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Interpolation in propositional Horn logic

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Abstract

In this paper we give an abstract representation of propositional Horn logic (finite or infinitary) in terms of set-valued functions $\mathcal{F} : \wp(\mathbf{PR}) \rightarrow \wp(\mathbf{PR})$; here $\mathbf{PR}$ is the collection of atomic propositions. We investigate the properties of set-valued functions and establish that they satisfy the axioms of *weak lazy Kleene algebra*. These axioms and other properties are used to prove uniform left interpolation for propositional Horn logic. Then we introduce *thin* set-valued functions, which enable us to prove polynomial interpolation for propositional Horn logic. For the infinite case, the Axiom of Choice is used.

Keywords: propositional Horn logic, uniform interpolation, polynomial interpolation, lazy Kleene algebra.

1 Introduction

In logic and logical reasoning, conditional statements are omnipresent. They appear on different levels:

- implications in the object language, e.g. $p \rightarrow q$;
- entailments in the metalanguage, e.g. sequents like $\varphi, \psi \vdash \varphi \wedge \psi$;
- proof rules involving sequents, e.g. *if* $\Gamma \vdash A$ *and* $\Gamma, B \vdash C$ *then* $\Gamma, A \rightarrow B \vdash C$.

Horn logic is the pure logic of conditional statements. In this paper, we present an abstract representation of the formulae of propositional Horn logic which we use to obtain some interpolation results. These formulae may be finite or infinite, without restriction on their size. These results may be generalized to universal Horn logic and other logics, e.g. (conditional) equational logic, and may also be transferred to proof systems involving sequents, which are structurally similar to Horn formulae.

In the next three sections, we introduce propositional Horn logic and interpolation, and we survey related literature. In Section 5 an abstract representation of the formulae of propositional Horn logic by set-valued functions $\mathcal{F}$ in $\mathbb{F}(\mathbf{PR}) = (\wp(\mathbf{PR}) \rightarrow \wp(\mathbf{PR}))$ is presented. The underlying idea is to represent conditionality by functionality: the formula $P \rightarrow Q$ expressing that P is a condition for Q can be modelled by a function $\mathcal{F}$ that maps P to Q . Elementary properties of $\mathbb{F}(\mathbf{PR})$ are discussed in Section 6, focusing on monotonicity and continuity. The iterative closure $\mathcal{F}^*$ of $\mathcal{F} \in \mathbb{F}(\mathbf{PR})$ is introduced in Section 7, where we prove in Lemma 6 that it corresponds with entailment in propositional Horn logic: $\mathcal{F} \models \mathcal{G}$ iff $\mathcal{G}$ is contained in $\mathcal{F}^*$. This result leads to a completeness result with respect to a notion of derivability introduced in Section 8.

In Section 9 we investigate $\mathbf{SF}(\mathbf{PR})$, the algebraic structure consisting of $\mathbb{F}(\mathbf{PR})$ and the operators $\sqcup$ (union, corresponding to conjunction), $\overset{\circ}{\circ}$ (function composition combined with monotonic closure) and $\cdot^*$ (iterative closure). Some but not all axioms of Kleene algebra hold for this structure: e.g. left distributivity $a \cdot (b + c) = a \cdot b + a \cdot c$ and right strictness $a \cdot 0 = 0$ fail. There is a better fit with the axioms of lazy Kleene algebra, a weakening of Kleene algebra introduced in [12] by

Bernhard Möller in 2007. But the fit is not perfect, and therefore we develop *weak* lazy Kleene algebra as a proper axiomatization of $\mathbf{SF}(\mathbf{PR})$. To deal with the parameter condition in the interpolation theorem, we introduce in Section 10 operations on set-valued functions that restrict their domain and range. Using properties of these operations and the axioms of weak lazy Kleene algebra, we prove uniform left interpolation for propositional Horn logic in Section 11. A generalization to propositional Horn logic with $\perp$ is given in Section 12.

For the proof of polynomial interpolation, we introduce in Section 13 thin functions, corresponding with Horn formulae where the heads of the clauses are unique. With thin functions, we reduce the size of an interpolant to polynomial size in Section 14. For the infinitary case, we need the Axiom of Choice. Some concluding remarks are made in Section 15.

2 Propositional Horn Logic

We consider propositional Horn logic over an arbitrary collection $\mathbf{PR}$ of elementary propositions, without any cardinality restriction besides being non-zero. $\mathbf{PR}$ may contain the false proposition $\perp$, but for the moment $\perp$ will play no special role. We postpone the special treatment of $\perp$ to Section 12.

A *Horn clause* is a formula of the form $\bigwedge P \rightarrow p$ with $P \subseteq \mathbf{PR}$ and $p \in \mathbf{PR}$. Observe that P may be infinite: we impose no restriction on its size. When $p \in P$, the clause is called *trivial*; and when $Q \supseteq P$ we call $\bigwedge Q \rightarrow p$ a *weakening* of $\bigwedge P \rightarrow p$. By convention, $\bigwedge \emptyset$ equals $\top$ (true), so $\bigwedge P \rightarrow p$ is logically equivalent to p when P is empty.

A *Horn formula* is a conjunction of Horn clauses. So a Horn formula is of the form

$$\bigwedge_{i \in I} \left(\bigwedge P_i \rightarrow p_i \right)$$

where I is some index set with $P_i \in \wp(\mathbf{PR})$ and $p_i \in \mathbf{PR}$ for every $i \in I$. When P_i is finite for every $i \in I$, we have a *clause-finite* Horn formula.

For Horn formulae $\varphi = \bigwedge_{i \in I} (\bigwedge P_i \rightarrow p_i)$, we define the collections $\mathbf{pos}(\varphi)$, $\mathbf{neg}(\varphi)$ of positive resp. negative propositions occurring in φ by

$$\begin{aligned} \mathbf{pos}(\varphi) &= \{p_i \mid i \in I\} \\ \mathbf{neg}(\varphi) &= \bigcup \{P_i \mid i \in I\} \end{aligned}$$

We also define $\mathbf{par}(\varphi)$, the parameters of φ , as the pair $(\mathbf{neg}(\varphi), \mathbf{pos}(\varphi))$. Union, intersection and inclusion are straightforwardly extended to pairs (N, P) of subsets of $\mathbf{PR}$. Moreover, we define $\mathbf{par}^{-1}(\varphi) = (\mathbf{pos}(\varphi), \mathbf{neg}(\varphi))$.

Horn logic is the logic of Horn formulae. Its semantics is formulated in terms of models $M \subseteq \mathbf{PR}$: atomic proposition $p \in \mathbf{PR}$ holds in M iff $p \in M$. As indicated above, we postpone the special treatment of $\perp$ to Section 12, so now $\perp$ may be an element of models M (provided that $\perp \in \mathbf{PR}$).

With the standard interpretation of $\rightarrow$, this yields for clauses $P \rightarrow p$ that $M \models P \rightarrow p$ iff $(P \subseteq M$ implies $p \in M)$. So for Horn formulae we have

$$M \models \bigwedge_{i \in I} \left(\bigwedge P_i \rightarrow p_i \right) \text{ iff } \forall i \in I (P_i \subseteq M \Rightarrow p_i \in M)$$

Semantic entailment of Horn formulae φ, ψ is defined by

$$\varphi \models \psi \quad \text{iff} \quad \forall P \subseteq \mathbf{PR} (P \models \varphi \Rightarrow P \models \psi)$$

We define the size $\|\varphi\|$ of Horn formula φ as the number of occurrences of elementary propositions, so

$$\left\| \bigwedge_{i \in I} (P_i \rightarrow p_i) \right\| = \#I + \sum_{i \in I} \#P_i$$

Here $\#X$ denotes the cardinality of X .

3 Interpolation

Informally speaking, an interpolant for $\varphi \wedge \chi \models \psi$ is a formula θ that expresses, in the common language of φ and $\chi \rightarrow \psi$, the ‘information’ or ‘semantic content’ that is needed to infer $\chi \rightarrow \psi$ from φ . More formally, we define

DEFINITION 1 (interpolation and polynomial interpolation)

Let φ, χ, ψ be Horn formulae with $\varphi \wedge \chi \models \psi$. An interpolant for φ, χ, ψ is a Horn formula θ with

1. $\varphi \models \theta$ and $\theta \wedge \chi \models \psi$;
2. $\text{par}(\theta) \subseteq \text{par}(\varphi) \cap (\text{par}^{-1}(\chi) \cup \text{par}(\psi))$.

Interpolation holds when every triple φ, χ, ψ with $\varphi \wedge \chi \models \psi$ has an interpolant. *Polynomial interpolation* holds when, moreover, the size of θ is polynomial in the size of φ, χ and ψ .

The parameter condition (2) is called the *Lyndon condition* (after [10]). A weaker condition is the *Craig condition* (named after the first formulation and proof of the interpolation property by Craig in [2]), where the distinction between positive and negative occurrences is ignored, i.e. where par is defined by $\text{par}(\varphi) = \text{pos}(\varphi) \cup \text{neg}(\varphi)$.

The relation ‘polynomial in’ between sizes usually refers to finite sizes, i.e. natural numbers. Here we generalize to arbitrary cardinals as sizes. Observe that, for infinite cardinals λ , the expression ‘ κ is polynomial in λ ’ is equivalent with $\kappa \leq \lambda$ since $\lambda^n = \lambda$ for all $n \in \mathbb{N}^+$.

The notion of interpolant can be strengthened by requiring that it does not depend on all of φ, χ and ψ . We consider two versions of *uniform* interpolation.

DEFINITION 2 (Uniform left and right interpolation)

1. A *uniform left interpolant* for φ and $(N, P) \subseteq \text{par}(\varphi)$ is a formula θ with $\text{par}(\theta) \subseteq (N, P)$ such that, for all χ, ψ with $\text{par}(\varphi) \cap (\text{par}^{-1}(\chi) \cup \text{par}(\psi)) \subseteq (N, P)$ we have

$$\varphi \wedge \chi \models \psi \Leftrightarrow \theta \wedge \chi \models \psi \quad (1)$$

2. A *uniform right interpolant* for χ, ψ and $(N, P) \subseteq \text{par}^{-1}(\chi) \cup \text{par}(\psi)$ is a formula θ with $\text{par}(\theta) \subseteq (N, P)$ such that, for all φ with $\text{par}(\varphi) \cap (\text{par}^{-1}(\chi) \cup \text{par}(\psi)) \subseteq (N, P)$ we have

$$\varphi \wedge \chi \models \psi \Leftrightarrow \varphi \models \theta$$

Observe that a uniform left interpolant θ for φ and $(N, P) \subseteq \text{par}(\varphi)$ corresponds to $\exists \bar{p} \varphi$, where $\bar{p}$ lists all propositions in $\text{par}(\varphi) - (N, P)$. Analogously, a uniform right interpolant θ for χ, ψ and $(N, P) \subseteq \text{par}^{-1}(\chi) \cup \text{par}(\psi)$ corresponds to $\forall \bar{p} (\chi \rightarrow \psi)$, where $\bar{p}$ lists all propositions in $\text{par}^{-1}(\chi) \cup \text{par}(\psi) - (N, P)$.

Both versions of uniform interpolation imply ordinary interpolation. To see this for uniform left interpolation, assume $\varphi \wedge \chi \models \psi$ and take $(N, P) = \text{par}(\varphi) \cap (\text{par}^{-1}(\chi) \cup \text{par}(\psi))$, so $(N, P) \subseteq \text{par}(\varphi)$. Now the uniform left interpolant θ satisfies $\theta \wedge \chi \models \psi$, a consequence of (1) and $\varphi \wedge \chi \models \psi$. Moreover, when we instantiate (1) with $\chi := \top$, $\psi := \theta$ we obtain $\varphi \wedge \top \models \theta \Leftrightarrow \theta \wedge \top \models \theta$, so $\varphi \models \theta$. Analogously for right interpolation.

We shall show in Theorem 2 that uniform left interpolation holds for Horn logic. With this result we obtain polynomial interpolation for Horn logic (Theorem 4).

An example shows that this result cannot be strengthened to *polynomial* uniform left interpolation.

EXAMPLE 1

Let I be some index set with $\#I = \kappa$ and let $P = \{p_i \mid i \in I\}$, $Q = \{q_i \mid i \in I\}$ and $R = \{r_i \mid i \in I\}$ be mutually disjoint subsets of PR , with $s \in \text{PR}$ in none of them. Define φ by

$$\varphi = \bigwedge_{i \in I} (p_i \rightarrow r_i) \wedge \bigwedge_{i \in I} (q_i \rightarrow r_i) \wedge \left(\left(\bigwedge_{i \in I} r_i \right) \rightarrow s \right)$$

Then $\|\varphi\| = 5\kappa + 1$.

Claim: the minimal uniform left interpolant for φ and $(P \cup Q, \{s\})$ is

$$\theta = \bigwedge_{J \subseteq I} \left(\left(\bigwedge (\{p_j \mid j \in J\} \cup \{q_i \mid i \in I - J\}) \right) \rightarrow s \right)$$

To see this, observe that

$$\left\{ \bigwedge (\{p_j \mid j \in J\} \cup \{q_j \mid j \in J'\}) \rightarrow s \mid J \cup J' = I \right\}$$

equals the set of all Horn clauses with parameters in $(P \cup Q, \{s\})$ that follow from φ . All these clauses follow from the strongest clauses, i.e. the clauses with $J \cap J' = \emptyset$, so $J' = I - J$. Moreover, all strongest clauses are mutually independent, so they are all needed in the interpolant. We observe that $\|\theta\| = 2^\kappa(\kappa + 1)$, which is exponential in $\|\varphi\|$.

Open question: are there interesting conditions on φ that guarantee the existence of a polynomial uniform left interpolant?

For uniform right interpolation, the situation is different: it does not hold in Horn logic, as the following example shows.

COUNTEREXAMPLE 1

There is no uniform right interpolant for $\chi = (p \rightarrow r) \wedge (q \rightarrow r)$, $\psi = r$ and $(\emptyset, \{p, q\})$, i.e. there is no Horn formula θ with

1. $\text{neg}(\theta) = \emptyset$, $\text{pos}(\theta) \subseteq \{p, q\}$;
2. for all φ with $\text{pos}(\varphi) \subseteq \{p, q\}$ and $r \notin \text{pos}(\varphi) \cup \text{neg}(\varphi)$ we have

$$\varphi \models \theta \Leftrightarrow \varphi \wedge (p \rightarrow r) \wedge (q \rightarrow r) \models r$$

By the first condition, the only candidates for θ are $\top$, p , q and $p \wedge q$. The second condition implies (by taking $\varphi = p$ and $\varphi = q$) that $p \models \theta$ and $q \models \theta$, which is satisfied only by candidate $\theta = \top$. However, by taking $\varphi = \theta = \top$ in the second condition, we see that $(p \rightarrow r) \wedge (q \rightarrow r) \models r$ should hold, which is not the case. So there is no θ that meets both conditions.

Observe that $\theta = p \vee q$ would satisfy the conditions 1 and 2 given in the counterexample, but disjunction is not definable in Horn logic.

4 Related literature

In this section, we have a look at related literature on Horn logic and interpolation, reading entailment $\models$ in terms of models M not containing $\perp$.

In [4, Theorem 10], Dahlhaus, Israeli and Makowsky prove the following version of interpolation for finitary propositional Horn logic:

$$\text{if } \varphi \wedge \psi \models \perp \text{ then there is a } \theta \text{ with } \varphi \models \theta \text{ and } \theta \wedge \psi \models \perp. \quad (2)$$

Here $\varphi = \varphi(\bar{x}, \bar{y})$, $\psi = \psi(\bar{y}, \bar{z})$ and $\theta = \theta(\bar{y})$ are propositional Horn formulae with $\bar{x}, \bar{y}, \bar{z}$ disjoint sets of propositional variables. Observe that the implicit parameter condition in this formulation is the Craig condition, which comes down to: all propositional variables in θ occur in both φ and ψ . The proof in [4] runs via a polynomial algorithm that computes θ given φ and ψ . As a consequence, the interpolant has polynomial size.

Apart from the specific condition on the parameters, (2) looks weaker than the version given in Definition 1, i.e. with an arbitrary Horn formula as conclusion instead of $\perp$. However, the version of Definition 1 with the Craig condition instead of the Lyndon condition follows from (2). To see this, assume $\varphi \wedge \chi \models \psi$ with $\psi = \bigwedge_{i \in I} (\bigwedge P_i \rightarrow p_i)$. Then $\varphi \wedge \chi \wedge \bigwedge P_i \wedge (p_i \rightarrow \perp) \models \perp$ for all $i \in I$. Let θ_i ($i \in I$) be the interpolants for these entailments according to (2), so for all $i \in I$ we have $\varphi \models \theta_i$ and $\theta_i \wedge \chi \wedge \bigwedge P_i \wedge (p_i \rightarrow \perp) \models \perp$. The latter entailment implies $\theta_i \wedge \chi \wedge \bigwedge P_i \models p_i$ (using the fact that $\varphi \wedge (p \rightarrow \perp) \models \perp$ implies $\varphi \models p$). As a consequence, we see that $\theta = \bigwedge_{i \in I} \theta_i$ is an interpolant for $\varphi \wedge \chi \models \psi$.

In [13], Pynko investigates interpolation for infinitary universal Horn logic, where the language contains functions, relations and quantifiers. Here Horn formulae are of the form $\forall \bar{x} \bigwedge_{i \in I} (\bigwedge \Phi_i \rightarrow \varphi_i)$ where the φ_i and the elements of Φ_i are atomic formulae of predicate logic, so may contain variables and functions. Pynko proves interpolation with the Lyndon condition restricted to relation symbols, so there are no restrictions on the occurrence of functions.

Uniform interpolation is studied for many logics, e.g. modal logics, description logics, fuzzy logics. Surveys of this work can be found in [3, 9, 15]. However, we did not find any reference to the investigation of uniform interpolation in the context of infinitary logics.

We mention an interesting application of interpolation and Horn logic in the area of software verification. Here interpolants can be used to repair errors in software design. The idea is to take a proof that a software design fails to satisfy a requirement, to extract an appropriate interpolant from the proof and to use that interpolant to improve the design. The parameter condition of the interpolant enables to extract the essential information from the unsatisfiability proof: see [7]. In [14], Rümmer, Hojjat and Kuncak introduce the idea of disjunctive interpolant: a function from positions in a formula that generalizes the notion of interpolant. This is applied to solving sets of Horn clauses.

5 Mathematical representation of Horn formulae

There are several ways to give a mathematical representation of Horn formulae. An obvious approach is to consider a Horn clause $\bigwedge P \rightarrow p$ as a tuple (P, p) , and hence a Horn formula $\bigwedge_{i \in I} (\bigwedge P_i \rightarrow p_i)$ as a subset of $\wp(\text{PR}) \times \text{PR}$, i.e. as a relation between $\wp(\text{PR})$ and PR . Such relations have been studied

under the names *hypergraphs* and *multirelations*. In [6], Gallo and others introduce B-graphs as a special kind of directed hypergraphs. B-graphs contain hyperedges $\langle Y, z \rangle$ with $Y \subseteq X$, $z \in X$, so they correspond to subsets of $\wp(X) \times X$; they are used to model Horn formulae. In 2014, a Dagstuhl seminar was devoted to Horn formulas, directed hypergraphs and other related formalisms: see the seminar report [1].

Martin and Curtis investigate in [11] the algebra of multirelations, i.e. subsets of $X \times \wp(Y)$. The focus is on upclosed multirelations M which satisfy the monotonicity condition $(x, Z) \in M \ \& \ Z \subseteq Z' \Rightarrow (x, Z') \in M$. As Martin and Curtis observe, the collection $\wp(X \times \wp(Y))$ of multirelations is isomorphic to the collection of set-valued functions $\wp(X) \rightarrow \wp(Y)$. They prefer to work with multirelations, and motivate this by [11, p. 639]:

[. . .], multirelations have a simpler algebra because relations are generally considered easier to manipulate than set-valued functions.

Another advantage of multirelations, especially when considered as hypergraphs, is that they facilitate the transfer of methods and techniques from graph algorithms to algorithms involving Horn formulae.

In the present paper, however, we choose for set-valued functions $\wp(\mathbf{PR}) \rightarrow \wp(\mathbf{PR})$ instead of multirelations in $\wp(\wp(\mathbf{PR}) \times \mathbf{PR})$. The reason is that, in our eyes, set-valued functions *with equal domain and codomain* are rather easy to manipulate because function composition is available. Moreover, fixpoint operators—essential for the definition of derivability in Horn logic—can be introduced straightforwardly.

A Horn formula $\varphi = \bigwedge_{i \in I} (\bigwedge P_i \rightarrow p_i)$ is uniquely represented by $\mathcal{F} \in \mathbb{F}(\mathbf{PR})$ defined by

$$\mathcal{F} = \lambda Q. \{p_i \mid i \in I \ \& \ P_i = Q\}$$

Conversely, we define $\text{horn}(\mathcal{F})$, the Horn formula represented by $\mathcal{F} \in \mathbb{F}(\mathbf{PR})$:

$$\text{horn}(\mathcal{F}) = \bigwedge_{P \subseteq \mathbf{PR}} \bigwedge_{q \in \mathcal{F}(P)} \left(\bigwedge P \rightarrow q \right)$$

The size $\|\mathcal{F}\|$ of function $\mathcal{F}$ is defined by

$$\|\mathcal{F}\| = \sum \{\#P + \#\mathcal{F}(P) \mid P \subseteq \mathbf{PR}, \mathcal{F}(P) \neq \emptyset\}$$

We have $\|\mathcal{F}\| \leq \|\text{horn}(\mathcal{F})\|$, but equality does not hold in general. We do have $\|\text{horn}^+(\mathcal{F})\| = \|\mathcal{F}\|$ for another translation horn^+ from functions to *extended Horn formulae*, where clauses of the form $\bigwedge P \rightarrow \bigwedge Q$ are allowed:

$$\text{horn}^+(\mathcal{F}) = \bigwedge_{\mathcal{F}(P) \neq \emptyset} \left(\bigwedge P \rightarrow \bigwedge \mathcal{F}(P) \right)$$

It is evident that $\text{horn}(\mathcal{F})$ and $\text{horn}^+(\mathcal{F})$ are logically equivalent.

We turn to validity and entailment, and observe that $P \models \text{horn}(\mathcal{F})$ iff $\forall Q \subseteq P \ \mathcal{F}(Q) \subseteq P$. So when we define the *monotonic closure* $\mathcal{F}^m$ of $\mathcal{F}$ by

$$\mathcal{F}^m(P) = \bigcup_{Q \subseteq P} \mathcal{F}(Q)$$

we have $P \models \text{horn}(\mathcal{F})$ iff $\mathcal{F}^m(P) \subseteq P$. This leads us to define

$$P \models \mathcal{F} \text{ iff } \mathcal{F}^m(P) \subseteq P$$

so that we have $P \models \mathcal{F}$ iff $P \models \text{horn}(\mathcal{F})$. For entailment, we define

$$\mathcal{F} \models \mathcal{G} \text{ iff } \forall P \in \wp(\text{PR})(P \models \mathcal{F} \Rightarrow P \models \mathcal{G})$$

so $\mathcal{F} \models \mathcal{G}$ iff $\text{horn}(\mathcal{F}) \models \text{horn}(\mathcal{G})$. We also define $\text{dom}, \text{rg} : \mathbb{F}(\text{PR}) \rightarrow \wp(\text{PR})$ by

$$\begin{aligned} \text{dom}(\mathcal{F}) &= \bigcup_{\mathcal{F}(P) \neq \emptyset} P \\ \text{rg}(\mathcal{F}) &= \bigcup_{P \subseteq \text{PR}} \mathcal{F}(P) \end{aligned}$$

Observe that our definitions for dom and rg are non-standard, but they satisfy

$$\begin{aligned} \text{pos}(\text{horn}(\mathcal{F})) &= \text{rg}(\mathcal{F}) \\ \text{neg}(\text{horn}(\mathcal{F})) &= \text{dom}(\mathcal{F}) \end{aligned}$$

We also define $\text{Dom} : \mathbb{F}(\text{PR}) \rightarrow \wp(\wp(\text{PR}))$ by

$$\text{Dom}(\mathcal{F}) = \{P \mid \mathcal{F}(P) \neq \emptyset\}$$

so $\text{dom}(\mathcal{F}) = \bigcup \text{Dom}(\mathcal{F})$.

What is the effect of function composition on the Horn formulae represented by set-valued functions? We have: $\bigwedge P \rightarrow p$ is a clause in $\text{horn}(\mathcal{F} \circ \mathcal{G})$ iff $\text{horn}(\mathcal{F})$ contains the clause $\bigwedge Q \rightarrow p$ where $Q = \{q \mid \bigwedge P \rightarrow q \text{ is a clause in } \text{horn}(\mathcal{G})\}$. So the clauses in $\text{horn}(\mathcal{F} \circ \mathcal{G})$ logically follow from the clauses in $\text{horn}(\mathcal{G})$ and $\text{horn}(\mathcal{F})$ via the transitivity of implication. However, not all such clauses are in $\text{horn}(\mathcal{F} \circ \mathcal{G})$: e.g. if $\text{horn}(\mathcal{G}) = (p \rightarrow q_1) \wedge (p \rightarrow q_2)$ and $\text{horn}(\mathcal{F}) = (q_1 \rightarrow r)$, then $\text{horn}(\mathcal{F} \circ \mathcal{G})$ contains no clauses at all, for $\mathcal{F}(\mathcal{G}(\{p\})) = \mathcal{F}(\{q_1, q_2\}) = \emptyset$. To improve this situation, we will need the monotonic closure of set-valued functions, which will be discussed in the next section.

6 More about $\mathbb{F}(\text{PR})$

The function space $\mathbb{F}(\text{PR}) = \wp(\text{PR}) \rightarrow \wp(\text{PR})$ is a complete lattice, because its range $\wp(\text{PR})$ is a complete lattice. The ordering relation $\subseteq$ and the operations $\sqcap$ (meet) and $\sqcup$ (join) are lifted to $\mathbb{F}(\text{PR})$ straightforwardly, e.g. by: $\mathcal{F} \subseteq \mathcal{G}$ iff $\forall P \subseteq \text{PR} \mathcal{F}(P) \subseteq \mathcal{G}(P)$. Idem for the operators $\bigsqcap$ (infimum) and $\bigsqcup$ (supremum) of arbitrary collections of functions. Observe that $\mathcal{F} \subseteq \mathcal{G}$ iff every clause in $\text{horn}(\mathcal{F})$ is a clause in $\text{horn}(\mathcal{G})$.

We let $\underline{P}$ denote the constant function with value P , i.e. $\underline{P}(Q) = P$ for all Q . Then $\underline{\text{PR}}$ is the $\subseteq$ -largest function in $\mathbb{F}(\text{PR})$, and $\underline{\emptyset}$ the smallest. We write $\mathcal{I}$ for the identity function in $\mathbb{F}(\text{PR})$. So $\text{horn}(\underline{P})$ contains the clauses $\bigwedge Q \rightarrow p$ for all $Q \subseteq \text{PR}$ and all $p \in P$, while $\text{horn}(\mathcal{I})$ consists of all trivial clauses $\bigwedge P \rightarrow p$ with $p \in P \subseteq \text{PR}$.

Monotonicity

A function $\mathcal{F} \in \mathbb{F}(\text{PR})$ is *monotonic* when we have $\mathcal{F}(P) \subseteq \mathcal{F}(Q)$ for all P, Q with $P \subseteq Q \subseteq \text{PR}$. So if $\mathcal{F}$ is monotonic and $\bigwedge P \rightarrow p$ is a clause in $\text{horn}(\mathcal{F})$, then all weaker clauses $\bigwedge Q \rightarrow p$

with $Q \supseteq P$ are also in $\text{horn}(\mathcal{F})$. In general, monotonic functions (and hence the associated Horn clauses) are large: we have

$$\|\mathcal{F}\| \geq 2^{\#P} \text{ whenever } \mathcal{F} \text{ monotonic and } \text{PR} - P \in \text{Dom}(\mathcal{F}). \quad (3)$$

When $\mathcal{F}$ and $\mathcal{G}$ are monotonic, then so is $\mathcal{F} \circ \mathcal{G}$, and we have: $\bigwedge P \rightarrow p$ is a clause in $\text{horn}(\mathcal{F} \circ \mathcal{G})$ iff there is a Q such that $\bigwedge P \rightarrow q$ is a clause in $\text{horn}(\mathcal{G})$ for all $q \in Q$, and $\bigwedge Q \rightarrow p$ is a clause in $\text{horn}(\mathcal{F})$. So, for monotonic $\mathcal{F}$ and $\mathcal{G}$, $\text{horn}(\mathcal{F} \circ \mathcal{G})$ exactly contains all clauses that follow via an application of transitivity of implication between some clauses in $\text{horn}(\mathcal{G})$ and a clause in $\text{horn}(\mathcal{F})$.

We define $\mathbb{F}_m(\text{PR})$ as the collection of monotonic functions in $\mathbb{F}(\text{PR})$. Since $\sqcap$, $\sqcup$, $\sqcap$ and $\sqcup$ preserve monotonicity, we have that $\mathbb{F}_m(\text{PR})$ is a complete lattice, too. We observe that monotonicity is preserved under function composition: if $\mathcal{F}$ and $\mathcal{G}$ are monotonic, then so is $\mathcal{F} \circ \mathcal{G}$.

Recall the definition of monotonic closure in Section 5: $\mathcal{F}^m(P) = \bigcup \{\mathcal{F}(Q) \mid Q \subseteq P\}$. It is easily checked that $\mathcal{F}^m$ is the least monotonic $\mathcal{G} \in \mathbb{F}(\text{PR})$ with $\mathcal{F} \subseteq \mathcal{G}$, so we have $\mathcal{F}$ monotonic iff $\mathcal{F}^m = \mathcal{F}$. Moreover, $\cdot^m$ is idempotent, i.e. $(\mathcal{F}^m)^m = \mathcal{F}^m$, and it distributes over $\sqcup$ and $\sqcap$:

$$\left(\bigsqcup_{i \in I} \mathcal{F}_i \right)^m = \bigsqcup_{i \in I} \mathcal{F}_i^m \quad (4)$$

By property (3), the size of $\mathcal{F} \in \mathbb{F}(\text{PR})$ may increase exponentially under monotonic closure. However, $\text{horn}(\mathcal{F})$ and $\text{horn}(\mathcal{F}^m)$ are logically equivalent: the additional clauses in $\text{horn}(\mathcal{F}^m)$ are all weakenings of clauses in $\text{horn}(\mathcal{F})$.

For reasons of conciseness, a kind of inverse operator $\cdot^-$ of $\cdot^m$ might be useful, with the property

$$\mathcal{F}^- \text{ is a } \subseteq - \text{ minimal function } \mathcal{G} \text{ with } \mathcal{G}^m = \mathcal{F}^m$$

However, such an operator $\cdot^-$ cannot be defined in general, as we show now. Assume that PR is infinite and consider the monotonic function $\mathcal{F}$ with

$$\mathcal{F}(P) = \begin{cases} \{p\} & \text{if } P \subseteq \text{PR} \text{ is infinite} \\ \emptyset & \text{if } P \subseteq \text{PR} \text{ is finite} \end{cases}$$

then there is no minimal $\mathcal{G}$ with $\mathcal{G}^m = \mathcal{F}^m = \mathcal{F}$. To see this, assume that $\mathcal{G}$ is such a minimal function. Then $\mathcal{G}(P) = \{p\}$ for some P (for otherwise $\mathcal{G}(P) = \mathcal{G}^m(P) = \emptyset$ for all $P \subseteq \text{PR}$, so $\mathcal{G}^m \neq \mathcal{F}$). P must be infinite, so it contains a proper subset Q that is infinite, and $\mathcal{F}(Q) = \mathcal{G}^m(Q) = \{p\}$. This implies that Q contains a subset Q' with $\mathcal{G}(Q') = \{p\}$. But now we can define $\mathcal{G}'$ by $\mathcal{F}'(P) = \emptyset$ and $\mathcal{G}'(P') = \mathcal{G}(P)$ for all $P' \neq P$. We have $\mathcal{G}' \sqsubset \mathcal{G}$ and $\mathcal{G}'^m = \mathcal{G}^m = \mathcal{F}$, so $\mathcal{G}$ is not minimal.

An important subclass of functions that admits the definition of $\mathcal{F}^-$ is the class of *domain-finite* functions, i.e. $\mathcal{F}$ with $\text{Dom}(\mathcal{F}) \subseteq \wp^{\text{fin}}(\text{PR})$. Observe that domain-finite functions correspond to the clause-finite Horn formulae. For these $\mathcal{F}$, $\mathcal{F}^-$ is defined by

$$\mathcal{F}^-(P) = \{p \mid P \text{ is minimal in } \{Q \mid p \in \mathcal{F}(Q)\}\}$$

This works, because every Q with $p \in \mathcal{F}(Q)$ is finite and hence the superset of some Q_0 that is a minimal element of $\{Q \mid p \in \mathcal{F}(Q)\}$.

Continuity

Let β be an ordinal and assume that $P_\alpha \subseteq \mathbf{PR}$ for all ordinals $\alpha < \beta$. We call $(P_\alpha | \alpha < \beta)$ a *chain* when $P_\alpha \subseteq P_{\alpha'}$ for all $\alpha, \alpha' < \beta$ with $\alpha \leq \alpha'$. A function $\mathcal{F}$ is called *continuous* whenever

$$\mathcal{F}\left(\bigcup_{\alpha < \beta} P_\alpha\right) = \bigcup_{\alpha < \beta} \mathcal{F}(P_\alpha) \text{ for all chains } (P_\alpha | \alpha < \beta)$$

The $\supseteq$ part of this equation of continuity is equivalent with monotonicity. We have:

LEMMA 1

$\mathcal{F}$ is continuous iff there is a domain-finite $\mathcal{G}$ with $\mathcal{F} = \mathcal{G}^m$.

PROOF. For the ‘if’ part, it suffices to show $\mathcal{G}^m\left(\bigcup_{\alpha < \beta} P_\alpha\right) \subseteq \bigcup_{\alpha < \beta} \mathcal{G}^m(P_\alpha)$ for domain-finite $\mathcal{G}$ and chain $(P_\alpha | \alpha < \beta)$. So let $p \in \mathcal{G}^m\left(\bigcup_{\alpha < \beta} P_\alpha\right)$: we shall show that $p \in \bigcup_{\alpha < \beta} \mathcal{G}^m(P_\alpha)$. $p \in \mathcal{G}^m\left(\bigcup_{\alpha < \beta} P_\alpha\right)$ implies that there is a finite $P = \{p_1, \dots, p_n\} \subseteq \bigcup_{\alpha < \beta} P_\alpha$ with $p \in \mathcal{G}(P)$. So for every $i \leq n$ we have $p_i \in P_{\alpha_i}$ for some $\alpha_i < \beta$. Define $\alpha = \max_{i \leq n} \alpha_i$, then $\alpha < \beta$ and $P \subseteq P_\alpha$, so $p \in \mathcal{G}^m(P_\alpha) \subseteq \bigcup_{\alpha < \beta} \mathcal{G}^m(P_\alpha)$.

For the ‘only if’ part, assume that $\mathcal{F}$ is continuous. We define $\mathcal{G}$ by $\mathcal{G}(P) = \mathcal{F}(P)$ if P finite, otherwise $\mathcal{G}(P) = \emptyset$. So $\mathcal{G}$ is domain-finite. We shall show $\mathcal{G}^m = \mathcal{F}$, i.e. for all $P \subseteq \mathbf{PR}$

$$\mathcal{F}(P) = \bigcup \{\mathcal{F}(Q) \mid Q \subseteq P, Q \text{ finite}\} \quad (5)$$

We prove (5) with induction over $\gamma = \#P$. When γ is finite, (5) is evident, so we assume that γ is infinite. Let $P = \{p_\alpha | \alpha < \gamma\}$. Define $P_\alpha = \{p_\beta | \beta < \alpha\}$ for $\alpha < \gamma$, then $(P_\alpha | \alpha < \gamma)$ is a chain with $P = \bigcup_{\alpha < \gamma} P_\alpha$ (thanks to P being infinite, hence γ is a limit ordinal). Now

$$\begin{aligned} & \mathcal{F}(P) \\ = & \{P = \bigcup_{\alpha < \gamma} P_\alpha, \mathcal{F} \text{ is continuous}\} \\ & \bigcup_{\alpha < \gamma} \mathcal{F}(P_\alpha) \\ = & \{\text{induction hypothesis, using that } \#P_\alpha = \alpha < \gamma\} \\ & \bigcup_{\alpha < \gamma} \bigcup \{\mathcal{F}(Q) \mid Q \subseteq P_\alpha, Q \text{ finite}\} \\ = & \{Q \text{ finite, so } Q \subseteq P \Leftrightarrow \exists \alpha < \gamma \ Q \subseteq P_\alpha\} \\ & \bigcup \{\mathcal{F}(Q) \mid Q \subseteq P, Q \text{ finite}\} \end{aligned}$$

so we proved (5). □

Monotonicity of operators on $\mathbb{F}(\mathbf{PR})$

We also consider monotonicity on the level of operators on $\mathbb{F}(\mathbf{PR})$. To distinguish it from monotonicity w.r.t. $\subseteq$, we call this notion $\sqsubseteq$ -monotonicity. It is not hard to see that dom , rg , Dom , $\cdot^m$ and $\|\cdot\|$ are $\sqsubseteq$ -monotonic, i.e. if $\mathcal{F} \sqsubseteq \mathcal{G}$ then $\text{Dom}(\mathcal{F}) \subseteq \text{Dom}(\mathcal{G})$, $\text{dom}(\mathcal{F}) \subseteq \text{dom}(\mathcal{G})$, $\text{rg}(\mathcal{F}) \subseteq \text{rg}(\mathcal{G})$, $\mathcal{F}^m \sqsubseteq \mathcal{G}^m$ and $\|\mathcal{F}\| \leq \|\mathcal{G}\|$.

For the composition operator $\circ$ on $\mathbb{F}(\mathbf{PR})$, the situation is different. On the one hand, $\circ$ is $\sqsubseteq$ -monotonic in its first argument:

$$\mathcal{F} \sqsubseteq \mathcal{G} \Rightarrow \mathcal{F} \circ \mathcal{H} \sqsubseteq \mathcal{G} \circ \mathcal{H} \quad (6)$$

However, $\circ$ is *not* $\sqsubseteq$ -monotonic in its second argument: $\mathcal{G} \sqsubseteq \mathcal{H}$ does not always imply $\mathcal{F} \circ \mathcal{G} \sqsubseteq \mathcal{F} \circ \mathcal{H}$. To see this, consider $\mathcal{F}, \mathcal{G}, \mathcal{H}$ with $\mathcal{F}(\{q\}) = \{r\}$, $\mathcal{G}(\{p\}) = \{q\}$, $\mathcal{H}(\{p\}) = \{q, r\}$ while $\mathcal{F}, \mathcal{G}, \mathcal{H}$ yield $\emptyset$ on all other arguments. Now $\mathcal{G} \sqsubseteq \mathcal{H}$ but $\mathcal{F}(\mathcal{G}(\{p\})) = \{r\}$ and $\mathcal{F}(\mathcal{H}(\{p\})) = \emptyset$, so $\mathcal{F} \circ \mathcal{G} \not\sqsubseteq \mathcal{F} \circ \mathcal{H}$. But we do have that left-composing with a *monotonic* function is $\sqsubseteq$ -monotonic, i.e.

$$\mathcal{F} \text{ monotonic and } \mathcal{G} \sqsubseteq \mathcal{H} \Rightarrow \mathcal{F} \circ \mathcal{G} \sqsubseteq \mathcal{F} \circ \mathcal{H} \quad (7)$$

The $\sqsubseteq$ -monotonic closure of $\circ$, i.e. the least $\sqsubseteq$ -monotonic operator $\diamond$ such that $\mathcal{F} \circ \mathcal{G} \sqsubseteq \mathcal{F} \diamond \mathcal{G}$ for all $\mathcal{F}, \mathcal{G}$, is $\diamond$ with $\mathcal{F} \diamond \mathcal{G} = \mathcal{F}^m \circ \mathcal{G}$. However, $\diamond$ is not associative. Therefore we define *monotonic function composition* $\overset{m}{\circ}$ by $\mathcal{F} \overset{m}{\circ} \mathcal{G} = \mathcal{F}^m \circ \mathcal{G}^m$. By the $\sqsubseteq$ -monotonicity of $\circ$, (6) and (7), $\overset{m}{\circ}$ is $\sqsubseteq$ -monotonic. It is also associative:

$$(\mathcal{F} \overset{m}{\circ} \mathcal{G}) \overset{m}{\circ} \mathcal{H} = \mathcal{F} \overset{m}{\circ} (\mathcal{G} \overset{m}{\circ} \mathcal{H}) \quad (8)$$

for $(\mathcal{F}^m \circ \mathcal{G}^m)^m = \mathcal{F}^m \circ \mathcal{G}^m$, so $(\mathcal{F} \overset{m}{\circ} \mathcal{G}) \overset{m}{\circ} \mathcal{H} = \mathcal{F}^m \circ \mathcal{G}^m \circ \mathcal{H}^m = \mathcal{F} \overset{m}{\circ} (\mathcal{G} \overset{m}{\circ} \mathcal{H})$. From the idempotency of $\overset{m}{\circ}$ it follows that $\mathcal{F} \circ \mathcal{G} = \mathcal{F} \overset{m}{\circ} \mathcal{G}$ whenever $\mathcal{F}$ and $\mathcal{G}$ are monotonic. In the sequel, we will often write $\mathcal{F} \circ \mathcal{G}$ for $\mathcal{F} \overset{m}{\circ} \mathcal{G}$ when it is evident that $\mathcal{F}$ and $\mathcal{G}$ are monotonic.

Another property that fails is left distributivity of $\circ$ and $\overset{m}{\circ}$ over $\sqcup$, while right distributivity holds:

$$\left(\bigsqcup_{i \in I} \mathcal{F}_i \right) \circ \mathcal{G} = \bigsqcup_{i \in I} (\mathcal{F}_i \circ \mathcal{G}) \quad \left(\bigsqcup_{i \in I} \mathcal{F}_i \right) \overset{m}{\circ} \mathcal{G} = \bigsqcup_{i \in I} (\mathcal{F}_i \overset{m}{\circ} \mathcal{G}) \quad (9)$$

7 Fixpoints in $\wp(\text{PR})$ and in $\mathbb{F}(\text{PR})$

$\wp(\text{PR})$ is a complete lattice, so according to the Knaster–Tarski fixed point theorem, any monotonic $\mathcal{F} \in \mathbb{F}(\text{PR})$ has a least fixpoint $\mu(\mathcal{F})$. We make μ a total function on $\mathbb{F}(\text{PR})$ by defining

$$\mu(\mathcal{F}) = \bigcap \{P \mid \mathcal{F}^m(P) \subseteq P\}$$

and we claim

$$\begin{array}{ll} \mathcal{F}^m(\mu(\mathcal{F})) = \mu(\mathcal{F}) & \text{defining property of } \mu \\ \mathcal{F}^m(P) \subseteq P \Rightarrow \mu(\mathcal{F}) \subseteq P & \text{inductive property of } \mu \end{array}$$

As is well known, $\mu(\mathcal{F})$ can also be defined ‘from below’ by

$$\mu(\mathcal{F}) = \bigcup \{(\mathcal{F})_\alpha \mid \alpha \text{ an ordinal number}\}$$

where $(\mathcal{F})_\alpha \subseteq \text{PR}$ is defined by

$$\begin{array}{ll} (\mathcal{F})_0 &= \emptyset \\ (\mathcal{F})_{\alpha+1} &= \mathcal{F}^m((\mathcal{F})_\alpha) \\ (\mathcal{F})_\alpha &= \bigcup \{(\mathcal{F})_\beta \mid \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal} \end{array}$$

We have $\mu\mathcal{F} = (\mathcal{F})_\alpha$ whenever $\alpha \geq 2^{\#\text{PR}}$. In semantical terms, $\mu\mathcal{F}$ is the Herbrand model of $\mathcal{F}$, i.e. the smallest M with $M \models \mathcal{F}$.

We go one level higher and consider fixpoints in $\mathbb{F}(\text{PR})$. Since $(\mathbb{F}(\text{PR}), \sqsubseteq)$ is a complete lattice, every $\sqsubseteq$ -monotonic function $f : \mathbb{F}(\text{PR}) \rightarrow \mathbb{F}(\text{PR})$ has a least fixpoint. We are especially interested in the iterative closure of $\mathcal{F}$. It can be obtained as a least fixpoint of the operator $\lambda\mathcal{G}.(\mathcal{I} \sqcup \mathcal{F} \circ \mathcal{G})$, *provided that this operator is $\sqsubseteq$ -monotonic*. This is only the case when $\mathcal{F}$ is monotonic. Therefore

we use the operator $\lambda \mathcal{G}.(\mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{G})$ which is indeed $\sqsubseteq$ -monotonic. So we define the *iterative closure* $\mathcal{F}^*$ of $\mathcal{F}$ by

$$\mathcal{F}^* = \bigcap \{ \mathcal{G} \mid \mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{G} \subseteq \mathcal{G} \}$$

The alternative definition ‘from below’ reads

$$\begin{aligned} \mathcal{F}^* &= \bigcup \{ \mathcal{F}^\alpha \mid \alpha \text{ an ordinal} \}, \text{ where} \\ \mathcal{F}^0 &= \emptyset \\ \mathcal{F}^{\alpha+1} &= \mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{F}^\alpha \\ \mathcal{F}^\alpha &= \bigcup \{ \mathcal{F}^\beta \mid \beta < \alpha \} \text{ if } \alpha \text{ is a limit ordinal} \end{aligned}$$

We have

$$\begin{aligned} \mathcal{F}^* &= \mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{F}^* && \text{defining property of } \mathcal{F}^* \\ \mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{G} \subseteq \mathcal{G} &\Rightarrow \mathcal{F}^* \subseteq \mathcal{G} && \text{inductive property of } \mathcal{F}^* \end{aligned}$$

For later use, we observe

$$\text{all } \mathcal{F}^\alpha \text{ and } \mathcal{F}^* \text{ are monotonic.} \quad (10)$$

For the $\mathcal{F}^\alpha$, this follows easily with induction over α ; the monotonicity of $\mathcal{F}^*$ then follows from the fact that monotonicity is preserved by unions.

If $\mathcal{F}$ is continuous, then $\mathcal{F}^* = \mathcal{F}^\omega = \bigcup_{n < \omega} \mathcal{F}^n$, and all $\mathcal{F}^n$ and $\mathcal{F}^*$ are continuous, too. The first statement follows from $\mathcal{F}^{\omega+1} = \mathcal{F}^\omega$, a consequence of continuity. The continuity of the $\mathcal{F}^n$ is proved with induction over n , and then the continuity of $\mathcal{F}^*$ readily follows.

We have the following property relating fixpoints in $\wp(\mathbf{PR})$ and $\mathbb{F}(\mathbf{PR})$.

LEMMA 2

$\mathcal{F}^* = \lambda P. \mu(\mathcal{F} \sqcup P)$ and, as a consequence, $\mu(\mathcal{F}) = \mathcal{F}^*(\emptyset)$.

PROOF. Straightforward: $\mathcal{F}^* \subseteq \lambda P. \mu(\mathcal{F} \sqcup P)$ follows from the inductive property of $\mathcal{F}^*$, and $\mu(\mathcal{F} \sqcup P) \subseteq \mathcal{F}^*(P)$ follows from the inductive property of μ . $\square$

We have the following characterization of $\models$ in terms of fixpoints.

LEMMA 3

$\mathcal{F} \models \mathcal{G} \Leftrightarrow \mathcal{G} \subseteq \mathcal{F}^*$

PROOF. We have

$$\begin{aligned}
& \mathcal{F} \models \mathcal{G} \\
\Leftrightarrow & \quad \{\text{definition of } \models\} \\
& \forall P (\mathcal{F}^m(P) \subseteq P \Rightarrow \mathcal{G}^m(P) \subseteq P) \\
\Leftrightarrow & \quad \{\Rightarrow: \text{use that } \mathcal{G}^m \text{ is monotonic}; \Leftarrow: \text{take } Q := P\} \\
& \forall PQ (\mathcal{F}^m(P) \cup Q \subseteq P \Rightarrow \mathcal{G}^m(Q) \subseteq P) \\
\Leftrightarrow & \quad \{\text{elementary}\} \\
& \forall Q (\mathcal{G}^m(Q) \subseteq \bigcap \{P \mid \mathcal{F}^m(P) \cup Q \subseteq P\}) \\
\Leftrightarrow & \quad \{\text{definition of } \mu\} \\
& \forall Q (\mathcal{G}^m(Q) \subseteq \mu(\mathcal{F} \sqcup Q)) \\
\Leftrightarrow & \quad \{\text{Lemma 2}\} \\
& \forall Q (\mathcal{G}^m(Q) \subseteq \mathcal{F}^*(Q)) \\
\Leftrightarrow & \quad \{\text{definition of } \sqsubseteq\} \\
& \mathcal{G}^m \subseteq \mathcal{F}^* \\
\Leftrightarrow & \quad \{\Rightarrow: \mathcal{G} \subseteq \mathcal{G}^m; \Leftarrow: \mathcal{F}^* \text{ is monotonic}\} \\
& \mathcal{G} \subseteq \mathcal{F}^*
\end{aligned}$$

□

As a consequence of Lemma 3, we have that $\text{horn}(\mathcal{F}^*)$ consists of all Horn clauses that are entailed by $\text{horn}(\mathcal{F})$.

8 Derivability

The form of Lemma 3 suggests a completeness result for a notion of derivability $\mathcal{F} \vdash \mathcal{G}$ that corresponds with $\mathcal{G} \subseteq \mathcal{F}^*$. In this section, we present such a derivability relation. $\mathcal{F} \vdash \mathcal{G}$ is defined as the least relation on $\mathbb{F}(\text{PR})$ closed under the following rules.

$$\begin{aligned}
& \mathbf{id} \quad \mathcal{F} \vdash \mathcal{I} \\
& \mathbf{comp} \quad \frac{\mathcal{F} \vdash \mathcal{G}}{\mathcal{F} \vdash \mathcal{F}^m \circ \mathcal{G}} \\
& \mathbf{sub} \quad \frac{\mathcal{F} \vdash \mathcal{G} \sqcup \mathcal{H}}{\mathcal{F} \vdash \mathcal{G}} \\
& \mathbf{union} \quad \frac{\mathcal{F} \vdash \mathcal{G}_i \text{ for all } i \in I}{\mathcal{F} \vdash \bigsqcup_{i \in I} \mathcal{G}_i}
\end{aligned}$$

We claim

$$\mathcal{F} \vdash \mathcal{G} \Leftrightarrow \mathcal{G} \subseteq \mathcal{F}^* \tag{11}$$

i.e. $\mathcal{F}^*$ contains exactly all $\mathcal{G}$ that are derivable from $\mathcal{F}$.

The $\Rightarrow$ direction of (11) is proved straightforwardly with induction over the size of a derivation of $\mathcal{F} \vdash \mathcal{G}$. For the proof of the $\Leftarrow$ direction of (11), it suffices to show that $\mathcal{F} \vdash \mathcal{F}^*$, for then $\mathcal{F} \vdash \mathcal{G}$ for all $\mathcal{G} \subseteq \mathcal{F}^*$ follows with **sub**. In order to prove $\mathcal{F} \vdash \mathcal{F}^*$, we use the function $Pr(\mathcal{F})$ defined by $Pr(\mathcal{F}) = \bigsqcup \{\mathcal{G} \mid \mathcal{F} \vdash \mathcal{G}\}$. By **union**, we have that $\mathcal{F} \vdash Pr(\mathcal{F})$. By **sub**, it now suffices to show that $\mathcal{F}^* \subseteq Pr(\mathcal{F})$. This follows via the inductive property of $\mathcal{F}^*$ from $\mathcal{I} \sqcup \mathcal{F}^m \circ Pr(\mathcal{F}) \subseteq Pr(\mathcal{F})$, which is a consequence of **id**, $\mathcal{F} \vdash Pr(\mathcal{F})$, **comp** and **union**.

So by Lemma 3 and (11) we now have soundness and completeness:

$$\mathcal{F} \vdash \mathcal{G} \Leftrightarrow \mathcal{F} \models \mathcal{G}$$

9 Lazy Kleene algebra

In this section we consider the structures

$$\mathbf{SF}(\mathbf{PR}) = \langle \mathbb{F}(\mathbf{PR}), \sqcup, \overset{\mathbf{m}}{\circ}, *, \underline{0}, \mathcal{I} \rangle \quad (12)$$

$$\mathbf{SF}_m(\mathbf{PR}) = \langle \mathbb{F}_m(\mathbf{PR}), \sqcup, \circ, *, \underline{0}, \mathcal{I} \rangle \quad (13)$$

(recall that $\mathbb{F}_m(\mathbf{PR})$ denotes the collection of monotonic functions in $\mathbb{F}(\mathbf{PR})$). As left distributivity of $\overset{\mathbf{m}}{\circ}$ and $\circ$ over $+$ fails, we see that $\mathbf{SF}(\mathbf{PR})$ and $\mathbf{SF}_m(\mathbf{PR})$ are not Kleene algebras. An interesting relaxation of Kleene algebra is introduced by Möller in [12], under the name *lazy Kleene algebra*. The introduction of lazy Kleene algebra is motivated in [12] by the desire to enable a natural treatment of objects like infinite traces and infinite computations which behave as left annihilators, i.e. objects i with $i \cdot a = i$ for all a . Somewhat surprisingly, lazy Kleene algebra almost axiomatizes the structure $\mathbf{SF}(\mathbf{PR})$ of set-values functions, as we shall show here. A lazy Kleene algebra is a structure $\langle A, +, \cdot, *, 0, 1 \rangle$ axiomatized by

- (A1) $+$ is commutative: $a + b = b + a$
- (A2) $+$ is associative: $a + (b + c) = (a + b) + c$
- (A3) $+$ is idempotent: $a + a = a$
- (A4) $+$ has unit element 0: $a + 0 = a$
- (A5) $\cdot$ is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- (A6) $\cdot$ has unit element 1: $a \cdot 1 = 1 \cdot a = a$
- (A7) $\cdot$ is right-distributive over $+$: $(a + b) \cdot c = a \cdot c + b \cdot c$
- (A8) if $b \leq c$ then $a \cdot b \leq a \cdot c$
- (A9) $\cdot$ is left strict w.r.t. 0: $0 \cdot a = 0$
- (A10) $1 + a \cdot a^* \leq a^*$
- (A11) if $a \cdot b \leq b$ then $a^* \cdot b \leq b$

Here $\leq$ is defined by $a \leq b$ iff $a + b = b$. By (A1)–(A4), $\leq$ is a partial order with least element 0.

This axiomatization is obtained from the usual Kleene algebra axiomatization by replacing left distributivity by the weaker axiom (A8), and by removing the axioms $a \cdot 0 = 0$ (right strictness), $1 + a^* \cdot a \leq a^*$ and $a \cdot b \leq a \Rightarrow a \cdot b^* \leq a$.

It turns out that $\mathbf{SF}_m(\mathbf{PR})$ is indeed a lazy Kleene algebra. However, $\mathbf{SF}(\mathbf{PR})$ is not: we have $\mathcal{F} \overset{\mathbf{m}}{\circ} \mathcal{I} = \mathcal{F}^m$ which is not equal to $\mathcal{F}$ in general, so (A6) fails. Moreover, we have $\underline{0} \overset{\mathbf{m}}{\circ} \mathcal{G} = \underline{0} \sqsubseteq \mathcal{G}$ for all $\mathcal{G}$, but $\underline{0}^* \overset{\mathbf{m}}{\circ} \mathcal{G} = \mathcal{I} \overset{\mathbf{m}}{\circ} \mathcal{G} = \mathcal{G}^m \sqsubseteq \mathcal{G}$ is false for nonmonotonic $\mathcal{G}$. So in $\mathbf{SF}(\mathbf{PR})$, (A11) fails too. Therefore we introduce *weak* lazy Kleene algebras by replacing (A6) and (A11) by

- (A6⁻) $a \leq a \cdot 1 = 1 \cdot a$, $1 \cdot 1 = 1$, $a \cdot b \cdot 1 = a \cdot b$ and $(a \cdot 1)^* = a^* \cdot 1 = a^*$
- (A11⁻) if $a \cdot b \leq b$ then $a^* \cdot b \leq b \cdot 1$

We have that every weak lazy Kleene algebra contains a lazy Kleene algebra.

LEMMA 4

Let $\langle A, +, \cdot, *, 0, 1 \rangle$ be a weak lazy Kleene algebra. Then $\langle A_1, +, \cdot, *, 0, 1 \rangle$ with $A_1 = \{a \cdot 1 \mid a \in A\}$ is a lazy Kleene algebra.

PROOF. By (A5), (A6⁻), (A7) and (A9) we see that $0, 1 \in A_1$ and that A_1 is closed under $+$, $\cdot$ and $*$. (A6) for A_1 follows from (A6⁻); (A11) for A_1 follows from (A11⁻) and (A6⁻). The other axioms transfer directly from A to A_1 . $\square$

THEOREM 1

1. **SF(PR)** is a weak lazy Kleene algebra.
2. **SF_m(PR)** is a lazy Kleene algebra.

PROOF.

1. The axioms (A1)–(A4) and (A9) are evident.

(A5) follows from (8).

(A6⁻) from $\mathcal{I} \circ^m \mathcal{F} = \mathcal{F} \circ^m \mathcal{I} = \mathcal{F}^m$, $\mathcal{F} \subseteq \mathcal{F}^m$, $(\mathcal{F}^m)^* = \mathcal{F}^*$ and (10).

(A7) from (4) and (9).

(A8) from (7) and the $\subseteq$ -monotonicity of $\cdot^m$.

(A10) from the defining property for $\mathcal{F}^*$.

For (A11⁻) we reason as follows. Assume that $\mathcal{F} \circ^m \mathcal{G} \subseteq \mathcal{G}$ holds, i.e. $\mathcal{F}^m \circ \mathcal{G}^m \subseteq \mathcal{G}$. This implies $\mathcal{F}^m \circ \mathcal{G}^m \subseteq \mathcal{G}^m$, for $\mathcal{G} \subseteq \mathcal{G}^m$. We shall demonstrate $\mathcal{F}^\alpha \circ \mathcal{G}^m \subseteq \mathcal{G}^m$ for all ordinals α .

$\alpha = 0$: evident, for $\mathcal{F}^0 = \emptyset$.

$\alpha = \beta + 1$: we have as induction hypothesis $\mathcal{F}^\beta \circ \mathcal{G}^m \subseteq \mathcal{G}^m$. Now

$$\mathcal{F}^\beta \circ \mathcal{G}^m \subseteq \mathcal{G}^m$$

$$\Rightarrow \{(7)\}$$

$$\mathcal{F}^m \circ \mathcal{F}^\beta \circ \mathcal{G}^m \subseteq \mathcal{F}^m \circ \mathcal{G}^m$$

$$\Rightarrow \{\mathcal{G} \subseteq \mathcal{G}^m \text{ and the assumption } \mathcal{F} \circ^m \mathcal{G} \subseteq \mathcal{G}\}$$

$$\mathcal{G}^m \sqcup \mathcal{F}^m \circ \mathcal{F}^\beta \circ \mathcal{G}^m \subseteq \mathcal{G}^m$$

$$\Rightarrow \{(9)\}$$

$$(\mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{F}^\beta) \circ \mathcal{G}^m \subseteq \mathcal{G}^m$$

$$\Rightarrow \{\text{definition of } \mathcal{F}^{\beta+1}\}$$

$$\mathcal{F}^{\beta+1} \circ \mathcal{G}^m \subseteq \mathcal{G}^m$$

i.e. $\mathcal{F}^\alpha \circ \mathcal{G}^m \subseteq \mathcal{G}^m$.

α is a limit ordinal: we have as induction hypothesis that $\mathcal{F}^\beta \circ \mathcal{G}^m \subseteq \mathcal{G}^m$ for all $\beta < \alpha$.

This implies $\bigsqcup_{\beta < \alpha} (\mathcal{F}^\beta \circ \mathcal{G}^m) \subseteq \mathcal{G}^m$. With (9), this yields $(\bigsqcup_{\beta < \alpha} \mathcal{F}^\beta) \circ^m \mathcal{G}^m \subseteq \mathcal{G}^m$, i.e.

$\mathcal{F}^\alpha \circ^m \mathcal{G}^m \subseteq \mathcal{G}^m$, and we are done.

So indeed $\mathcal{F}^\alpha \circ \mathcal{G}^m \subseteq \mathcal{G}^m$ for all α . This implies $\mathcal{F}^* \circ \mathcal{G}^m \subseteq \mathcal{G}^m$, i.e. $\mathcal{F}^* \circ^m \mathcal{G} \subseteq \mathcal{G} \circ^m \mathcal{I}$, the conclusion of (A11⁻).

2. Follows from Lemma 4, for $\{\mathcal{F} \circ^m \mathcal{I} \mid \mathcal{F} \in \mathbb{F}(\text{PR})\} = \mathbb{F}_m(\text{PR})$. $\square$

We shall use the following properties of weak lazy Kleene algebras.

LEMMA 5

In a weak lazy Kleene algebra, we have:

1. $+$ is monotonic in both arguments: if $a \leq b$ then $a + c \leq b + c$ and $c + a \leq c + b$.
2. $\cdot$ is monotonic in both arguments: if $a \leq b$ then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.
3. if $c + a \cdot b \leq b$ then $a^* \cdot c \leq b \cdot 1$

4. $*$ is monotonic: if $a \leq b$ then $a^* \leq b^*$.
5. $a \cdot b + a \cdot c \leq a \cdot (b + c)$ (weak left distributivity)
6. $1 + a \cdot a^* = a^*$
7. $1 \leq a^*$ and $a \leq a^*$
8. $a^* \cdot a^* = a^*$
9. $(a^*)^* = a^*$
10. $(a+b)^* = a^* \cdot (b \cdot a^*)^*$
11. $(a \cdot b)^* \cdot a \leq a \cdot (b \cdot a^*)^*$
12. $(a \cdot b)^* \cdot a = a \cdot (b \cdot a^*)^*$ if a is left distributive, i.e. $a \cdot (c + d) \leq a \cdot c + a \cdot d$ for all c, d .

PROOF. Several of these properties are proved for lazy Kleene algebras in [12]. We consider the parts 1, 2, 4 and 7 of this Lemma as elementary facts about weak lazy Kleene algebras: when they are proved, they will be used without further reference.

1. Follows from (A1), (A2), (A3) and the definition of $\leq$.
2. Follows from (A7) and (A8).
3. $c + a \cdot b \leq b$ implies $c \leq b$ and $a \cdot b \leq b$; with (A11⁻) this leads to $a^* \cdot c \leq a^* \cdot b \leq b \cdot 1$.
4. By (A10) and $a \leq b$ we have $1 + a \cdot b^* \leq b^*$. With part 3 of this Lemma, we obtain $a^* \cdot 1 \leq b^* \cdot 1$. By (A6⁻), this implies $a^* \leq b^*$.
5. By (A3). $b \leq b + c$ and $c \leq b + c$, so $a \cdot b \leq a \cdot (b + c)$ and $a \cdot c \leq a \cdot (b + c)$, hence (again (A3)) $a \cdot b + a \cdot c \leq a \cdot (b + c)$.
6. By (A10), we only have to prove $a^* \leq 1 + a \cdot a^*$. Now by (A10) we have $1 + a \cdot (1 + a \cdot a^*) \leq 1 + a \cdot a^*$. With part 3 of this Lemma, this yields $a^* \cdot 1 \leq (1 + a \cdot a^*) \cdot 1$. By (A6⁻), (A7) and (A5), this comes down to $a^* \leq 1 + a \cdot a^*$.
7. $1 \leq a^*$ follows from (A10). It implies $a \cdot 1 \leq a \cdot a^*$, so with (A10) we obtain $a \cdot 1 \leq a^*$, which leads via (A6⁻) to $a \leq a^*$.
8. $a^* \leq a^* \cdot a^*$ follows from $1 \leq a^*$ and (A6⁻). The other direction $a^* \cdot a^* \leq a^*$ follows from $a \cdot a^* \leq a^*$ (a consequence of (A10)), (A11⁻) and (A6⁻).
9. $a^* \leq (a^*)^*$ is an instance of $a \leq a^*$. The other direction $(a^*)^* \leq a^*$ follows via part 3 of the Lemma and (A6⁻) from $1 + a^* \cdot a^* \leq a^*$, which is a consequence of part 7 and 8 of this Lemma.
10. We have $a^* \cdot (b \cdot a^*)^* \leq (a + b)^* \cdot ((a + b) \cdot (a + b)^*)^* \leq (a + b)^*$ by part 8 and 9 of this Lemma. The other inequality $(a + b)^* \leq a^* \cdot (b \cdot a^*)^*$ follows via (A11⁻) and (A6⁻) from

$$(a + b) \cdot a^* \cdot (b \cdot a^*)^* \leq a^* \cdot (b \cdot a^*)^*$$

which via (A7) splits up in two inequalities: $a \cdot a^* \cdot (b \cdot a^*)^* \leq a^* \cdot (b \cdot a^*)^*$ follows from $a \cdot a^* \leq a^*$ (a consequence of (A10)), while $b \cdot a^* \cdot (b \cdot a^*)^* \leq (b \cdot a^*)^* \leq a^* \cdot (b \cdot a^*)^*$ follows from (A10) and $1 \leq a^*$.

11. By part 5 of this Lemma and (A10) we have $a + a \cdot b \cdot a \cdot (b \cdot a^*)^* = a \cdot (1 + b \cdot a \cdot (b \cdot a^*)^*) \leq a \cdot (b \cdot a^*)^*$; with part 3 of this Lemma we now get $(a \cdot b)^* \cdot a \leq a \cdot (b \cdot a^*)^* \cdot 1$. Using (A6⁻) we obtain the claimed inequality.
12. The $\leq$ part is part 11 of this Lemma. For the $\geq$ part, we argue as follows. First observe that for all c we have

$$a \cdot (1 + c \cdot a) = (1 + a \cdot c) \cdot a \tag{14}$$

which follows from (A7) and left distributivity of a . Now

$$\begin{aligned}
& a \cdot (b \cdot a)^* \leq (a \cdot b)^* \cdot a \\
\Leftarrow & \quad \{(A10)\} \\
& a \cdot (b \cdot a)^* \leq (1 + a \cdot b \cdot (a \cdot b)^*) \cdot a \\
\Leftarrow & \quad \{(14)\} \\
& (b \cdot a)^* \leq 1 + b \cdot (a \cdot b)^* \cdot a \\
\Leftarrow & \quad \{(A6^-), (A7)\} \\
& (b \cdot a)^* \cdot 1 \leq (1 + b \cdot (a \cdot b)^* \cdot a) \cdot 1 \\
\Leftarrow & \quad \{\text{part 3 of this Lemma}\} \\
& 1 + b \cdot a \cdot (1 + b \cdot (a \cdot b)^* \cdot a) \leq 1 + b \cdot (a \cdot b)^* \cdot a \\
\Leftarrow & \quad \{(14)\} \\
& 1 + a \cdot b \cdot (a \cdot b)^* \leq (a \cdot b)^* \\
\Leftarrow & \quad \{(A10)\} \\
& \text{true}
\end{aligned}$$

□

10 Projection and restriction

In the proof of the interpolation theorem, we want to restrict functions to subfunctions with a given domain and range. For that purpose, we shall use projection functions $\mathcal{I}_P$ and the restriction operator $\cdot \upharpoonright P$, defined as follows.

$$\begin{aligned}
\mathcal{I}_P(Q) &= Q \cap P \\
(\mathcal{F} \upharpoonright P)(Q) &= \mathcal{F}(Q) \quad \text{if } Q \subseteq P \\
&= \emptyset \quad \text{if } Q \not\subseteq P
\end{aligned}$$

We have

$$\begin{aligned}
\mathcal{F} \upharpoonright P &\subseteq \mathcal{F} \text{ and } \text{dom}(\mathcal{F} \upharpoonright P) \subseteq \text{dom}(\mathcal{F}) \cap P \\
\mathcal{I}_P \circ \mathcal{F} &\subseteq \mathcal{F} \text{ and } \text{rg}(\mathcal{I}_P \circ \mathcal{F}) = P \cap \text{rg}(\mathcal{F})
\end{aligned}$$

so $\mathcal{F} \upharpoonright P$ and $\mathcal{I}_P \circ \mathcal{F}$ are indeed restrictions of $\mathcal{F}$ with respect to its domain and range. Observe that, unlike range restriction, domain restriction cannot be obtained via composition. An attempt to realize domain restriction to P by $\mathcal{F} \circ \nabla_P$ where $\nabla_P(Q) = Q$ if $Q \subseteq P$ and $\nabla_P(Q) = \emptyset$ otherwise, only works in case $\mathcal{F}(\emptyset) = \emptyset$, but not generally.

Under monotonic closure, $\mathcal{F} \upharpoonright P$ and $\mathcal{F} \circ \mathcal{I}_P$ are equal:

$$(\mathcal{F} \upharpoonright P)^m = (\mathcal{F} \circ \mathcal{I}_P)^m = \mathcal{F}^m \circ \mathcal{I}_P \quad (15)$$

Moreover, projections $\mathcal{I}_P$ are monotonic and left-distributive:

$$\mathcal{I}_P \circ \bigsqcup_{i \in I} \mathcal{F}_i = \bigsqcup_{i \in I} (\mathcal{I}_P \circ \mathcal{F}_i) \quad (16)$$

so by Lemma 5.12 we have

$$(\mathcal{I}_P \circ \mathcal{F})^* \circ \mathcal{I}_P = \mathcal{I}_P \circ (\mathcal{F} \circ \mathcal{I}_P)^* \quad (17)$$

We list some other properties of projections:

$$\mathcal{I}_P \circ \mathcal{I}_Q = \mathcal{I}_{P \cap Q} \quad (18)$$

$$\mathcal{F} = \mathcal{I}_{\text{rg}(\mathcal{F})} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{I}_{\text{dom}(\mathcal{F})} \quad (19)$$

$$\mathcal{F}^m = \mathcal{I}_{\text{rg}(\mathcal{F})} \circ \mathcal{F}^m = \mathcal{F}^m \circ \mathcal{I}_{\text{dom}(\mathcal{F})} \quad (20)$$

11 Uniform left interpolation

With the axioms for weak lazy Kleene algebra and the properties of the projection and restriction operators given in Section 10, we now can prove uniform left interpolation for Horn logic. The idea is as follows. As a first approximation of the uniform left interpolant of $\mathcal{F}$ we take $\mathcal{F}^*$, which contains by Lemma 3 all consequences of $\mathcal{F}$. However, $\text{dom}(\mathcal{F}^*) = \text{rg}(\mathcal{F}^*) = \text{PR}$ so the parameter condition $(\text{dom}(\mathcal{J}), \text{rg}(\mathcal{J})) \subseteq (N, P)$ is not met. Therefore we take as interpolant $\mathcal{J} = (\mathcal{I}_P \circ \mathcal{F}^*) \upharpoonright N$, which meets the parameter condition (and even the Lyndon condition). It is also evident that $\mathcal{F} \models \mathcal{J}$, so it remains to show that $\mathcal{J} \sqcup \mathcal{G} \models \mathcal{H}$ for all $\mathcal{G}, \mathcal{H}$ meeting the requirements formulated in Definition 2. For that purpose, we prove Lemma 6 that tells us under which conditions certain occurrences of $\mathcal{F}$ can be replaced by $\mathcal{I}_P \circ \mathcal{F}^*$ or by $\mathcal{F}^* \circ \mathcal{I}_N$.

LEMMA 6

Let $X, Y, N, P \subseteq \text{PR}$. Then:

1. $\text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup X) \subseteq P$ implies $\mathcal{I}_X \circ (\mathcal{F} \sqcup \mathcal{G})^* = \mathcal{I}_X \circ (\mathcal{I}_P \circ \mathcal{F}^* \sqcup \mathcal{G})^*$;
2. $\text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup Y) \subseteq N$ implies $(\mathcal{F} \sqcup \mathcal{G})^* \circ \mathcal{I}_Y = (\mathcal{F}^* \circ \mathcal{I}_N \sqcup \mathcal{G})^* \circ \mathcal{I}_Y$.

PROOF. Because of monotonicity, it suffices to prove the equalities only for the case that P equals $\text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup X)$ and N equals $\text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup Y)$, i.e.

$$\mathcal{I}_X \circ (\mathcal{F} \sqcup \mathcal{G})^* = \mathcal{I}_X \circ (\mathcal{I}_{\text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup X)} \circ \mathcal{F}^* \sqcup \mathcal{G})^* \quad (21)$$

$$(\mathcal{F} \sqcup \mathcal{G})^* \circ \mathcal{I}_Y = (\mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup Y)} \sqcup \mathcal{G})^* \circ \mathcal{I}_Y \quad (22)$$

1. The $\supseteq$ part of (21) follows directly from monotonicity. To prove the $\subseteq$ part, we shall use

$$\mathcal{I}_X \circ \mathcal{F}^* = \mathcal{I}_X \circ (\mathcal{I}_{\text{rg}(\mathcal{F}) \cap X} \circ \mathcal{F}^*)^* \quad (23)$$

The $\supseteq$ part of (23) follows from Lemma 5.9 and monotonicity. Proof of the $\subseteq$ part:

$$\begin{aligned} & \mathcal{I}_X \circ \mathcal{F}^* \\ = & \{\text{Lemma 5.6}\} \\ & \mathcal{I}_X \circ (\mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{F}^*) \\ = & \{(20), (18), \text{idempotency and right distributivity of } \mathcal{I}_X\} \\ & \mathcal{I}_X \circ (\mathcal{I} \sqcup \mathcal{I}_{\text{rg}(\mathcal{F}) \cap X} \circ \mathcal{F}^m \circ \mathcal{F}^*) \\ \sqsubseteq & \{\text{monotonicity}\} \\ & \mathcal{I}_X \circ (\mathcal{I} \sqcup \mathcal{I}_{\text{rg}(\mathcal{F}) \cap X} \circ (\mathcal{I} \sqcup \mathcal{F}^m \circ \mathcal{F}^*)) \\ = & \{\text{Lemma 5.6}\} \\ & \mathcal{I}_X \circ (\mathcal{I} \sqcup \mathcal{I}_{\text{rg}(\mathcal{F}) \cap X} \circ \mathcal{F}^*) \\ \sqsubseteq & \{\text{Lemma 5.7}\} \\ & \mathcal{I}_X \circ (\mathcal{I}_{\text{rg}(\mathcal{F}) \cap X} \circ \mathcal{F}^*)^* \end{aligned}$$

Now we prove the $\sqsubseteq$ part of (21), using P to abbreviate $\text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup X)$.

$$\begin{aligned}
& \mathcal{I}_X \circ (\mathcal{F} \sqcup \mathcal{G})^* \\
= & \quad \{\text{Lemma 5.10}\} \\
& \mathcal{I}_X \circ \mathcal{F}^* \circ (\mathcal{G}^m \circ \mathcal{F}^*)^* \\
= & \quad \{(20)\} \\
& \mathcal{I}_X \circ \mathcal{F}^* \circ (\mathcal{G}^m \circ \mathcal{I}_{\text{dom}(\mathcal{G})} \circ \mathcal{F}^*)^* \\
= & \quad \{(23), \text{twice}\} \\
& \mathcal{I}_X \circ (\mathcal{I}_{\text{rg}(\mathcal{F}) \cap X} \circ \mathcal{F}^*)^* \circ (\mathcal{G}^m \circ \mathcal{I}_{\text{dom}(\mathcal{G})} \circ (\mathcal{I}_{\text{rg}(\mathcal{F}) \cap \text{dom}(\mathcal{G})} \circ \mathcal{F}^*)^*)^* \\
\sqsubseteq & \quad \{P = \text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup X), \text{monotonicity and (20)}\} \\
& \mathcal{I}_X \circ (\mathcal{I}_P \circ \mathcal{F}^*)^* \circ (\mathcal{G}^m \circ (\mathcal{I}_P \circ \mathcal{F}^*)^*)^* \\
= & \quad \{\text{Lemma 5.10}\} \\
& \mathcal{I}_X \circ (\mathcal{I}_P \circ \mathcal{F}^* \sqcup \mathcal{G})^*
\end{aligned}$$

2. To prove (22), we shall use

$$\mathcal{F}^* = \mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F})} \quad (24)$$

$$\mathcal{F}^* \circ \mathcal{I}_Y = (\mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F}) \cap Y})^* \circ \mathcal{I}_Y \quad (25)$$

The $\sqsupseteq$ part of (24) follows from $\mathcal{I} \sqsubseteq \mathcal{F}^*$ and $\mathcal{I}_{\text{dom}(\mathcal{F})} \sqsubseteq \mathcal{I}$, and the $\sqsubseteq$ part is proved as follows:

$$\begin{aligned}
& \mathcal{F}^* \sqsubseteq \mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F})} \\
\Leftarrow & \quad \{\text{Lemma 5.3 and (A6}^-\}\} \\
& \mathcal{I} \sqcup \mathcal{F}^m \circ (\mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F})}) \sqsubseteq \mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F})} \\
\Leftrightarrow & \quad \{(20)\} \\
& \mathcal{F}^m \circ \mathcal{I}_{\text{dom}(\mathcal{F})} \circ (\mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F})}) \sqsubseteq \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F})} \\
\Leftarrow & \quad \{(17)\} \\
& \mathcal{F}^m \circ (\mathcal{I} \sqcup \mathcal{I}_{\text{dom}(\mathcal{F})} \circ \mathcal{F}^*) \sqsubseteq \mathcal{F}^* \\
\Leftrightarrow & \quad \{\text{Lemma 5.8, } \mathcal{F}^m \sqsubseteq \mathcal{F}^*, \mathcal{I} \sqsubseteq \mathcal{F}^* \text{ and } \mathcal{I}_{\text{dom}(\mathcal{F})} \sqsubseteq \mathcal{I}\} \\
& \text{true}
\end{aligned}$$

The $\sqsupseteq$ part of (25) follows from Lemma 5.9, and the $\sqsubseteq$ part is proved as follows:

$$\begin{aligned}
& \mathcal{F}^* \circ \mathcal{I}_Y \\
= & \quad \{(24)\} \\
& (\mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F})}) \circ \mathcal{I}_Y \\
= & \quad \{(18)\} \\
& (\mathcal{I} \sqcup \mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F}) \cap Y}) \circ \mathcal{I}_Y \\
\sqsubseteq & \quad \{\text{Lemma 5.7 and monotonicity of } \circ\} \\
& (\mathcal{F}^* \circ \mathcal{I}_{\text{dom}(\mathcal{F}) \cap Y})^* \circ \mathcal{I}_Y
\end{aligned}$$

For the proof of (22) below, we abbreviate $\text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup Y)$ to N .

$$\begin{aligned}
 & (\mathcal{F} \sqcup \mathcal{G})^* \circ \mathcal{I}_Y \\
 = & \quad \{\text{Lemma 5.10}\} \\
 & \mathcal{F}^* \circ (\mathcal{G}^m \circ \mathcal{F}^*)^* \circ \mathcal{I}_Y \\
 = & \quad \{(20)\} \\
 & \mathcal{F}^* \circ (\mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y} \circ \mathcal{G}^m \circ \mathcal{F}^*)^* \circ \mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y} \circ \mathcal{I}_Y \\
 = & \quad \{(17)\} \\
 & \mathcal{F}^* \circ \mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y} \circ (\mathcal{G}^m \circ \mathcal{F}^* \circ \mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y})^* \circ \mathcal{I}_Y \\
 = & \quad \{(25), \text{twice}; N = \text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup Y)\} \\
 & (\mathcal{F}^* \circ \mathcal{I}_N)^* \circ \mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y} \circ (\mathcal{G}^m \circ (\mathcal{F}^* \circ \mathcal{I}_N)^* \circ \mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y})^* \circ \mathcal{I}_Y \\
 = & \quad \{(17)\} \\
 & (\mathcal{F}^* \circ \mathcal{I}_N)^* \circ (\mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y} \circ \mathcal{G}^m \circ (\mathcal{F}^* \circ \mathcal{I}_N)^*)^* \circ \mathcal{I}_{\text{rg}(\mathcal{G}) \cup Y} \circ \mathcal{I}_Y \\
 = & \quad \{(20)\} \\
 & (\mathcal{F}^* \circ \mathcal{I}_N)^* \circ (\mathcal{G}^m \circ (\mathcal{F}^* \circ \mathcal{I}_N)^*)^* \circ \mathcal{I}_Y \\
 = & \quad \{\text{Lemma 5.10}\} \\
 & ((\mathcal{F}^* \circ \mathcal{I}_N) \sqcup \mathcal{G})^* \circ \mathcal{I}_Y
 \end{aligned}$$

□

Now we are ready to prove uniform left interpolation.

THEOREM 2

Let $(N, P) \subseteq \text{par}(\mathcal{F})$. Then

$$\mathcal{J} = (\mathcal{I}_P \circ \mathcal{F}^*) \upharpoonright N$$

is a uniform left interpolant for $\mathcal{F}$ and (N, P) . So $\text{par}(\mathcal{J}) \subseteq (N, P)$ and for all $\mathcal{G}, \mathcal{H}$ we have

$$\text{if } \text{par}(\mathcal{F}) \cap (\text{par}^{-1}(\mathcal{G}) \cup \text{par}(\mathcal{H})) \subseteq (N, P) \text{ then } \mathcal{F} \sqcup \mathcal{G} \models \mathcal{H} \Leftrightarrow \mathcal{J} \sqcup \mathcal{G} \models \mathcal{H}.$$

PROOF. $\text{par}(\mathcal{J}) \subseteq (N, P)$ follows directly from the properties of projection and restriction mentioned in Section 10. Assume that $\text{par}(\mathcal{F}) \cap (\text{par}^{-1}(\mathcal{G}) \cup \text{par}(\mathcal{H})) \subseteq (N, P)$. This comes down to

$$\text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup \text{dom}(\mathcal{H})) \subseteq N \tag{26}$$

$$\text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup \text{rg}(\mathcal{H})) \subseteq P \tag{27}$$

We shall use the property

$$(\mathcal{I}_P \circ \mathcal{F}^*)^* = \mathcal{I} \sqcup \mathcal{I}_P \circ \mathcal{F}^* \tag{28}$$

The $\sqsubseteq$ direction is straightforward, and for the other direction we reason as follows:

$$\begin{aligned}
& (\mathcal{I}_P \circ \mathcal{F}^*)^* \sqsubseteq \mathcal{I} \sqcup \mathcal{I}_P \circ \mathcal{F}^* \\
\Leftarrow & \quad \{\text{Lemma 5.3 and (A6}^-\}) \\
& \mathcal{I} \sqcup \mathcal{I}_P \circ \mathcal{F}^* \circ (\mathcal{I} \sqcup \mathcal{I}_P \circ \mathcal{F}^*) \sqsubseteq \mathcal{I} \sqcup \mathcal{I}_P \circ \mathcal{F}^* \\
\Leftarrow & \quad \{\text{monotonicity}\} \\
& \mathcal{F}^* \circ (\mathcal{I} \sqcup \mathcal{I}_P \circ \mathcal{F}^*) \sqsubseteq \mathcal{F}^* \\
\Leftarrow & \quad \{\mathcal{I} \sqsubseteq \mathcal{F}^*, \mathcal{I}_P \sqsubseteq \mathcal{I}\} \\
& \mathcal{F}^* \circ \mathcal{F}^* \sqsubseteq \mathcal{F}^* \\
\Leftarrow & \quad \{\text{Lemma 5.8}\} \\
& \text{true}
\end{aligned}$$

Now we have

$$\begin{aligned}
& \mathcal{F} \sqcup \mathcal{G} \models \mathcal{H} \\
\Leftarrow & \quad \{\text{Lemma 3}\} \\
& \mathcal{H} \sqsubseteq (\mathcal{F} \sqcup \mathcal{G})^* \\
\Leftarrow & \quad \{(19)\} \\
& \mathcal{H} \sqsubseteq \mathcal{I}_{\text{rg}(\mathcal{H})} \circ (\mathcal{F} \sqcup \mathcal{G})^* \circ \mathcal{I}_{\text{dom}(\mathcal{H})} \\
\Leftarrow & \quad \{\text{Lemma 6.1 with } X := \text{rg}(\mathcal{H}), \text{ using (27)}\} \\
& \mathcal{H} \sqsubseteq \mathcal{I}_{\text{rg}(\mathcal{H})} \circ (\mathcal{I}_P \circ \mathcal{F}^* \sqcup \mathcal{G})^* \circ \mathcal{I}_{\text{dom}(\mathcal{H})} \\
\Leftarrow & \quad \{\text{Lemma 6.2 with } Y := \text{dom}(\mathcal{H}), \text{ using (26)}\} \\
& \mathcal{H} \sqsubseteq \mathcal{I}_{\text{rg}(\mathcal{H})} \circ ((\mathcal{I}_P \circ \mathcal{F}^*)^* \circ \mathcal{I}_N \sqcup \mathcal{G})^* \circ \mathcal{I}_{\text{dom}(\mathcal{H})} \\
\Leftarrow & \quad \{(19)\} \\
& \mathcal{H} \sqsubseteq ((\mathcal{I}_P \circ \mathcal{F}^*)^* \circ \mathcal{I}_N \sqcup \mathcal{G})^* \\
\Leftarrow & \quad \{(28)\} \\
& \mathcal{H} \sqsubseteq (\mathcal{I}_N \sqcup \mathcal{I}_P \circ \mathcal{F}^* \circ \mathcal{I}_N \sqcup \mathcal{G})^* \\
\Leftarrow & \quad \{(\mathcal{I}_N \sqcup \mathcal{K})^* = \mathcal{K}^* \text{ for all } \mathcal{K}\} \\
& \mathcal{H} \sqsubseteq (\mathcal{I}_P \circ \mathcal{F}^* \circ \mathcal{I}_N \sqcup \mathcal{G})^* \\
\Leftarrow & \quad \{(15), \text{ using that } \mathcal{I}_P \circ \mathcal{F}^* \circ \mathcal{I}_N \text{ is monotonic}\} \\
& \mathcal{H} \sqsubseteq (((\mathcal{I}_P \circ \mathcal{F}^*) \upharpoonright N)^m \sqcup \mathcal{G})^* \\
\Leftarrow & \quad \{\text{Lemma 5.8}\} \\
& \mathcal{H} \sqsubseteq ((\mathcal{I}_P \circ \mathcal{F}^*) \upharpoonright N \sqcup \mathcal{G})^* \\
\Leftarrow & \quad \{\text{definition of } \mathcal{J}\} \\
& \mathcal{H} \sqsubseteq (\mathcal{J} \sqcup \mathcal{G})^* \\
\Leftarrow & \quad \{\text{Lemma 3}\} \\
& \mathcal{J} \sqcup \mathcal{G} \models \mathcal{H}
\end{aligned}$$

□

REMARK

The uniform interpolant in Theorem 2 is defined by $\mathcal{J} = (\mathcal{I}_P \circ \mathcal{F}^*) \upharpoonright N$. This expression contains the domain restriction operator $\cdot \upharpoonright P$ which cannot be obtained via composition, as we showed in Section 10. This leads to the question: is it possible to define a uniform interpolant using only the operators in the (weak) lazy Kleene algebras $\mathbf{SF}(\mathbf{PR})$ and $\mathbf{SF}_m(\mathbf{PR})$, and not the restriction operator?

12 Adding the false proposition $\perp$

In this section, we let $\perp \in \text{PR}$ act as a false proposition by admitting only models $M \subseteq \text{PR}$ where $\perp$ is false, i.e. with $\perp \notin M$. The corresponding semantical entailment relation $\models_n$ is defined by

$$\mathcal{F} \models_n \mathcal{G} \text{ iff } \forall M \subseteq \text{PR} - \{\perp\} (M \models \mathcal{F} \Rightarrow M \models \mathcal{G})$$

We will transfer Theorem 2 to $\models_n$, via reduction of $\models_n$ to $\models$. For that purpose, we introduce the function $\mathcal{N} \in \mathbb{F}(\text{PR})$ which corresponds with the Horn formula $\bigwedge_{p \in \text{PR}} (\perp \rightarrow p)$: if $\perp$ holds then everything is true.

Here and in the proof of Theorem 3, we use the notation $(A \mapsto B)$ for the function in $\mathbb{F}(\text{PR})$ that maps $A \subseteq \text{PR}$ to $B \subseteq \text{PR}$, and any $C \subseteq \text{PR}$ different from A to $\emptyset$. Now $\mathcal{N}$ is defined by

$$\mathcal{N} = (\{\perp\} \mapsto \text{PR})^m \sqcup \mathcal{I}$$

So $\mathcal{N}(P) = \text{PR}$ if $\perp \in P$, otherwise $\mathcal{N}(P) = P$.

We also define $\mathcal{G}_{\mathcal{F}}$, the $\perp$ -modification of $\mathcal{G}$ with respect to $\mathcal{F}$, by

$$\begin{aligned} \mathcal{G}_{\mathcal{F}}(P) &= \{\perp\} && \text{if } P \in \text{Dom}(\mathcal{G}) \text{ and } \perp \in \mathcal{F}^*(P) \\ &= \mathcal{G}(P) && \text{otherwise} \end{aligned}$$

Now we have a reduction of $\models_n$ to $\models$.

LEMMA 7

$$\mathcal{F} \models_n \mathcal{G} \Leftrightarrow \mathcal{F} \models \mathcal{G}_{\mathcal{F}}$$

PROOF. Follows from Lemma 3 and the following properties of $\mathcal{N}$ and $\mathcal{G}_{\mathcal{F}}$:

$$\mathcal{F} \models_n \mathcal{G} \Leftrightarrow \mathcal{F} \sqcup \mathcal{N} \models \mathcal{G} \quad (29)$$

$$(\mathcal{F} \sqcup \mathcal{N})^* = \mathcal{N} \circ \mathcal{F}^* \quad (30)$$

$$\mathcal{G} \subseteq \mathcal{N} \circ \mathcal{F}^* \Leftrightarrow \mathcal{G}_{\mathcal{F}} \subseteq \mathcal{F}^* \quad (31)$$

Here (29) follows easily from $M \models \mathcal{N} \Leftrightarrow (\perp \notin M \text{ or } M = \text{PR})$. The $\supseteq$ direction of (30) follows from properties of $\circ$, while the $\subseteq$ direction is proved as follows.

$$\begin{aligned} &(\mathcal{F} \sqcup \mathcal{N})^* \subseteq \mathcal{N} \circ \mathcal{F}^* \\ \Leftrightarrow &\{\text{Lemma 5.3, (A7)}\} \\ &\mathcal{I} \sqcup \mathcal{F} \circ \mathcal{N} \circ \mathcal{F}^* \sqcup \mathcal{N} \circ \mathcal{N} \circ \mathcal{F}^* \subseteq \mathcal{N} \circ \mathcal{F}^* \\ \Leftrightarrow &\{\mathcal{I} \subseteq \mathcal{N} \text{ and } \mathcal{N} \circ \mathcal{N} = \mathcal{N}\} \\ &\mathcal{F} \circ \mathcal{N} \circ \mathcal{F}^* \subseteq \mathcal{N} \circ \mathcal{F}^* \\ \Leftrightarrow &\{\text{definition of } \subseteq\} \\ &\forall P \mathcal{F}(\mathcal{N}(\mathcal{F}^*(P))) \subseteq \mathcal{N}(\mathcal{F}^*(P)) \\ \Leftrightarrow &\{\text{definition of } \mathcal{N}\} \\ &\forall P (\perp \notin \mathcal{F}^*(P) \Rightarrow \mathcal{F}(\mathcal{F}^*(P)) \subseteq \mathcal{F}^*(P)) \\ \Leftrightarrow &\{\mathcal{F} \circ \mathcal{F}^* \subseteq \mathcal{F}^*\} \\ &\text{true} \end{aligned}$$

Finally, (31) follows directly from the definition of $\mathcal{N}$ and $\mathcal{G}_{\mathcal{F}}$. □

We now shall establish uniform left interpolation for $\models_n$. We consider $\perp$ as a constant, not belonging to the parameters of a formula. So par_n is defined by

$$\text{par}_n(\mathcal{F}) = \text{par}(\mathcal{F}) - (\{\perp\}, \{\perp\})$$

It is easily checked that $\text{par}_n(\mathcal{G}_{\mathcal{F}}) \subseteq \text{par}(\mathcal{G})$.

THEOREM 3

Let $(N, P) \subseteq \text{par}_n(\mathcal{F})$. Then there is a uniform left interpolant $\mathcal{J}$ w.r.t. $\models_n$ for $\mathcal{F}$ and (N, P) , i.e. $\text{par}_n(\mathcal{J}) \subseteq (N, P)$ and for all $\mathcal{G}, \mathcal{H}$ with $\text{par}_n(\mathcal{F}) \cap (\text{par}_n^{-1}(\mathcal{G}) \cup \text{par}_n(\mathcal{H})) \subseteq (N, P)$ we have

$$\mathcal{F} \sqcup \mathcal{G} \models_n \mathcal{H} \Leftrightarrow \mathcal{J} \sqcup \mathcal{G} \models_n \mathcal{H}. \quad (32)$$

PROOF. Let $(N, P) \subseteq \text{par}_n(\mathcal{F})$. Define $\mathcal{F}' = \mathcal{F} \sqcup (\{\perp\} \mapsto \{\perp\})$, $N' = N \cup \{\perp\}$, $P' = P \cup \{\perp\}$. Then $(N', P') \subseteq \text{par}(\mathcal{F}')$. Let $\mathcal{J}$ be the interpolant provided by Theorem 2 for $\mathcal{F}'$, N' and P' . Then $\text{par}_n(\mathcal{J}) \subseteq (N, P)$. Now let $\mathcal{G}, \mathcal{H}$ be two functions with

$$\text{par}_n(\mathcal{F}) \cap (\text{par}_n^{-1}(\mathcal{G}) \cup \text{par}_n(\mathcal{H})) \subseteq (N, P) \quad (33)$$

We claim that (32) holds. By Lemma 7, it suffices to show

$$\mathcal{F} \sqcup \mathcal{G} \models \mathcal{H}_{\mathcal{F} \sqcup \mathcal{G}} \Leftrightarrow \mathcal{J} \sqcup \mathcal{G} \models \mathcal{H}_{\mathcal{J} \sqcup \mathcal{G}} \quad (34)$$

By uniform left interpolation (Theorem 2) for $\mathcal{F}'$, N' and P' we have

$$\mathcal{F}' \sqcup \mathcal{G} \models \mathcal{H}' \Leftrightarrow \mathcal{J} \sqcup \mathcal{G} \models \mathcal{H}' \quad (35)$$

for every $\mathcal{H}'$ with

$$\text{par}(\mathcal{F}') \cap (\text{par}^{-1}(\mathcal{G}) \cup \text{par}(\mathcal{H}')) \subseteq (N', P') \quad (36)$$

We observe that, by (33) and $\text{par}_n(\mathcal{H}_{\mathcal{F} \sqcup \mathcal{G}}) \subseteq \text{par}(\mathcal{H})$, $\mathcal{H}' := \mathcal{H}_{\mathcal{F} \sqcup \mathcal{G}}$ satisfies (36), so by (35) we get

$$\mathcal{F}' \sqcup \mathcal{G} \models \mathcal{H}_{\mathcal{F} \sqcup \mathcal{G}} \Leftrightarrow \mathcal{J} \sqcup \mathcal{G} \models \mathcal{H}_{\mathcal{F} \sqcup \mathcal{G}} \quad (37)$$

so in order to obtain (34) it suffices to show that $\mathcal{H}_{\mathcal{F} \sqcup \mathcal{G}} = \mathcal{H}_{\mathcal{J} \sqcup \mathcal{G}}$, which follows from $\perp \in (\mathcal{F} \sqcup \mathcal{G})^*(P) \Leftrightarrow \perp \in (\mathcal{J} \sqcup \mathcal{G})^*(P)$ for all $P \in \text{Dom}(\mathcal{H})$, i.e.

$$\mathcal{F} \sqcup \mathcal{G} \models (P \mapsto \{\perp\}) \Leftrightarrow \mathcal{J} \sqcup \mathcal{G} \models (P \mapsto \{\perp\})$$

This is a consequence of (35) with $\mathcal{H}' := (P \mapsto \{\perp\})$, since $\text{par}_n(P \mapsto \{\perp\}) \subseteq \text{par}_n(\mathcal{H})$ for all $P \in \text{Dom}(\mathcal{H})$. $\square$

13 Thin functions

As observed in Section 3, uniform interpolation leads to ordinary interpolation. So by Theorem 2 we have interpolation for propositional Horn logic: if $\mathcal{F} \sqcup \mathcal{G} \models \mathcal{H}$, we have an interpolant $\mathcal{J} = (\mathcal{I}_P \circ \mathcal{F}^*) \upharpoonright N$ with $P = \text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup \text{rg}(\mathcal{H}))$ and $N = \text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup \text{dom}(\mathcal{H}))$. The interpolant $\mathcal{J}$ is the restriction of a monotonic function: as a consequence of (3), its size is in general exponential in the size of the largest $Q \subseteq N$ with $N - Q \in \text{Dom}(\mathcal{J})$, i.e. with $\mathcal{J}(N - Q) \neq \emptyset$.

In the next section, we shall show that propositional Horn logic satisfies polynomial interpolation. For that purpose, we introduce an interesting subclass of functions with size polynomial in the number of elementary propositions occurring in it. We say that $\mathcal{F} \in \mathbb{F}(\text{PR})$ is *thin* when

- P and $\mathcal{F}(P)$ are disjoint, for every $P \subseteq \text{PR}$;
- for every $p \in \text{rg}(\mathcal{F})$, there is *exactly one* $Q \subseteq \text{PR}$ with $p \in \mathcal{F}(Q)$.

In fact, the first condition is not essential and only added for aesthetic reasons. We observe that thin functions correspond with Horn formulae consisting of nontrivial clauses with unique heads. It is straightforward to check that

$$\mathcal{F} \text{ thin} \Rightarrow \|\mathcal{F}\| \leq \#rg(\mathcal{F}) \cdot (\#dom(\mathcal{F}) + 1) \quad (38)$$

We shall use thin functions for the definition of polynomial interpolants. For that purpose, we introduce a thinning operator τ that transforms any $\mathcal{F} \in \mathbb{F}$ into a thin function $\tau(\mathcal{F})$ that is contained in $\mathcal{F}$ and has the same fixpoint.

The idea behind τ is the following property of Horn formulae φ :

when $\varphi \models p$, there is a subformula φ' of φ with unique heads of clauses such that $\varphi' \models p$.

Intuitively, this can be seen as follows. Consider the steps in a derivation of $\varphi \vdash p$ in some derivation system and mark, whenever an atom q is entailed for the first time, the clause with head q that was used in that step. Then compose the subformula from the clauses that are marked.

To transfer this idea to set-valued functions, recall that $\mu(\mathcal{F}) = \bigcup \{(\mathcal{F})_\alpha \mid \alpha \text{ an ordinal}\}$ where $((\mathcal{F})_\alpha \mid \alpha \text{ an ordinal})$ is a chain with $(\mathcal{F})_{\alpha+1} = \mathcal{F}^m((\mathcal{F})_\alpha)$. To obtain $\mu(\tau(\mathcal{F})) = \mu(\mathcal{F})$, we want $\tau(\mathcal{F})$ to be a minimal function such that $(\tau(\mathcal{F}))_\alpha = (\mathcal{F})_\alpha$ for all ordinals α . So, for each $p \in (\mathcal{F})_{\alpha+1} - (\mathcal{F})_\alpha$ there has to be a $P \subseteq (\mathcal{F})_\alpha$ such that $p \in \tau(\mathcal{F})(P)$, and this P should be unique to guarantee thinness. To find this P , we use a choice function to select P from the candidate sets $Q \subseteq (\mathcal{F})_\alpha$ with $p \in \mathcal{F}(Q)$.

DEFINITION 3

Let $\chi : \wp^+(\wp(\mathbf{PR})) \rightarrow \wp(\mathbf{PR})$ be a choice function that satisfies $\chi(P) \in P$ for every nonempty $P \subseteq \wp(\mathbf{PR})$. With help of χ , the *thinning operator* $\tau : \mathbb{F}(\mathbf{PR}) \rightarrow \mathbb{F}(\mathbf{PR})$ is defined by

$$\tau(\mathcal{F})(P) = \{p \mid \exists \alpha (p \in (\mathcal{F})_{\alpha+1} - (\mathcal{F})_\alpha \ \& \ P = \chi(\{Q \subseteq (\mathcal{F})_\alpha \mid p \in \mathcal{F}(Q)\})\}$$

where α ranges over the ordinals.

Observe that $\tau(\mathcal{F})$ is well-defined, for $p \in (\mathcal{F})_{\alpha+1} - (\mathcal{F})_\alpha$ implies that $p \in \mathcal{F}(Q)$ for some $Q \subseteq (\mathcal{F})_\alpha$, so the argument $\{Q \subseteq (\mathcal{F})_\alpha \mid p \in \mathcal{F}(Q)\}$ of χ is a nonempty set. We have

LEMMA 8

1. $\tau(\mathcal{F}) \sqsubseteq \mathcal{F}$;
2. $\tau(\mathcal{F})$ is thin;
3. $\mu(\tau(\mathcal{F})) = \mu(\mathcal{F})$.

PROOF.

1. We observe

$$p \in \tau(\mathcal{F})(P) \Rightarrow p \in \mathcal{F}(P) \ \& \ \exists \alpha (p \in (\mathcal{F})_{\alpha+1} - (\mathcal{F})_\alpha \ \& \ P \subseteq (\mathcal{F})_\alpha) \quad (39)$$

which directly implies $\tau(\mathcal{F}) \sqsubseteq \mathcal{F}$.

2. Evidently, (39) implies $P \cap \tau(\mathcal{F})(P) = \emptyset$. Moreover, the definition of $\tau(\mathcal{F})$ entails that, given $p \in \mathbf{PR}$, there is at most one $Q \subseteq \mathbf{PR}$ with $p \in \tau(\mathcal{F})(Q)$.

3. This follows from $(\mathcal{F})_\alpha = (\tau(\mathcal{F}))_\alpha$ for all ordinals α . The inclusion $(\tau(\mathcal{F}))_\alpha \subseteq (\mathcal{F})_\alpha$ follows from part 1 of this Lemma. We demonstrate the other inclusion $(\mathcal{F})_\alpha \subseteq (\tau(\mathcal{F}))_\alpha$ with induction over α .

$\alpha = 0$: evident, since $(\mathcal{F})_0 = \emptyset$.

$\alpha = \beta + 1$: let $p \in (\mathcal{F})_\alpha = (\mathcal{F})_{\beta+1}$. When $p \in (\mathcal{F})_\beta$, then $p \in (\tau(\mathcal{F}))_\beta$ by induction hypothesis, so $p \in (\tau(\mathcal{F}))_{\beta+1} = (\tau(\mathcal{F}))_\alpha$. When $p \notin (\mathcal{F})_\beta$, we have $p \in (\mathcal{F})_{\beta+1} - (\mathcal{F})_\beta$, so $p \in \tau(\mathcal{F})(P)$ with $P = \chi(\{Q \subseteq (\mathcal{F})_\beta \mid p \in \mathcal{F}(Q)\})$. This implies $P \subseteq (\mathcal{F})_\beta$, so by induction hypothesis $P \subseteq (\tau(\mathcal{F}))_\beta$, hence $p \in \tau(\mathcal{F})^m((\tau(\mathcal{F}))_\beta) = (\tau(\mathcal{F}))_{\beta+1} = (\tau(\mathcal{F}))_\alpha$. So indeed $(\mathcal{F})_\alpha \subseteq (\tau(\mathcal{F}))_\alpha$.

α a limit ordinal: now the induction hypothesis yields $(\mathcal{F})_\beta \subseteq (\tau(\mathcal{F}))_\beta$ for all $\beta < \alpha$, which implies $(\mathcal{F})_\alpha = \bigcup_{\beta < \alpha} (\mathcal{F})_\beta \subseteq \bigcup_{\beta < \alpha} (\tau(\mathcal{F}))_\beta = (\tau(\mathcal{F}))_\alpha$. $\square$

14 Polynomial interpolation

To obtain a polynomial interpolant, we define two functions derived from τ .

$$\tau_{\mathcal{G}}(\mathcal{F}) = \tau(\mathcal{F} \sqcup \mathcal{G}) \sqcap \mathcal{F}$$

$$\tau_{\mathcal{G}}^{\mathcal{H}}(\mathcal{F}) = \bigsqcup_{P \in \text{Dom}(\mathcal{H})} \tau_{\mathcal{G} \sqcup \bar{P}}(\mathcal{F})$$

$\tau_{\mathcal{G}}$ and $\tau_{\mathcal{G}}^{\mathcal{H}}$ have the following properties.

LEMMA 9

1. $\tau_{\mathcal{G}}(\mathcal{F}) \subseteq \mathcal{F}$, $\tau_{\mathcal{G}}^{\mathcal{H}}(\mathcal{F}) \subseteq \mathcal{F}$ and $\tau_{\mathcal{G}}(\mathcal{F})$ is thin
2. $\mu(\tau_{\mathcal{G}}(\mathcal{F}) \sqcup \mathcal{G}) = \mu(\mathcal{F} \sqcup \mathcal{G})$
3. $\|\tau_{\mathcal{G}}(\mathcal{F})\| \leq \#\text{rg}(\mathcal{F}) \cdot (\#\text{dom}(\mathcal{F}) + 1)$
4. $\|\tau_{\mathcal{G}}^{\mathcal{H}}(\mathcal{F})\| \leq \#\text{rg}(\mathcal{F}) \cdot (\#\text{dom}(\mathcal{F}) + 1) \cdot \#\text{Dom}(\mathcal{H})$

PROOF.

1. Follows from the definition of $\tau_{\mathcal{G}}$ and the fact that thinness is preserved by subfunctions: if $\mathcal{F}$ is thin and $\mathcal{F}' \subseteq \mathcal{F}$ then $\mathcal{F}'$ is thin.
2. $\mu(\tau_{\mathcal{G}}(\mathcal{F}) \sqcup \mathcal{G}) \subseteq \mu(\mathcal{F} \sqcup \mathcal{G})$ follows from $\tau_{\mathcal{G}}(\mathcal{F}) \subseteq \mathcal{F}$ (part 1 of this Lemma) and the monotonicity of μ . For the other inclusion $\mu(\mathcal{F} \sqcup \mathcal{G}) \subseteq \mu(\tau_{\mathcal{G}}(\mathcal{F}) \sqcup \mathcal{G})$ we reason as follows. As a consequence of $\tau(\mathcal{F} \sqcup \mathcal{G}) \subseteq \mathcal{F} \sqcup \mathcal{G}$ (Lemma 8.1), we have $\tau(\mathcal{F} \sqcup \mathcal{G}) \subseteq (\tau(\mathcal{F} \sqcup \mathcal{G}) \sqcap \mathcal{F}) \sqcup \mathcal{G} = \tau_{\mathcal{G}}(\mathcal{F}) \sqcup \mathcal{G}$, so (monotonicity of μ) $\mu(\tau(\mathcal{F} \sqcup \mathcal{G})) \subseteq \mu(\tau_{\mathcal{G}}(\mathcal{F}) \sqcup \mathcal{G})$. Now the result follows from $\mu(\tau(\mathcal{F} \sqcup \mathcal{G})) = \mu(\mathcal{F} \sqcup \mathcal{G})$ (Lemma 8.3).
3. Follows from (38) and part 1 of this Lemma.
4. Follows from the definition of $\tau_{\mathcal{G}}^{\mathcal{H}}$ and part 3 of this Lemma. $\square$

THEOREM 4

Horn logic has polynomial interpolation.

PROOF. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be given with $\mathcal{F} \sqcup \mathcal{G} \models \mathcal{H}$. We define $P = \text{rg}(\mathcal{F}) \cap (\text{dom}(\mathcal{G}) \cup \text{rg}(\mathcal{H}))$ and $N = \text{dom}(\mathcal{F}) \cap (\text{rg}(\mathcal{G}) \cup \text{dom}(\mathcal{H}))$. Theorem 2 yields the interpolant $\mathcal{I} = (\mathcal{I}_P \circ \mathcal{F}^*) \upharpoonright N$ with

$\mathcal{F} \models \mathcal{J}$ and $\mathcal{J} \sqcup \mathcal{G} \models \mathcal{H}$. We claim that $\mathcal{J}_p = \tau_{\mathcal{G}}^{\mathcal{H}}(\mathcal{J})$ is an interpolant, too. $\mathcal{F} \models \mathcal{J}_p$ follows directly from $\mathcal{J}_p \sqsubseteq \mathcal{J}$. Furthermore

$$\begin{aligned}
& \mathcal{F} \sqcup \mathcal{G} \models \mathcal{H} \\
& \Leftrightarrow \{\text{Lemma 3}\} \\
& \mathcal{H} \sqsubseteq (\mathcal{F} \sqcup \mathcal{G})^* \\
& \Leftrightarrow \{\text{definition of } \sqsubseteq\} \\
& \forall P \in \text{Dom}(\mathcal{H}) \mathcal{H}(P) \subseteq (\mathcal{F} \sqcup \mathcal{G})^*(P) \\
& \Leftrightarrow \{\text{Lemma 2}\} \\
& \forall P \in \text{Dom}(\mathcal{H}) \mathcal{H}(P) \subseteq \mu(\mathcal{F} \sqcup \mathcal{G} \sqcup \underline{P}) \\
& \Leftrightarrow \{\text{Lemma 9.2}\} \\
& \forall P \in \text{Dom}(\mathcal{H}) \mathcal{H}(P) \subseteq \mu(\tau_{\mathcal{G} \sqcup \underline{P}}(\mathcal{F}) \sqcup \mathcal{G} \sqcup \underline{P}) \\
& \Rightarrow \{\text{monotonicity of } \mu\} \\
& \forall P \in \text{Dom}(\mathcal{H}) \mathcal{H}(P) \subseteq \mu(\bigsqcup_{Q \in \text{Dom}(\mathcal{H})} \tau_{\mathcal{G} \sqcup \underline{Q}}(\mathcal{F}) \sqcup \mathcal{G} \sqcup \underline{P}) \\
& \Leftrightarrow \{\text{Lemma 2}\} \\
& \forall P \in \text{Dom}(\mathcal{H}) \mathcal{H}(P) \subseteq (\bigsqcup_{Q \in \text{Dom}(\mathcal{H})} \tau_{\mathcal{G} \sqcup \underline{Q}}(\mathcal{F}) \sqcup \mathcal{G})^*(P) \\
& \Leftrightarrow \{\text{definition of } \sqsubseteq\} \\
& \mathcal{H} \sqsubseteq (\bigsqcup_{Q \in \text{Dom}(\mathcal{H})} \tau_{\mathcal{G} \sqcup \underline{Q}}(\mathcal{F}) \sqcup \mathcal{G})^* \\
& \Leftrightarrow \{\text{definition of } \mathcal{J}_p\} \\
& \mathcal{H} \sqsubseteq (\mathcal{J}_p \sqcup \mathcal{G})^* \\
& \Leftrightarrow \{\text{Lemma 3}\} \\
& \mathcal{J}_p \sqcup \mathcal{G} \models \mathcal{H}
\end{aligned}$$

So $\mathcal{J}_p$ is indeed an interpolant. Moreover, we have

$$\|\mathcal{J}_p\| \leq \#\text{rg}(\mathcal{F}) \cdot (\#\text{dom}(\mathcal{F}) + 1) \cdot \#\text{Dom}(\mathcal{H}) \leq \|\mathcal{F}\|^2 \cdot \|\mathcal{H}\| \quad (40)$$

as a direct consequence of Lemma 9.4. Observe that $\mathcal{G}$ has no influence on the size of the interpolant. $\square$

As we observed in Section 5, the size $\|\mathcal{F}\|$ of a function does not correspond exactly with the size of the Horn formula $\text{horn}(\mathcal{F})$, but with the size of the extended Horn formula $\text{horn}^+(\mathcal{F})$. In the context of extended Horn formulae, (40) translates to the following upper bound for an interpolant θ of $\varphi, \chi \models \psi$:

$$\|\theta\| \leq \#\text{pos}(\varphi) \cdot (\#\text{neg}(\varphi) + 1) \cdot \#\text{Cl}(\psi) \leq \|\varphi\|^2 \cdot \|\psi\| \quad (41)$$

Here $\text{Cl}(\psi)$ denotes the collection of clauses of ψ . The following example from [16] shows that, for ordinary Horn formulae, $\|\varphi\|^2 \cdot \#\text{Cl}(\psi)$ is a tight upper bound for the size of an interpolant.

EXAMPLE 2

Let I be an index set with $\#I = \kappa$, and let $P = \{p_i | i \in I\}$, $Q = \{q_i | i \in I\}$ and $S = \{s_i | i \in I\}$ be disjoint subsets of PR with $r, t \in \text{PR}$ in none of them. Define φ, χ and ψ by

$$\begin{aligned}
\varphi &= \bigwedge_{i \in I} (p_i \rightarrow q_i) \wedge ((\bigwedge_{i \in I} q_i) \rightarrow r) \wedge \bigwedge_{i \in I} (r \rightarrow s_i) \\
\chi &= (\bigwedge_{i \in I} s_i) \rightarrow t \\
\psi &= \bigwedge_{i \in I} \left(\left(\left(\bigwedge_{j \in I - \{i\}} p_j \right) \wedge q_i \right) \rightarrow t \right)
\end{aligned}$$

Then $\varphi, \chi \models \psi$, $\|\varphi\| = 5\kappa + 1$ and $\#\text{Cl}(\psi) = \kappa$. The minimal Horn interpolant for $\varphi, \chi \models \psi$ equals

$$\theta = \bigwedge_{k \in I} \bigwedge_{i \in I} \left(\left(\left(\bigwedge_{j \in I - \{i\}} p_j \right) \wedge q_i \right) \rightarrow s_k \right)$$

We have that $\|\theta\| = \kappa^3 + \kappa$ which is in $\Theta(\kappa^3) = \Theta(\|\varphi\|^2 \cdot \#\text{Cl}(\psi))$.

Observe that this does not apply for extended Horn formulae, for then the minimal interpolant is

$$\theta' = \bigwedge_{i \in I} \left(\left(\left(\bigwedge_{j \in I - \{i\}} p_j \right) \wedge q_i \right) \rightarrow \left(\bigwedge_{k \in I} s_k \right) \right)$$

with $\|\theta'\| = 2\kappa^2$.

We end this section with an open question related to an alternative complexity perspective on interpolation. Consider the interpolant $\mathcal{J} = (\mathcal{I}_p \circ \mathcal{F}^*) \upharpoonright N$ defined in Theorem 2 as the specification of a computation. What is, in the finite case, the complexity of this computation? Does this complexity match the size of the corresponding Horn formula?

15 Concluding remarks

We developed a functional representation of propositional Horn formulae, axiomatized by a weakening of lazy Kleene algebra. It enabled us to prove uniform left interpolation via some subtle reasoning with the axioms of weak lazy Kleene algebra and properties of projection and restriction. This result was transferred to propositional Horn logic with $\perp$. Using the Axiom of Choice, we managed to reduce the size of interpolants, resulting in polynomial interpolation for propositional Horn logic.

Several generalisations come to mind. A first idea is to consider *universal* Horn logic, where the elementary propositions (the elements of PR) are atomic predicate formulae $p(\vec{t})$ where $\vec{t}$ is a sequence of terms. Universal quantification is implicit: the formulae in universal Horn logic are to be read universally quantified. In other words: a formula φ entails all its substitution instances $\sigma(\varphi)$. So for this generalisation, the representation developed in Section 5 has to be extended with substitutions. An interesting sublogic of universal Horn logic is conditional equational logic, where all atomic propositions are equations $s = t$. In fact, the desire to extend the interpolation result in [5] for equational logic to conditional equational logic gave the impetus to develop the theory presented in this paper.

Another direction for generalisation concerns propositional logics. The idea is to take propositional formulae $A, B, \dots$ as elements of PR, and to consider sequents $A_1, A_2, \dots \vdash B$ as Horn formulae. Then entailments $\bigwedge_{i \in I} \varphi_i \models \psi$ between Horn formulae correspond with proof rules like *if $\Gamma_i \vdash A_i$ for all $i \in I$ then $\Gamma \vdash A$* . An interesting challenge is to use this approach to find proof-theoretic proofs for results on admissible rules as obtained e.g. in [8].

Finally we observe that we considered a notion of derivability in Section 8, but not the notion of derivation or proof as a tree-like structure. Intuitively, there is a relation between proof trees that prove a Horn formula $\text{horn}(\mathcal{F})$ and set-valued functions $\mathcal{G}$ with $\mathcal{G} \models \mathcal{F}$ (i.e. $\mathcal{F} \subseteq \mathcal{G}^*$) that deserves to be investigated further. A related question is whether the existence of polynomial interpolants relates to polynomial-size proofs.

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