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Stability Analysis of a Nonlinear Opinion Dynamics Model for Biased Assimilation

Weiguo Xia, Ji Liu, Ming Cao, and Xi-Ming Sun

Abstract—This paper studies the stability of equilibria of a nonlinear opinion dynamics model proposed in [1] for biased assimilation, which generalizes the DeGroot model by introducing a bias parameter. When the bias parameter is zero, the model reduces to the original DeGroot model. A positive value of this parameter reflects the degree of how biased an agent is. The opinions of the agents lie between 0 and 1. When the bias parameter is positive, it is shown that the equilibria with all elements equal identically to the extreme value 0 or 1 is locally stable, while the equilibrium with all elements equal to the intermediate consensus value $\frac{1}{2}$ is unstable. For the equilibrium consisting of both extreme values 0 and 1, which corresponds to opinion polarization according to the model, it is shown that the equilibrium is locally stable if the bias parameter is greater than one for two-island networks, becomes unstable if the bias parameter is less than one, and its stability heavily depends on the network topology when the parameter equals one, in which case the limiting behavior of the model is established for certain initial conditions. It is also shown that for a small negative bias parameter, with which the agents can be regarded as anti-biased, the equilibrium with all elements equal to $\frac{1}{2}$ is locally stable.

I. INTRODUCTION

There has been a persistent interest in theoretical sociology over the last decades in the study of opinion formation processes [2]. Various models have been proposed to understand how the opinions of an interconnected social group evolve and may reach a consensus, polarize, or split into multiple segregated subgroups. The well-known DeGroot model [3] has been proposed to explain how consensus is reached; in this model, each agent repeatedly updates its opinion to a weighted average of those of its neighbors, reflecting the human cognitive capability of taking convex combinations when processing new information [4]. The states of the individuals reach a consensus as long as the network characterizing the interrelationship among individuals is connected in some way.

Along this research line, a couple of variants of the DeGroot model have been established and studied extensively [5]–[11]. The Hegselmann-Krause model [6], [12]–[14] introduces a confidence bound quantized by a scalar for each individual to capture the social psychological phenomenon

that individuals tend to interact more with like-minded individuals. Consequently, each individual only assimilates the opinions of those whose opinions are within his confidence bound. The network becomes state-dependent and consensus, polarization, and a fragmentation of the patterns of opinions could be a result of the opinion evolution depending on the value of confidence bound. Another popular theory, called structural balance theory, uses signed graphs to describe the trust and mistrust relationships among the interacting social agents; the theory plays a major role in studying the dynamical process of opinion separation. This theory combined with a variant of the DeGroot opinion updating rule can also lead to several different results of opinion evolution [7], [15]–[18].

A new way to generalize the DeGroot model accounts for the biased assimilation phenomenon in social psychology [1]. The biased assimilation phenomenon indicates that individuals tend to process new information with a bias and consequently they tend to accept confirming evidence while evaluating disconfirming evidence critically. A bias parameter is introduced and is the key factor for opinion polarization. For a model of homophilous networks, it has been shown that under some specific conditions, polarization arises if the individuals are sufficiently biased and consensus is reached under some specific initial condition for a small bias parameter close to zero [1]. However, the situation that the system converges to consensus is rarely observed for other initial conditions.

In this paper, we further examine the stability of the equilibria of the model proposed in [1] to gain more insight into dynamical behavior of evolving opinions with biased assimilation. The opinion of each individual takes a value between 0 and 1 and the bias parameter is represented by a scalar b . We show that the equilibria with all elements equal identically to the extreme values 0 or 1 is locally stable under a connected graph when $b > 0$. The equilibrium with all elements equal to an intermediate consensus value $\frac{1}{2}$ is unstable when $b > 0$, while it is locally stable for a negative b close to 0. For the equilibrium consisting of both extreme values, which corresponds to opinion polarization of the system, it is shown that if the bias parameter is greater than 1, then this equilibrium is locally stable under a two-island network, while it is unstable if the bias parameter is less than 1. For the case $b = 1$, the equilibrium consisting of both extreme values is unstable under a complete graph or a star graph. In this case, we also establish the limiting behavior of the system for some special initial conditions. In addition, the equilibria with all-ones or all-zeros are the

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only stable equilibria of the model under a star graph.

The rest of the paper is organized as follows. Section II introduces the biased opinion dynamics model. Section III analyzes the equilibria and their stability of the model under a general graph topology and discusses some classes of specific graphs with the proofs given in Section IV. Section V performs several simulations to show a rich set of dynamic behaviors of the model in different graph topologies. Section VI concludes the paper.

Notation: For a positive integer N , let $\mathbf{1}_N$ and $\mathbf{0}_N$ denote the N -dimensional all-one vector and all-zero vector, respectively. Let $I_{N \times N}$ and $O_{N \times N}$ denote the $N \times N$ identity matrix and zero matrix, respectively. We will use the terms “individual” and “agent” interchangeably.

II. THE MODEL

Consider a group of N agents labeled by 1 to N . The neighbor relationships among the N agents are characterized by an N -node simple directed graph denoted by $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ is the node set and $\mathcal{E}$ is the edge set. The graph is associated with a weight matrix $W = (w_{ij})_{N \times N}$ where the self-weight $w_{ii} \geq 0$, and if $(i, j) \in \mathcal{E}$, then $w_{ji} > 0$. Let $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ be the set of neighbors of agent i representing the agents that have influence on i . Note that no self-loop is allowed in the simple graph $\mathbb{G}$ and therefore $i \notin \mathcal{N}_i$ for $i = 1, \dots, N$, but the self-weight w_{ii} can be positive.

Each agent i has a time-dependent, real-valued opinion $x_i(t) \in \mathbb{R}$ whose initial value $x_i(0)$ is assumed to be in the interval $[0, 1]$. Here 0 and 1 are extreme opinions that represent opposing points of view on one issue. The value $x_i(t)$ can be regarded as agent i 's degree of support at time t for the extreme opinion represented by 1, and so correspondingly $1 - x_i(t)$ can be regarded as the degree of support for the extreme opinion represented by 0 [1]. At each discrete time t , each agent i updates its opinion by setting

$$x_i(t+1) = \frac{w_{ii}x_i(t) + (x_i(t))^{b_i}s_i(t)}{w_{ii} + (x_i(t))^{b_i}s_i(t) + (1 - x_i(t))^{b_i}(d_i - s_i(t))}, \quad (1)$$

where $w_{ij}, 1 \leq i, j \leq N$, are the elements in the weight matrix W associated with $\mathbb{G}$ representing the influence weights, $d_i = \sum_{j \in \mathcal{N}_i} w_{ij}$, $s_i(t) = \sum_{j \in \mathcal{N}_i} w_{ij}x_j(t)$, and $b_i \geq 0$ is the bias parameter of agent i .

The dynamics (1) was first proposed in [1] to account for the biased assimilation phenomenon studied in sociology. Note that when $b_i = 0$ for all $i \in \mathcal{V}$, the dynamics (1) simplifies to the well-known DeGroot model [3], in which agent i updates its opinion at time t according to

$$x_i(t+1) = \frac{w_{ii}x_i(t) + \sum_{j \in \mathcal{N}_i} w_{ij}x_j(t)}{w_{ii} + \sum_{j \in \mathcal{N}_i} w_{ij}}.$$

Biased assimilation is well-known in social psychology [19], which shows that individuals usually process new information with bias in the sense that they are inclined to accept confirming evidence while evaluate disconfirming evidence critically. The model takes into account the biased

assimilation by introducing a bias parameter b_i with which each agent assimilates its neighbors' opinions. For detailed motivation and validation of the model, see [1].

In (1), agent i weights neighbor j 's support for 1 by an additional factor $(x_i(t))^{b_i}$ so that $x_i(t+1) \propto (x_i(t))^{b_i}w_{ij}x_j(t)$ and i weights j 's support for 0 by $(1 - x_i(t))^{b_i}$ so that $(1 - x_i(t+1)) \propto (1 - x_i(t))^{b_i}w_{ij}(1 - x_j(t))$.

Throughout this paper, we assume that $b_i = b$ for all $i = 1, \dots, N$, meaning that all agents have the same degree of bias. System (1) reduces to

$$x_i(t+1) = \frac{w_{ii}x_i(t) + (x_i(t))^b s_i(t)}{w_{ii} + (x_i(t))^b s_i(t) + (1 - x_i(t))^b (d_i - s_i(t))}. \quad (2)$$

We will study the equilibria and their stability of the system (2) for both $b > 0$ and $b < 0$. It turns out that this is a very challenging problem in general.

III. MAIN RESULTS

In this section, we present the main results of the paper.

For system (2) with $b > 0$, note that if the opinion of an agent is 1 at some time instant, then it will remain 1 from then on. Similarly, if the opinion of an agent is 0 at some time instant, then it will remain 0 from then on. It can be verified that for $b > 0$, we have that $\mathbf{0}_N$, $\mathbf{1}_N$, and $\frac{1}{2}\mathbf{1}_N$ are equilibria of system (2). Any vector whose elements are either zero or one is also an equilibrium; without loss of generality, we denote such an equilibrium as $[\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$ with $n_1 + n_2 = N$. Besides the above equilibria, there may exist other equilibria of the system depending on the graph $\mathbb{G}$, the value of the bias parameter b , and the weights w_{ij} . If the bias parameter $b < 0$ but close to 0, $\frac{1}{2}\mathbf{1}_N$ is an equilibrium of system (2).

Theorem 1: Suppose that the neighbor graph $\mathbb{G}$ is strongly connected¹ and $b > 0$. Then, $\mathbf{0}_N$ and $\mathbf{1}_N$ are locally asymptotic stable equilibria and $\frac{1}{2}\mathbf{1}_N$ is an unstable equilibrium of the system (2).

Theorem 2: Suppose that the neighbor graph $\mathbb{G}$ is strongly connected, $w_{ii} > 0$ for all $1 \leq i \leq N$, and $b < 0$ but close to 0. Then, $\frac{1}{2}\mathbf{1}_N$ is a locally asymptotic stable equilibrium of the system (2).

For some specific graph topology and some specific value of b , more results can be said about the stability of the equilibria.

Theorem 3: Let $b = 1$. For an undirected complete graph with weights $w_{ij} = 1, i \neq j, i, j = 1, \dots, N$, the equilibrium $[\mathbf{0}_n^T, \mathbf{1}_{N-n}^T]^T$ is unstable for any $1 < n < N$.

Theorem 4: Let $b = 1$. Consider an undirected star graph with the weight of each edge equal to 1 and $w_{ii} = 0, i = 1, \dots, N$. Without loss of generality, suppose that node 1 is the center node. The equilibria of system (2) with such a star graph include those vectors whose elements are either zero or one, and $[\frac{1}{2}, a_2, \dots, a_N]^T$ with $a_i \in [0, 1]$ and $\sum_{j=2}^N a_i = \frac{N-1}{2}$. If N is odd, the system has additional equilibria of the form $[c, \mathbf{0}_{\frac{N-1}{2}}^T, \mathbf{1}_{\frac{N-1}{2}}^T]^T$ with $c \in (0, 1)$. Among these equilibria, only $\frac{1}{2}\mathbf{1}_N$ and $\mathbf{0}_N$ are locally stable.

¹A directed graph is said to be strongly connected if there is a directed path from every node to every other node in this graph.

Next we introduce the two-island network model studied in [1], which is used to model a *homophilous* network. Consider an undirected network in which the nodes in $\mathcal{V}$ are partitioned into two types, say τ_1, τ_2 . Let $\mathcal{V}_i$ denote the set of nodes of type τ_i and $|\mathcal{V}_i| = n_i$, $i = 1, 2$. Without loss of generality, assume that $\mathcal{V}_1 = \{1, \dots, n_1\}$ and $\mathcal{V}_2 = \{n_1 + 1, \dots, N\}$. Assume that each node in $\mathcal{V}_1$ has $n_1 p_s$ neighbors in $\mathcal{V}_1$ and $n_1 p_d$ neighbors in $\mathcal{V}_2$, and each node in $\mathcal{V}_2$ has $n_2 p_s$ neighbors in $\mathcal{V}_2$ and $n_2 p_d$ neighbors in $\mathcal{V}_1$, where $p_s, p_d \in (0, 1)$ and $n_1 p_s, n_1 p_d, n_2 p_s, n_2 p_d$ are all integers.

Theorem 5: Suppose that the neighbor graph $\mathbb{G}$ is a two-island network and is strongly connected. Then, $[0_{n_1}^T, \mathbf{1}_{n_2}^T]^T$ is an unstable equilibrium when $0 < b < 1$, and is a stable equilibrium when $b \geq 1$.

For the limiting behavior of system (2), we have the following result with the system starting from some restricted initial conditions.

Theorem 6: Let $b = 1$. Suppose that the neighbor graph $\mathbb{G}$ is strongly connected.

- 1) If $x_i(0) \geq 0.5$ for all $1 \leq i \leq N$ and there exists at least one $1 \leq j \leq N$ such that $x_j(0) > 0.5$, then $x_i(t)$ will asymptotically converge to 1 for all $1 \leq i \leq N$.
- 2) If $x_i(0) \leq 0.5$ for all $1 \leq i \leq N$ and there exists at least one $1 \leq k \leq N$ such that $x_k(0) < 0.5$, then $x_i(t)$ will asymptotically converge to 0 for all $1 \leq i \leq N$.

IV. ANALYSIS

In this section, we prove Theorems 1-3 and 5 in the previous section and the proofs of Theorems 4 and 6 are omitted due to space limitation. Let the equilibria of system (2) be denoted by

$$\begin{aligned} x_a^* &= \mathbf{0}_N, & x_c^* &= \mathbf{1}_N, \\ x_d^* &= \frac{1}{2} \mathbf{1}_N, & x_e^* &= [0_{n_1}^T, \mathbf{1}_{n_2}^T]^T. \end{aligned}$$

We linearize the system (2) to discuss the local stability of the equilibria $x_a^*, x_c^*, x_d^*, x_e^*$.

Let

$$f_i(x(t)) \triangleq \frac{w_{ii}x_i(t) + (x_i(t))^b s_i(t)}{w_{ii} + (x_i(t))^b s_i(t) + (1 - x_i(t))^b (d_i - s_i(t))}.$$

The update of the opinions of all N individuals in the network is rewritten as follows

$$x(t+1) = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_N(t+1) \end{bmatrix} = \begin{bmatrix} f_1(x(t)) \\ f_2(x(t)) \\ \vdots \\ f_N(x(t)) \end{bmatrix} \triangleq F(x(t)). \quad (3)$$

The Jacobian matrix of the function $F(x(t)) = [f_1(x(t)), \dots, f_N(x(t))]^T$ is

$$\frac{\partial F}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \frac{\partial f_N}{\partial x_2} & \dots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}.$$

In the following, we neglect the argument t for brevity. Let

$$g_i(x) \triangleq w_{ii} + x_i^b s_i + (1 - x_i)^b (d_i - s_i),$$

for $i = 1, \dots, N$. By calculation, one can show that

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} &= \frac{1}{g_i^2(x)} \{ (w_{ii} + b x_i^{b-1} s_i) g_i(x) \\ &\quad - (w_{ii} x_i + x_i^b s_i) [b x_i^{b-1} s_i - b(1 - x_i)^{b-1} (d_i - s_i)] \} \end{aligned} \quad (4)$$

and

$$\begin{aligned} \frac{\partial f_i}{\partial x_k} &= \frac{1}{g_i^2(x)} \left\{ x_i^b w_{ik} g_i(x) \right. \\ &\quad \left. - (w_{ii} x_i + x_i^b s_i) (x_i^b w_{ik} - (1 - x_i)^b w_{ik}) \right\} \end{aligned} \quad (5)$$

for $k \neq i$ and $i, k = 1, \dots, N$.

Before proving the theorems, the following lemma is first introduced that will be used later.

Let $M = (m_{ij})_{N \times N}$ be a real matrix. M is called a *nonnegative* matrix if $m_{ij} \geq 0$, $i, j = 1, \dots, N$.

Lemma 1: ([20]) Suppose $M \in \mathbb{R}^{N \times N}$ and M is a nonnegative matrix. Then $\rho(M)$ is an eigenvalue of M and

$$\min_{1 \leq i \leq N} \sum_{j=1}^N m_{ij} \leq \rho(M) \leq \max_{1 \leq i \leq N} \sum_{j=1}^N m_{ij}. \quad (6)$$

Proof of Theorem 1: Consider the equilibrium $x_a^* = \mathbf{0}_N$. For all $i = 1, \dots, N$, one knows that $s_i = 0$ and

$$g_i(x_a^*) = w_{ii} + d_i.$$

It is easy to derive that

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} \Big|_{x=x_a^*} &= \frac{1}{g_i^2(x_a^*)} [w_{ii} g_i(x_a^*) - 0] = \frac{w_{ii}}{g_i(x_a^*)}, \\ \frac{\partial f_i}{\partial x_k} \Big|_{x=x_a^*} &= 0, \text{ for } k \neq i. \end{aligned}$$

Thus the Jacobian matrix at the equilibrium $x_a^* = \mathbf{0}_N$ becomes

$$P \triangleq \frac{\partial F}{\partial x} \Big|_{x=x_a^*} = \text{diag} \left\{ \frac{w_{11}}{g_1(x_a^*)}, \frac{w_{22}}{g_2(x_a^*)}, \dots, \frac{w_{NN}}{g_N(x_a^*)} \right\}.$$

Note that $g_i(x_a^*) = w_{ii} + \sum_{j \in \mathcal{N}_i} w_{ij}$. The eigenvalues of P are $\frac{w_{ii}}{g_i(x_a^*)}$, $i = 1, \dots, N$, which lie in the interval $[0, 1)$ as long as each agent has at least one neighbor. Since the graph is strongly connected, $\rho(P) < 1$ and thus the equilibrium $x_a^* = \mathbf{0}_N$ is locally stable.

For the equilibrium $x_c^* = \mathbf{1}_N$, it is true that for $i = 1, \dots, N$,

$$g_i(x_c^*) = w_{ii} + d_i.$$

and

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} \Big|_{x=x_c^*} &= \frac{1}{g_i^2(x_c^*)} [(w_{ii} + b d_i) g_i(x_c^*) - (w_{ii} + d_i) b d_i] \\ &= \frac{w_{ii}}{g_i(x_c^*)} \\ \frac{\partial f_i}{\partial x_k} \Big|_{x=x_c^*} &= \frac{1}{g_i^2(x_c^*)} [w_{ik} g_i(x_c^*) - w_{ik} (w_{ii} + d_i)] \\ &= 0, \end{aligned}$$

for $k \neq i$. Thus the Jacobian matrix at the equilibrium $x_c^* = \mathbf{1}_N$ becomes

$$P \triangleq \frac{\partial F}{\partial x} \Big|_{x=x_c^*} = \text{diag} \left\{ \frac{w_{11}}{g_1(x_c^*)}, \frac{w_{22}}{g_2(x_c^*)}, \dots, \frac{w_{NN}}{g_N(x_c^*)} \right\}.$$

The eigenvalues of P are $\frac{w_{ii}}{g_i(x_c^*)}$, $i = 1, \dots, N$, which lie in the interval $[0, 1)$ as in the previous case. Thus the equilibrium $x_c^* = \mathbf{1}_N^T$ is locally stable.

For the equilibrium $x_d^* = \frac{1}{2}\mathbf{1}_N$, one has that

$$g_i(x_d^*) = w_{ii} + \frac{1}{2^b}d_i,$$

for all $i = 1, \dots, N$. It can be further calculated that

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} \Big|_{x=x_d^*} &= \frac{1}{g_i^2(x_d^*)} \left[\left(w_{ii} + b \frac{1}{2^b} d_i \right) g_i(x_d^*) \right. \\ &\quad \left. - \frac{1}{2} \left(w_{ii} + \frac{1}{2^b} d_i \right) \left(b \frac{1}{2^b} d_i g_i(x_d^*) - b \frac{1}{2^b} d_i g_i(x_d^*) \right) \right] \\ &= \frac{w_{ii} + b \frac{1}{2^b} d_i}{g_i(x_d^*)}, \end{aligned}$$

and for $k \neq i$,

$$\begin{aligned} \frac{\partial f_i}{\partial x_k} \Big|_{x=x_d^*} &= \frac{1}{g_i^2(x_d^*)} \left[\frac{1}{2^b} w_{ik} g_i(x_d^*) \right. \\ &\quad \left. - \frac{1}{2} \left(w_{ii} + \frac{1}{2^b} d_i \right) \left(\frac{1}{2^b} w_{ik} - \frac{1}{2^b} w_{ik} \right) \right] \\ &= \frac{\frac{1}{2^b} w_{ik}}{g_i(x_d^*)}. \end{aligned}$$

Hence the Jacobian matrix at the equilibrium $x_d^* = \frac{1}{2}\mathbf{1}_N$ is

$$P \triangleq \frac{\partial F}{\partial x} \Big|_{x=x_d^*} = \begin{bmatrix} \frac{w_{11} + b \frac{1}{2^b} d_1}{g_1(x_d^*)} & \frac{\frac{1}{2^b} w_{12}}{g_1(x_d^*)} & \dots & \frac{\frac{1}{2^b} w_{1N}}{g_1(x_d^*)} \\ \frac{\frac{1}{2^b} w_{21}}{g_2(x_d^*)} & \frac{w_{22} + b \frac{1}{2^b} d_2}{g_2(x_d^*)} & \dots & \frac{\frac{1}{2^b} w_{2N}}{g_2(x_d^*)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\frac{1}{2^b} w_{N1}}{g_N(x_d^*)} & \frac{\frac{1}{2^b} w_{N2}}{g_N(x_d^*)} & \dots & \frac{w_{NN} + b \frac{1}{2^b} d_N}{g_N(x_d^*)} \end{bmatrix}. \quad (7)$$

Since P is a nonnegative matrix, it follows from Lemma 1 that the spectral radius $\rho(P)$ of P satisfies that

$$\begin{aligned} \rho(P) &\geq \min_{i=1, \dots, N} \sum_{j=1}^N p_{ij} \\ &= 1 + \min_{i=1, \dots, N} \frac{b \frac{1}{2^b} d_i}{w_{ii} + \frac{1}{2^b} d_i} > 1. \end{aligned} \quad (8)$$

Thus $x_d^* = \frac{1}{2}\mathbf{1}_N$ is an unstable equilibrium. $\square$

Proof of Theorem 2: Similar calculation to the proof of Theorem 1 shows that the Jacobian matrix at the equilibrium $x_d^* = \frac{1}{2}\mathbf{1}_N$ is the same as (7). The off-diagonal elements of

P is nonnegative. Since $w_{ii} > 0$ for all $i = 1, \dots, N$, for $b < 0$ and is close to 0, one has

$$\frac{w_{ii} + b \frac{1}{2^b} d_i}{g_i(x_d^*)} = \frac{w_{ii} + b \frac{1}{2^b} d_i}{w_{ii} + \frac{1}{2^b} d_i} \geq 0$$

for $1 \leq i \leq N$, and hence P is a nonnegative matrix. By Lemma 1, the spectral radius of P satisfies that

$$\rho(P) \leq \max_{i=1, \dots, N} \sum_{j=1}^N p_{ij} = 1 + \max_{i=1, \dots, N} \frac{b \frac{1}{2^b} \sum_{j \in \mathcal{N}_i} w_{ij}}{g_i(x_d^*)} < 1.$$

Therefore $x_d^* = \frac{1}{2}\mathbf{1}_N$ is a locally stable equilibrium of system (2) for $b < 0$ that is close to 0. $\square$

Before proving Theorems 3, we calculate the Jacobian matrix of system (2) at the equilibrium $x_e^* = [\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$.

Let x_{ei}^* be the i -th element of the equilibrium x_e^* . Let $\mathcal{N}_i^{(0)} = \{j : j \in \mathcal{N}_i, x_{ej}^* = 0\}$ and $\mathcal{N}_i^{(1)} = \{j : j \in \mathcal{N}_i, x_{ej}^* = 1\}$ for $i = 1, \dots, N$, and let

$$d_i^{(0)} = \sum_{j \in \mathcal{N}_i^{(0)}} w_{ij}, \quad d_i^{(1)} = \sum_{j \in \mathcal{N}_i^{(1)}} w_{ij}.$$

For agent i that satisfies $x_{ei}^* = 0$, it is easy to see that

$$g_i(x_e^*) = w_{ii} + d_i^{(0)}.$$

Simple calculation shows that

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} \Big|_{x=x_e^*} &= \frac{1}{g_i^2(x_e^*)} \left[(w_{ii} + x_{ei}^{*b-1} b d_i^{(1)}) g_i(x_e^*) \right. \\ &\quad \left. - x_{ei}^{*b} (x_{ei}^{*b-1} b d_i^{(1)} - b d_i^{(0)}) \right] \\ &= \frac{1}{g_i^2(x_e^*)} \left[(w_{ii} + x_{ei}^{*b-1} b d_i^{(1)}) g_i(x_e^*) \right. \\ &\quad \left. - x_{ei}^{*2b-1} b d_i^{(1)} \right]. \end{aligned}$$

When $0 < b < 1$,

$$\frac{\partial f_i}{\partial x_i} \Big|_{x=x_e^*} = \frac{w_{ii}}{g_i(x_e^*)} + \frac{x_{ei}^{*b-1} b d_i^{(1)}}{g_i^2(x_e^*)} (g_i(x_e^*) - x_{ei}^{*b}) = +\infty.$$

When $b = 1$,

$$\frac{\partial f_i}{\partial x_i} \Big|_{x=x_e^*} = \frac{w_{ii} + d_i^{(1)}}{w_{ii} + d_i^{(0)}}.$$

When $b > 1$,

$$\frac{\partial f_i}{\partial x_i} \Big|_{x=x_e^*} = \frac{w_{ii}}{g_i(x_e^*)}.$$

In addition, for $k \neq i$,

$$\frac{\partial f_i}{\partial x_k} \Big|_{x=x_e^*} = \frac{1}{g_i^2(x_e^*)} x_{ei}^{*b} = 0.$$

For agent i that satisfies $x_{ei}^* = 1$, one has

$$g_i(x_e^*) = w_{ii} + d_i^{(1)}.$$

Simple calculation shows that

$$\begin{aligned}\left.\frac{\partial f_i}{\partial x_i}\right|_{x=x_e^*} &= \frac{1}{g_i^2(x_e^*)} \left[(w_{ii} + bd_i^{(1)})g_i(x_e^*) \right. \\ &\quad \left. - (w_{ii} + d_i^{(1)})(bd_i^{(1)} - (1 - x_{ei}^*)^{b-1}bd_i^{(1)}) \right] \\ &= \frac{1}{g_i(x_e^*)} \left[w_{ii} + (1 - x_{ei}^*)^{b-1}bd_i^{(0)} \right].\end{aligned}$$

When $0 < b < 1$,

$$\left.\frac{\partial f_i}{\partial x_i}\right|_{x=x_e^*} = +\infty.$$

When $b = 1$,

$$\left.\frac{\partial f_i}{\partial x_i}\right|_{x=x_e^*} = \frac{w_{ii} + d_i^{(0)}}{w_{ii} + d_i^{(1)}}.$$

When $b > 1$,

$$\left.\frac{\partial f_i}{\partial x_i}\right|_{x=x_e^*} = \frac{w_{ii}}{g_i(x_e^*)}.$$

In addition, for $k \neq i$,

$$\left.\frac{\partial f_i}{\partial x_k}\right|_{x=x_e^*} = \frac{1}{g_i^2(x_e^*)} [w_{ik}g_i(x_e^*) - w_{ik}(w_{ii} + d_i^{(1)})] = 0.$$

From the above calculations, we can see that for $0 < b < 1$, the Jacobian matrix at the equilibrium $x_e^* = [\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$ is

$$P \triangleq \left.\frac{\partial F}{\partial x}\right|_{x=x_e^*} = \text{diag}\{+\infty, +\infty, \dots, +\infty\}, \quad (9)$$

which is unstable.

For $b = 1$, the Jacobian matrix at the equilibrium is

$$\begin{aligned}P &= \left.\frac{\partial F}{\partial x}\right|_{x=x_e^*} \\ &= \text{diag}\left\{\frac{w_{11} + d_1^{(1)}}{w_{11} + d_1^{(0)}}, \frac{w_{22} + d_2^{(1)}}{w_{22} + d_2^{(0)}}, \dots, \frac{w_{NN} + d_N^{(0)}}{w_{NN} + d_N^{(1)}}\right\}.\end{aligned} \quad (10)$$

For $b > 1$, the Jacobian matrix at the equilibrium is

$$\begin{aligned}P &= \left.\frac{\partial F}{\partial x}\right|_{x=x_e^*} \\ &= \text{diag}\left\{\frac{w_{11}}{w_{11} + d_1^{(0)}}, \frac{w_{22}}{w_{22} + d_2^{(0)}}, \dots, \frac{w_{NN}}{w_{NN} + d_N^{(1)}}\right\}.\end{aligned} \quad (11)$$

Now we are ready to prove Theorems 3.

Proof of Theorem 3: For the equilibrium $x_e^* = [\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$, suppose that $n_1 \geq n_2$. For the i -th agent with $x_{ei}^* = 1$, in view of (10), the i -th diagonal element of the matrix P is given by

$$p_{ii} = \frac{w_{ii} + d_i^{(0)}}{w_{ii} + d_i^{(1)}} = \frac{w_{ii} + n_1}{w_{ii} + n_2 - 1} > 1.$$

If $n_1 < n_2$, one can similarly show that there exists a diagonal element of P that is greater than 1. In both cases, the matrix P has an eigenvalue greater than 1. Therefore the equilibrium $x_e^* = [\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$ is unstable. $\square$

Proof of Theorem 5: For $0 < b < 1$, the conclusion that $[\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$ is an unstable equilibrium can be seen from (9). For $b = 1$, the following inequalities hold from the definition of the two-island network model

$$d_1^{(0)} > d_1^{(1)}, d_2^{(0)} > d_2^{(1)}, \dots, d_N^{(1)} > d_N^{(0)}. \quad (12)$$

Then the eigenvalues of P given in (10) lie in the interval $[0, 1)$ and thus the equilibrium $x_e^* = [\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$ is locally stable.

For $b > 1$, the Jacobian matrix at the equilibrium is given by (11). Thus for the two-island network, the eigenvalues of P lie in the interval $[0, 1)$. Hence the equilibrium $x_e^* = [\mathbf{0}_{n_1}^T, \mathbf{1}_{n_2}^T]^T$ is locally stable. $\square$

V. NUMERICAL SIMULATIONS

In this section, we perform several simulations to show the rich asymptotic behaviors of the system (2). In each of the following simulations, the graph consists of $N = 1000$ nodes and is undirected. The weight of each edge is 1, and the self-weight w_{ii} is set to 0 for all $1 \leq i \leq 1000$. As the evolution of the agents' opinions is affected by the time-varying environment disturbance in reality, random noise is added to the system (2) to observe its dynamic behavior.

We first consider the case $b > 1$ and take $b = 1.01$. Let the graph be a 40-regular graph. If the initial conditions of the agents are chosen uniformly randomly in the interval $[0, 1]$, the evolution of the states of the agents is illustrated in Fig. 1, from which one can see that the state of the system polarizes. For network topologies like ring graphs and two-island networks, similar simulation results can be observed, while for complete graphs and random graphs, all the agents' states finally evolve either close to 1 or 0.

Fig. 1. The system state under a 40-regular graph and noise with $b = 1.01$.

For the case $b = 1$ and a 40-regular graph, if the initial condition is randomly chosen in $[0, 1]$, then Fig. 2 illustrates that the states of most of the agents stay either close to 0 or 1 and the states of the remaining agents lie in between. For path graphs and two-island networks, similar results can be observed, while for complete graphs and random graphs, all the agents' states finally evolve either close to 1 or 0.

Fig. 2. The system state under a 40-regular graph and noise with $b = 1$ and a random initial condition chosen in $[0, 1]$.

Now consider the case $0 < b < 1$. Consider a two-island network model with each island consisting of 500 nodes. For each node, the number of neighbors of the same type is 40 and the number of neighbors of the other type is 20. When b is close to 1, for some specific initial condition, the states of the agents can converge to two different values whose sum is 1 under the noise-free case, but neither of them is the extreme value. For example, we consider $b = 0.8$ under the two-island network, and the initial states of agents 1 to 500 are randomly chosen from $[0.1, 0.15]$ and the remaining agents take values from $[0.85, 0.9]$. If there is no noise, then the system state converges to $[a\mathbf{1}_{500}^T, (1-a)\mathbf{1}_{500}^T]^T$ with some constant $a \in (0, \frac{1}{2})$. Under the influence of noise, Fig. 3 shows that the individuals' opinions evolve to two clusters with one cluster close to 0 and the other close to 1.

Fig. 3. The system state under a two-island network and noise with $b = 0.8$ and the initial condition $x_i(0) \in [0.1, 0.15]$, $i = 1, \dots, 500$ and $x_i(0) \in [0.85, 0.9]$, $i = 501, \dots, 1000$.

VI. CONCLUSION

In this paper, we have studied the equilibria and their stability for a recently proposed nonlinear opinion dynamics model with biased assimilation in which each agent is associated with a bias parameter. We have shown that, with a homogeneous bias parameter, the stability of equilibria depends on the value of the degree of bias and the topology of the neighbor relationships among the agents. Both theoretical analyses and numerical simulations have shown that the value of the bias parameter plays a key role in determining the limiting opinions distributed in the network.

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