr>
<tr>
<td>9</td>
<td>I</td>
<td>Always</td>
<td>-0.17%</td>
<td>-0.56%</td>
</tr>
<tr>
<td>10</td>
<td>P</td>
<td>Gradual</td>
<td>-0.32%</td>
<td>-0.71%</td>
</tr>
<tr>
<td>11</td>
<td>Z</td>
<td>Gradual</td>
<td>-0.34%</td>
<td>-0.73%</td>
</tr>
<tr>
<td>12</td>
<td>I</td>
<td>Gradual</td>
<td>-0.32%</td>
<td>-0.70%</td>
</tr>
<tr>
<td>13</td>
<td>P</td>
<td>Always</td>
<td>-0.33%</td>
<td>-0.71%</td>
</tr>
<tr>
<td>14</td>
<td>Z</td>
<td>Always</td>
<td>-0.32%</td>
<td>-0.71%</td>
</tr>
<tr>
<td>15</td>
<td>I</td>
<td>Always</td>
<td>-0.30%</td>
<td>-0.69%</td>
</tr>
<tr>
<td>16</td>
<td>P</td>
<td>Gradual</td>
<td>-0.26%</td>
<td>-0.65%</td>
</tr>
<tr>
<td>17</td>
<td>Z</td>
<td>Always</td>
<td>-0.15%</td>
<td>-0.54%</td>
</tr>
<tr>
<td>18</td>
<td>I</td>
<td>Gradual</td>
<td>-0.18%</td>
<td>-0.57%</td>
</tr>
<tr>
<td>19</td>
<td>P</td>
<td>Gradual</td>
<td>-0.25%</td>
<td>-0.64%</td>
</tr>
<tr>
<td>20</td>
<td>Z</td>
<td>Always</td>
<td>-0.21%</td>
<td>-0.60%</td>
</tr>
<tr>
<td>21</td>
<td>I</td>
<td>Always</td>
<td>-0.16%</td>
<td>-0.53%</td>
</tr>
<tr>
<td>22</td>
<td>P</td>
<td>Gradual</td>
<td>-0.22%</td>
<td>-0.60%</td>
</tr>
<tr>
<td>23</td>
<td>Z</td>
<td>Always</td>
<td>-0.27%</td>
<td>-0.66%</td>
</tr>
<tr>
<td>24</td>
<td>I</td>
<td>Gradual</td>
<td>-0.26%</td>
<td>-0.66%</td>
</tr>
<tr>
<td>25</td>
<td>P</td>
<td>Gradual</td>
<td>-0.30%</td>
<td>-0.68%</td>
</tr>
<tr>
<td>26</td>
<td>Z</td>
<td>Gradual</td>
<td>-0.27%</td>
<td>-0.66%</td>
</tr>
<tr>
<td>27</td>
<td>I</td>
<td>Gradual</td>
<td>-0.32%</td>
<td>-0.71%</td>
</tr>
<tr>
<td>28</td>
<td>P</td>
<td>Always</td>
<td>-0.33%</td>
<td>-0.71%</td>
</tr>
<tr>
<td>29</td>
<td>Z</td>
<td>Gradual</td>
<td>-0.35%</td>
<td>-0.74%</td>
</tr>
<tr>
<td>30</td>
<td>I</td>
<td>Gradual</td>
<td>-0.26%</td>
<td>-0.64%</td>
</tr>
<tr>
<td>31</td>
<td>P</td>
<td>Always</td>
<td>-0.18%</td>
<td>-0.57%</td>
</tr>
<tr>
<td>32</td>
<td>Z</td>
<td>Always</td>
<td>-0.08%</td>
<td>-0.46%</td>
</tr>
<tr>
<td>33</td>
<td>I</td>
<td>Gradual</td>
<td>-0.13%</td>
<td>-0.52%</td>
</tr>
</tbody>
</table>

### Table 3
Line resistances \( R^{-1} \). The last six lines (2–21, 32–12, 3–19, 6–26, 22–26, 24–33) were added to form a mesh topology.

<table>
<thead>
<tr>
<th>Line</th>
<th>Resistance</th>
<th>Line</th>
<th>Resistance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–2</td>
<td>0.3032 Ω</td>
<td>16–19</td>
<td>0.1198 Ω</td>
</tr>
<tr>
<td>2–3</td>
<td>0.1281 Ω</td>
<td>17–18</td>
<td>0.0958 Ω</td>
</tr>
<tr>
<td>4–10</td>
<td>0.1704 Ω</td>
<td>10–20</td>
<td>0.1155 Ω</td>
</tr>
<tr>
<td>5–24</td>
<td>0.1475 Ω</td>
<td>19–23</td>
<td>0.1475 Ω</td>
</tr>
<tr>
<td>6–28</td>
<td>0.1972 Ω</td>
<td>21–22</td>
<td>0.1475 Ω</td>
</tr>
<tr>
<td>7–32</td>
<td>0.0986 Ω</td>
<td>23–24</td>
<td>0.0753 Ω</td>
</tr>
<tr>
<td>7–28</td>
<td>0.3668 Ω</td>
<td>25–26</td>
<td>0.1281 Ω</td>
</tr>
<tr>
<td>10–11</td>
<td>0.1475 Ω</td>
<td>25–27</td>
<td>0.0793 Ω</td>
</tr>
<tr>
<td>10–13</td>
<td>0.1972 Ω</td>
<td>27–28</td>
<td>0.1382 Ω</td>
</tr>
<tr>
<td>11–12</td>
<td>0.1115 Ω</td>
<td>28–29</td>
<td>0.0802 Ω</td>
</tr>
<tr>
<td>13–14</td>
<td>0.0321 Ω</td>
<td>28–30</td>
<td>0.1576 Ω</td>
</tr>
<tr>
<td>13–15</td>
<td>0.1281 Ω</td>
<td>30–31</td>
<td>0.0986 Ω</td>
</tr>
<tr>
<td>15–16</td>
<td>0.0885 Ω</td>
<td>31–32</td>
<td>0.0986 Ω</td>
</tr>
<tr>
<td>16–17</td>
<td>0.1594 Ω</td>
<td>32–33</td>
<td>0.0802 Ω</td>
</tr>
<tr>
<td>2–21</td>
<td>0.2430 Ω</td>
<td>6–26</td>
<td>0.1843 Ω</td>
</tr>
<tr>
<td>3–12</td>
<td>0.1566 Ω</td>
<td>22–26</td>
<td>0.1281 Ω</td>
</tr>
<tr>
<td>3–19</td>
<td>0.2166 Ω</td>
<td>24–33</td>
<td>0.3456 Ω</td>
</tr>
</tbody>
</table>

**Fig. 5.** Comparison of voltages at node 1 for both controllers.
Acknowledgments

The authors wish to thank Rowan Plomp for his suggestion on the numerical study. The work of C. De Persis and E.R.A. Weitenberg is partially supported by NWO within the program “Uncertainty Reduction in Smart Energy Systems (URSES)” under the auspices of the project ENBARK, by the DST-NWO Indo-Dutch Cooperation on “Smart Grids” under the auspices of the project “Energy management strategies for interconnected smart microgrids” and by the STW Perspectief program “Robust Design of Cyber-physical Systems & Control” under the auspices of the project “Energy Autonomous Smart Microgrids”. The work of F. Dörfler is supported by ETH Zürich funds.

References


Claudio De Persis is a Professor at the Engineering and Technology Institute, Faculty of Science and Engineering, University of Groningen, the Netherlands. He received the Laurea degree in Electronic Engineering in 1996 and the Ph.D. degree in System Engineering in 2000 both from the University of Rome “La Sapienza”, Italy. Previously he held faculty positions at the Department of Mechanical Automation and Mechatronics, University of Twente and the Department of Computer, Control, and Management Engineering, University of Rome “La Sapienza”. He was a Research Associate at the Department of Systems Science and Mathematics, Washington University, St. Louis, MO, USA, in 2000–2001, and at the Department of Electrical Engineering, Yale University, New Haven, CT, USA, in 2001–2002. His main research interest is in control theory, and his recent research focuses on dynamical networks, cyberphysical systems, smart grids and resilient control. He

**Erik Weitenberg** received the B.Sc. degree in mathematics and the M.Sc. degree in mathematics with a specialization in algebra and cryptography from the University of Groningen, The Netherlands, in 2010 and 2012 respectively.

He is currently working toward the Ph.D. degree in control of cyber-physical systems at University of Groningen.

His current research interests include stability and robustness of networked and cyber-physical systems, with applications to power systems.

**Florian Dörfler** is an Assistant Professor in the Automatic Control Laboratory at ETH Zürich. He received his Ph.D. degree in Mechanical Engineering from the University of California at Santa Barbara in 2013, and a Diploma degree in Engineering Cybernetics from the University of Stuttgart in 2008. From 2013 to 2014 he was an Assistant Professor at the University of California Los Angeles. His primary research interests are centered around distributed control, complex networks, and cyber-physical systems currently with applications in energy systems. His students were winners or finalists for Best Student Paper awards at the European Control Conference (2013), and the American Control Conference (2016), and the PES PowerTech Conference 2017. His articles received the 2010 ACC Student Best Paper Award, the 2011 O. Hugo Schuck Best Paper Award, and the 2012–2014 Automatica Best Paper Award, and the 2016 IEEE Circuits and Systems Guillemin-Cauer Best Paper Award. He is a recipient of the 2009 Regents Special International Fellowship, the 2011 Peter J. Frenkel Foundation Fellowship, and the 2015 UCSB ME Best Ph.D. award.