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Resilience against Misbehaving Nodes in Self-triggered Coordination Networks

Danial Senejohnny, Shreyas Sundaram, Claudio De Persis, Pietro Tesi

Abstract—Network systems are one of the most active research areas in the engineering community as they feature a paradigm shift from centralized to distributed control and computation. When dealing with network systems, a fundamental challenge is to ensure their functioning even when some of the network nodes do not operate as intended due to faults or attacks. The objective of this paper is to address the problem of resilient consensus in a context where the nodes have their own clocks, possibly operating in an asynchronous way, and can make updates at arbitrary time instants. The results represent a first step towards the development of resilient event-triggered and self-triggered coordination protocols.

I. INTRODUCTION

Network systems are one of the most active research areas in the engineering community as they feature a paradigm shift from centralized to distributed control and computation. When dealing with network systems, a fundamental challenge is to ensure their functioning even when some of the network units (nodes) do not operate as intended due to faults or attacks. The main difficulty originates from the fact that normal (non-misbehaving) nodes can receive, process, and spread erroneous data coming from misbehaving nodes with the consequence that a failure in one point of the network can compromise the whole network functioning.

The prototypical problem to study resilience in the presence of misbehaving nodes is the so-called consensus problem [1], which forms the foundation for distributed computing. In *resilient consensus*, each node is assumed to be aware of only local information available from its neighbors and the goal is to make sure that normal nodes eventually reach a common value despite the presence of misbehaving nodes. The resilient consensus problem has a long history, and it has been investigated first by computer scientists [2], [3], usually under the hypothesis that the network graph is complete, that is assuming an all-to-all communication structure. More recently, thanks to the widespread of consensus-based applications, this problem has attracted a lot of interest also within the engineering community, mostly in connection with the goal of delineating the minimal connectivity hypotheses that are needed to secure consensus.

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In [4], the authors consider *mean subsequence reduced* (MSR) algorithms and define a graph-theoretic property, referred to as *network robustness*, which characterizes necessary and sufficient connectivity hypotheses under which normal nodes can reach consensus using only local information available from their neighbors. The results indicate that, while the communication graph should possess a certain degree of redundancy, completeness of the communication graph is not necessary even for very general types of misbehavior. The results of [4] have been extended in many venues. Examples include methods for handling time-varying networks [5], double-integrator systems [6], sparse communication graphs [7] as well as methods for identifying the robustness of specific classes of networks [8].

Most of the research works in this area assume that the network operates in perfect *synchrony*, in the sense that all the nodes, at least the normal ones, update at the same moment in time. Since this condition might be difficult to obtain, a parallel line of research has focused on methods for handling *asynchrony*, which is known to render consensus much more challenging to obtain [2]. Among many notable works, we mention [9], [10], [11], [12] which consider MSR-type algorithms supporting asynchrony. In these works, asynchrony refers to the property that the nodes are equipped with identical clocks, operating synchronously, but can make updates at different steps, that is at different multiples of the clock period.

The objective of this paper is to address the problem of resilient consensus in a context where the nodes have their own clocks, possibly operating in an asynchronous way, and can make updates at arbitrary time instants. Besides the practical difficulties in achieving a perfect clock synchronization, one main reason for considering independent clocks is related to developments in the area of networked control systems where, in order to enhance efficiency and flexibility, it is more and more required to have fully autonomous devices, which is the paradigm of *event-triggered* and *self-triggered* control [13]. In fact, our approach utilizes a self-triggered control scheme [14]. Each node is equipped with a clock that determines when the next update is scheduled. At the update instant, the node polls its neighbors, collects the data and determines whether it is necessary to modify its controls along with a bound on the next update instant.

The main result of this paper establishes *approximate* consensus under certain conditions on the connectivity of the communication graph and a maximum number of misbehaving nodes (Theorem 1). While [4], [9], [11], [12] achieve perfect consensus and require milder connectivity

conditions, the present results indicate that the resilient consensus problem can be approached without requiring that the nodes are equipped with identical clocks, even when the graph is not complete, a feature which is very appealing for networked control applications.

From a technical viewpoint, we adopt a control logic which removes “extreme” values (as in classic MSR-type algorithms) and then form an average from a subset of the remaining values through a *quantized sign* function, which saturates the control action applied at the node. This is as an approximation of the pure (non-quantized) control law introduced in [15], which, in the absence of misbehaving nodes, guarantees *max-min* consensus, the quantization being instrumental to avoid a continuous data flow among the nodes. Interestingly, the use of sign functions has been considered to solve consensus on the *median value* [16], which is inherently robust to outliers and thus to some types of misbehavior. Although our work is substantially different from [16] as we do not consider a continuous data flow, both the approaches suggest that saturating the controls can be beneficial for resilience since this limits the effect of an incorrect update choice resulting from erroneous data.

II. SYSTEM DEFINITION

Consider a network of $n \in \mathbb{N}$ nodes interconnected in accordance with a time-invariant undirected connected graph $\mathcal{G} := (\mathcal{I}, \mathcal{E})$, where $\mathcal{I}$ is the set of nodes while $\mathcal{E} \subseteq \mathcal{I} \times \mathcal{I}$ is the set of edges. We let $\mathcal{Q}_i$ denote the set of neighbors of $i \in \mathcal{I}$, and by d_i the cardinality of $\mathcal{Q}_i$, that is $d_i := |\mathcal{Q}_i|$. The set $\mathcal{Q}_i$ represents the set of nodes with which node i exchanges data. For every $i \in \mathcal{I}$, the dynamics are given by

$$\dot{x}_i(t) = u_i(t) \quad t \in \mathbb{R}_{\geq 0} \quad (1)$$

where $x_i \in \mathbb{R}$ is the state with $x_i(0)$ arbitrary; $u_i \in \mathbb{R}$ is the control action applied by node i . The variable $t \in \mathbb{R}_{\geq 0}$ is understood as the *absolute* time frame within which all the nodes carry out their operations in an asynchronous way.

The objective is to design a coordination protocol in such a way that normal (non-misbehaving) nodes eventually reach *approximate* consensus despite the presence of misbehaving nodes. We will specify later when a node is considered misbehaving. According to the usual notion of consensus [1], the network nodes should converge to an equilibrium point where all the nodes have the same value lying somewhere between the minimum and maximum of their initial values. The following definition formalizes the notion of *approximate* consensus considered in this paper.

Definition 1: The network is said to reach approximate consensus if, for every initial value of the nodes, normal nodes remain between the minimum and maximum of their initial values, and there exists a constant $c \in \mathbb{R}_{>0}$ such that

$$\limsup_{t \rightarrow \infty} |x_i(t) - x_j(t)| < c$$

for every pair (i, j) of normal nodes. ■

Network nodes carry out their operations by means of three main quantities:

- A parameter $\varepsilon \in \mathbb{R}_{>0}$, which determines the desired level of accuracy for consensus.
- A parameter $F \in \mathbb{N}$, which determines the maximum number of misbehaving nodes that the network is expected to encounter.
- A sequence $\{t_k^i\}_{k \in \mathbb{N}}$ of time instants at which node i requests data from its neighbors, where $t_0^i \in [0, t_{init}^i]$ defines the first time instant at which node i becomes active and $t_{init}^i \in \mathbb{R}_{\geq 0}$ denotes the first time instant at which all the nodes are active in the network. By convention, $0 = t_0^r$ where r is the first network node to become active and $x_i(t) = x_i(t_0^i)$ for every $i \in \mathcal{I}$ and for all $t \in [0, t_0^i]$.

It is implicit in the above definition of t_{init}^i that *all* the nodes become active in a finite time. We will also assume that all nodes remain active for the entire runtime. The analysis can be easily generalized to the case where some of the nodes never “wake up” or “die” during the network runtime.

A. Coordination protocol

Let $\mathcal{N}$ and $\mathcal{M}$ represent the sets of normal nodes and misbehaving nodes, respectively, which are assumed to be time-invariant (Assumption 1).

The scheme is based on the idea of discarding “extreme” values [2], which prevents normal nodes from processing potentially harmful information. For every $i \in \mathcal{I}$, let $\mathcal{D}_i(t) \subseteq \mathcal{Q}_i$ be the set of neighbors that are not discarded by i at $t \in \mathbb{R}_{\geq 0}$. For $i \in \mathcal{N}$ this set is determined as follows. Let $\mathcal{V}_i(t)$ be the ordered set formed by sorting the elements of $\mathcal{Q}_i$ in a non-decreasing order of value $x_j(t)$. An arbitrary ordering is pre-specified to classify elements with the same value. Consider the set $\mathcal{F}_i(t)$ formed by the first F elements of $\mathcal{V}_i(t)$, and let $\mathcal{E}_i(t)$ be the subset of $\mathcal{F}_i(t)$ consisting of all the elements of $\mathcal{F}_i(t)$ with associated value smaller than $x_i(t)$, that is $r \in \mathcal{E}_i(t)$ if and only if $r \in \mathcal{F}_i(t)$ and $x_r(t) < x_i(t)$. Similarly, let $\mathcal{L}_i(t)$ be the set formed by the last F elements of $\mathcal{V}_i(t)$, and let $\bar{\mathcal{E}}_i(t)$ be the subset of $\mathcal{L}_i(t)$ consisting of all the elements of $\mathcal{L}_i(t)$ with associated value larger than $x_i(t)$, that is $r \in \bar{\mathcal{E}}_i(t)$ if and only if $r \in \mathcal{L}_i(t)$ and $x_r(t) > x_i(t)$. By convention, $r \notin \mathcal{D}_i(t)$ if $t < t_0^r$, that is if node r is still not active at time t . For $i \in \mathcal{N}$, $\mathcal{D}_i(\cdot)$ satisfies

$$\mathcal{D}_i(t) = \mathcal{Q}_i \setminus (\mathcal{E}_i(t) \cup \bar{\mathcal{E}}_i(t)) \quad (2)$$

and the control action is given by

$$u_i(t) = \begin{cases} 0 & t \in [0, t_0^i) \\ \text{sign}_\varepsilon(\text{ave}_i(t_k^i)) & t \in [t_k^i, t_{k+1}^i) \end{cases} \quad (3)$$

where

$$\text{ave}_i(t) := \sum_{j \in \mathcal{D}_i(t)} (x_j(t) - x_i(t)) \quad (4)$$

and where, for every $\chi \in \mathbb{R}$,

$$\text{sign}_\varepsilon(\chi) := \begin{cases} 0 & \text{if } |\chi| < \varepsilon \\ \text{sign}(\chi) & \text{otherwise} \end{cases} \quad (5)$$

By convention, $\mathcal{D}_i(t) = \emptyset$ implies $\text{ave}_i(t) = 0$. The sign function in controller (3) results in $u_i(t) \in [-1, 0, 1]$. This

choice of controller is motivated by simplifying the agent dynamics to effectively design the self-triggering update policy [14].

Definition 2: A node is considered misbehaving when the control logic (3) is not satisfied for some $t \in \mathbb{R}_{\geq 0}$ and node i instead applies any arbitrary control logic. ■

In the relevant literature, this type of misbehavior is usually referred to as “malicious” [4].

For node $i \in \mathcal{I}$, the next round of operations is scheduled at time

$$t_{k+1}^i = t_k^i + \Delta_k^i \quad (6)$$

where

$$\begin{aligned} \Delta_k^i &\geq \underline{\Delta}_i \\ \Delta_k^i &\leq \frac{1}{4d_i} \max\{\varepsilon, |\text{ave}_i(t_k^i)|\} \end{aligned} \quad (7)$$

with $\underline{\Delta}_i \in \mathbb{R}_{>0}$ such that

$$\underline{\Delta}_i \leq \varepsilon/(4d_i).$$

Operations can be then periodic as well as aperiodic. The first condition avoids arbitrarily fast sampling (Zeno behavior), while the second one is needed to reach approximate consensus. For the sake of clarity a pseudo code of the resilient consensus protocol can be given as follow.

Resilient Consensus Protocol (for each $i \in \mathcal{N}$)
1: initialization: set $t_0^i \in [0, t_{init}]$ and $u_i(0) = 0$;
2: for $k \in \mathbb{Z}_{\geq 0}$ do
3: while $t < t_k^i$ do
4: i applies the control $u_i(t)$;
5: end while
6: if $t = t_k^i$ then
7: for all $j \in \mathcal{D}_i(t)$ do
8: i polls j and collects the information $x_j(t) - x_i(t)$;
9: end for
10: i computes $\text{ave}_i(t)$;
11: i computes $u_i(t)$ as in (3);
12: i computes t_{k+1}^i using (6) and (7);
13: end if
14: end for

B. Assumptions

Assumption 1: The set $\mathcal{M}$ of misbehaving nodes does not change over time and $|\mathcal{M}| \leq F$. ■

Assumption 2: For every $i \in \mathcal{M}$, $u_i(\cdot)$ is a locally integrable function. ■

Assumption (Design) 3: Every pair of normal nodes have at least $3F + 1$ neighbors in common. ■

Assumption 1 is what is usually referred to as the “F-total model” [4]. Assumption 2 ensures the existence of the solutions for all the nodes and for all time, that variables and functions are well defined. Assumption 3 deals with the graph connectivity properties and ensures that the normal nodes share sufficient “genuine” information for taking

control decisions. These assumptions hold, for instance, for classes of *strongly regular graphs* [17], though it is not needed that the graph is regular. These assumptions should be interpreted as design conditions when the graph topology can be assigned. Moreover, the connectivity condition in Assumption 3, for large values of n , leads to sparse network configurations. Consider a complete graph of $3F + 2$ nodes. Then add k more nodes, where each of these k nodes connects to all of the $3F + 2$ nodes in the clique. This graph with $n = 3F + 2 + k$ nodes has the property that every pair of nodes has at least $3F + 1$ nodes in common and we see that, for fixed F , the number of edges scales only linearly with n .

III. MAIN RESULT

In line with the definition of approximate consensus, i.e Definition 1, the below theorem represents the main result of this paper.

Theorem 1: Consider the network system (1)-(7), with the misbehaving nodes exhibiting an error in control update (3) (Definition 2). If Assumptions 1, 2 and 3 hold true, then all the normal nodes $i \in \mathcal{N}$ remain inside the convex hull containing their initial values. Moreover, there exists a finite time $T \in \mathbb{R}_{\geq 0}$ such that

$$|x_i(t) - x_j(t)| < 3\varepsilon$$

for all $t \geq T$ and $i, j \in \mathcal{N}$. ■

We define the maximum and minimum trajectories as

$$x_m(t) := \min_{i \in \mathcal{N}} x_i(t), \quad x_M(t) := \max_{i \in \mathcal{N}} x_i(t) \quad (8)$$

where $t \in \mathbb{R}_{\geq 0}$.

The sketch of the proof is as follow: In Lemma 1, we first show that $x_M(t)$ and $x_m(t)$ are monotonically non-increasing and non-decreasing, respectively. As a result of this Lemma, we can argue normal nodes remain in the convex hull containing their initial values. In addition, this Lemma implies that $x_m(\cdot)$ and $x_M(\cdot)$ admit a finite limit,

$$\underline{x} := \lim_{t \rightarrow \infty} x_m(t), \quad \bar{x} := \lim_{t \rightarrow \infty} x_M(t) \quad (9)$$

For the next developments, we strengthen Lemma 1 by showing that there exist normal nodes that settle on the minimum and maximum values in a finite time. Finally, in light of Assumption 3 the value $\bar{x} - \underline{x}$ can be upper bounded by 3ε .

Lemma 1: Consider the network system (1)-(7), and let Assumptions 1 and 2 hold. Then, $x_m(\cdot)$ and $x_M(\cdot)$ are monotonically non-decreasing and non-increasing, respectively. ■

Lemma 2: Consider the network system (1)-(7), and let Assumptions 1 and 2 hold. Then, there exist at least two indices $r, s \in \mathcal{N}$ and a finite time $T' \in \mathbb{R}_{\geq 0}$ such that

$$x_r(t) = \underline{x} \quad \text{and} \quad x_s(t) = \bar{x}$$

for all $t \geq T'$. In addition,

$$\min_{i \in \mathcal{N}} x_i(t) \geq \underline{x} \quad \text{and} \quad \max_{i \in \mathcal{N}} x_i(t) \leq \bar{x}$$

for all $t \geq T'$. ■

A. Convergence Analysis

Proof of Lemma 1. We prove the statement only for $x_m(\cdot)$ since the proof for $x_M(\cdot)$ is analogous. Suppose that the claim is false, and let τ be the first time instant at which there exists an index $i \in \mathcal{N}$ such that

$$\begin{cases} x_i(\tau) \leq x_j(\tau) & \forall j \in \mathcal{N} \\ u_i(\tau) < 0 \end{cases} \quad (10)$$

Clearly, there could be multiple nodes achieving (10) at time τ . In this case, i is any of such nodes. Notice that $\tau \geq t_0^i$ since $u_i(t) = 0$ for all $t \in [0, t_0^i)$.

Consider first the case where $\tau = t_k^i$ for some $k \in \mathbb{N}$. In order for $u_i(t_k^i) < 0$ we must have $\text{ave}_i(t_k^i) \leq -\varepsilon < 0$. However, this is not possible. In fact, any normal node j satisfies $x_j(t_k^i) \geq x_i(t_k^i)$ because i is the node of minimum at $\tau = t_k^i$. Hence, $x_j(t_k^i) < x_i(t_k^i)$ only if j is misbehaving. Since misbehaving nodes are not more than F by Assumption 1, if a misbehaving node j gives $x_j(t_k^i) < x_i(t_k^i)$ it is discarded by the control logic.

Consider next the case where τ is not an update time for node i . Let $t_k^i < \tau$ be the last update time for node i before τ . In order to have (10), there must exist a node $s \in \mathcal{N}$ such that

$$\begin{cases} x_s(t_k^i) \leq x_j(t_k^i) & \forall j \in \mathcal{N} \\ x_s(t_k^i) < x_i(t_k^i) \\ x_i(\tau) \leq x_j(\tau) & \forall j \in \mathcal{N} \\ u_i(\tau) < 0 \end{cases} \quad (11)$$

The first two conditions imply that i is not the node which takes on the minimum value at t_k^i , value which is instead attained by node s . Condition $x_s(t_k^i) < x_i(t_k^i)$ is needed otherwise $u_i(t_k^i) \geq 0$ in accordance with the previous arguments. The last three conditions mean that i becomes the minimum value at τ with $u_i(\tau) = u_i(t_k^i) < 0$. Let

$$\beta := x_i(t_k^i) - x_s(t_k^i).$$

Recall that normal nodes take controls in $\{-1, 0, 1\}$. Hence, $x_i(\tau) \leq x_j(\tau)$ for all $j \in \mathcal{N}$ only if $\tau - t_k^i \geq \beta/2$. However,

$$\begin{aligned} \Delta_k^i &\leq \frac{1}{4d_i} \sum_{j \in \mathcal{D}_i(t_k^i)} (x_i(t_k^i) - x_j(t_k^i)) \\ &\leq \frac{1}{4d_i} \sum_{j \in \mathcal{D}_i(t_k^i)} (x_i(t_k^i) - x_s(t_k^i)) \end{aligned} \quad (12)$$

The first equality follows from the fact that $u_i(t_k^i) < 0$ requires $\text{ave}_i(t_k^i) < 0$ so that $\Delta_k^i \leq \max\{\varepsilon, |\text{ave}_i(t_k^i)|\}/(4d_i)$. On the other hand, the second inequality follows since $x_j(t_k^i) < x_s(t_k^i)$ only if j is misbehaving, in which case it is discarded by node i in view of Assumption 1 and by the control logic. Since $|\mathcal{D}_i(t)| \leq d_i$ for all $t \in \mathbb{R}_{\geq 0}$ then $\Delta_k^i \leq \beta/4$. This leads to a contradiction since it implies

$$\frac{\beta}{2} \leq \tau - t_k^i < \Delta_k^i \leq \frac{\beta}{4}$$

with $\beta > 0$. ■

Proof of Lemma 2. We prove the statement only for $\underline{x}$ as the proof for $\bar{x}$ is analogous. Since $x_m(\cdot)$ converges to $\underline{x}$ and is continuous, for any $\delta \in \mathbb{R}_{>0}$ there exists a finite time $T_\delta \in \mathbb{R}_{\geq 0}$ such that $|x_m(t) - \underline{x}| < \delta$ for all $t > T_\delta$. Let

$$\underline{\Delta} := \min_{i \in \mathcal{N}} \underline{\Delta}_i$$

and pick

$$\delta = \underline{\Delta}/3.$$

Consider any $i \in \mathcal{N}$ and any update time t_k^i for node i such that $t_k^i \geq T_\delta$. Condition $t_k^i \geq T_\delta$ is well defined for any T_δ since by Lemma 1 normal nodes always remain in the convex hull containing their initial values so that Δ_k^i is bounded from above.

We claim that, for any $i \in \mathcal{N}$ and any $t_k^i \geq T_\delta$,

$$|x_i(t_k^i) - \underline{x}| \geq \delta \Rightarrow |x_i(t) - \underline{x}| \geq \delta \quad \forall t \geq t_k^i \quad (13)$$

In simple terms, this means that if $x_i(t_k^i)$ does not belong to

$$W := (\underline{x} - \delta, \underline{x} + \delta)$$

then $x_i(\cdot)$ can never enter W afterwards. The implication (13) is shown as follows. Since $|x_m(t) - \underline{x}| < \delta$ for all $t > T_\delta$ then we must also have $x_j(t) > \underline{x} - \delta$ for all $t > T_\delta$ and $j \in \mathcal{N}$. This means that condition $|x_i(t_k^i) - \underline{x}| \geq \delta$ implies $x_i(t_k^i) \geq \underline{x} + \delta$. The analysis is divided into two subcases.

Case 1. Assume $x_i(t_k^i) \in [\underline{x} + \delta, \underline{x} + 2\delta)$. In this case,

$$\begin{aligned} \text{ave}_i(t_k^i) &= \sum_{j \in \mathcal{D}_i(t_k^i)} (x_j(t_k^i) - x_i(t_k^i)) \\ &> \sum_{j \in \mathcal{D}_i(t_k^i)} (\underline{x} - \delta - x_i(t_k^i)) \\ &> -3\delta |\mathcal{D}_i(t_k^i)| \\ &> -\varepsilon \end{aligned} \quad (14)$$

The first inequality follows from the fact that $j \in \mathcal{D}_i(t_k^i)$ only if $x_j(t_k^i) > \underline{x} - \delta$. In fact, $x_j(t) > \underline{x} - \delta$ for all $t > T_\delta$ and $j \in \mathcal{N}$. Thus, nodes with an output value less than or equal to $\underline{x} - \delta$ are misbehaving, and they are discarded in view of Assumption 1 and by construction of the control logic. The second inequality follows because $x_i(t_k^i) < \underline{x} + 2\delta$ by hypothesis. The last inequality follows since $3\delta \leq \varepsilon/(4d_i)$ and $|\mathcal{D}_i(t)| \leq d_i$ for all $t \in \mathbb{R}_{\geq 0}$. Since $\text{ave}_i(t_k^i) > -\varepsilon$ implies $u_i(t) \in \{0, 1\}$ for all $t \in \bar{T}_k^i$ then $x_i(t) \notin W$ for all $t \in T_k^i$, and $x_i(t_{k+1}^i) \notin W$ by continuity of $x_i(\cdot)$.

Case 2. Assume $x_i(t_k^i) \geq \underline{x} + 2\delta$. In order for node i to decrease we must have $\text{ave}_i(t_k^i) \leq -\varepsilon$. Hence,

$$\begin{aligned} \Delta_k^i &\leq \frac{1}{4d_i} \sum_{j \in \mathcal{D}_i(t_k^i)} (x_i(t_k^i) - x_j(t_k^i)) \\ &< \frac{1}{4d_i} \sum_{j \in \mathcal{D}_i(t_k^i)} (x_i(t_k^i) - \underline{x} + \delta) \\ &\leq \frac{1}{4} (x_i(t_k^i) - \underline{x} + \delta) \end{aligned} \quad (15)$$

The second inequality follows because $j \in \mathcal{D}_i(t_k^i)$ only if $x_j(t_k^i) > \underline{x} - \delta$ according to the previous arguments. The third inequality follows since $|\mathcal{D}_i(t)| \leq d_i$ for all $t \in \mathbb{R}_{\geq 0}$. Since normal nodes take controls in $\{-1, 0, 1\}$, we obtain

$$x_i(t) \geq x_i(t_k^i) - \Delta_k^i \geq \underline{x} + \frac{5}{4}\delta \quad (16)$$

for all $t \in T_k^i$, where the last inequality follows from (15) and since $x_i(t_k^i) \geq \underline{x} + 2\delta$. Thus $x_i(t) \notin W$ for all $t \in T_k^i$, and $x_i(t_{k+1}^i) \notin W$ by continuity of $x_i(\cdot)$.

We conclude that if $x_i(t_k^i)$ does not belong to W for some $t_k^i \geq T_\delta$ then $x_i(\cdot)$ can never enter W afterwards. Moreover, for every $i \in \mathcal{N}$, if $x_i(t_k^i) \in W$ and $u_i(t_k^i) \neq 0$ then $x_i(t_{k+1}^i) \notin W$. In fact, in this case, node i must apply the same control input for a period not shorter than $\underline{\Delta}$. Thus, $x_i(t_{k+1}^i) \notin W$ since the control input is constant with unitary slope for at least $\underline{\Delta}$ time units and W has measure $2\delta = 2\underline{\Delta}/3$. Thus, since the number of nodes is finite, there exists a finite time $T'' \geq T_\delta$ starting from which the signal $x_i(\cdot)$, $i \in \mathcal{N}$, either persistently remains inside W or persistently remains outside W . Moreover, there exists at least one index $i \in \mathcal{N}$ for which $x_i(\cdot)$ persistently remains inside W since, by definition, $\underline{x} \in W$ is the limiting value of $x_m(\cdot)$.

Every $x_i(\cdot)$, $i \in \mathcal{N}$, that persistently remains outside W from T'' onwards satisfies $x_i(t) \geq \underline{x} + \delta$ for all $t \geq T''$. Consider next any $x_i(\cdot)$, $i \in \mathcal{N}$, that persistently remains inside W from T'' onwards. By the above arguments, $u_i(t_k^i) = 0$ for all $t_k^i \geq T''$. Moreover, the first sampling $t_k^i \geq T''$ must occur no later than

$$T' := T'' + \varepsilon/4.$$

This is because, either $u_i(T'') = 0$ so that $t_k^i - T'' \leq \varepsilon/(4d_i)$ by construction of the update times or $u_i(T'') \neq 0$ so that $t_k^i - T'' \leq 2\delta < \varepsilon/4$ otherwise $x_i(t_k^i) \notin W$ according to the previous arguments. Hence, every $x_i(\cdot)$, $i \in \mathcal{N}$, that persistently remains inside W from T'' onwards satisfies $x_i(t) = x_i(T')$ for all $t \geq T'$. Thus $\min_{i \in \mathcal{N}} x_i(t) \geq \underline{x}$ for all $t \geq T'$ and $x_r(T') = \underline{x}$ for some $r \in \mathcal{N}$ since $\underline{x}$ is the limiting value of $x_m(\cdot)$. ■

By Lemma 2, there exist at least two indices $r, s \in \mathcal{N}$ and a finite time T' such that $x_r(t) = \underline{x}$ and $x_s(t) = \bar{x}$ for all $t \geq T'$. We now show that under Assumption 3 the value $\bar{x} - \underline{x}$ is upper bounded by 3ε .

Proof of Theorem 1. The property that normal nodes always remain inside the convex hull containing their initial values has been shown in Lemma 1. Thus, we focus on the second part of the statement.

Let T' be as in Lemma 2, and denote by r and s any two indices belonging to $\mathcal{N}$ such that $x_r(t) = \underline{x}$ and $x_s(t) = \bar{x}$ for all $t \geq T'$. Consider now any update time $t_k^r \geq T'$ for node r . Since node r does not change its value from T' onwards, we must have $\text{ave}_r(t_k^r) < \varepsilon$. This implies that $z_i(t_k^r) - x_r(t_k^r) < \varepsilon$ for all $i \in \mathcal{D}_r(t_k^r)$. In fact, in order to have $z_i(t_k^r) - x_r(t_k^r) \geq \varepsilon$ for some $i \in \mathcal{D}_r(t_k^r)$ there should exist at least one index $j \in \mathcal{D}_r(t_k^r)$ such that $x_j(t_k^r) -$

Fig. 1. Network system considered in the numerical example. Normal nodes are depicted in grey, while the misbehaving node is depicted in orange. The graph satisfies Assumption 3 and is thus robust against misbehavior.

$x_r(t_k^r) < 0$. However, this is not possible. In fact, in view of Lemma 2, $x_j(t_k^r) \geq \underline{x}$ for all $j \in \mathcal{N}$ so that every node which takes on an output value smaller than $\underline{x}$ is misbehaving and it is discarded by node r in view of Assumption 1 and by construction of the control logic. Since every $i \in \mathcal{N}$ satisfies $z_i(\cdot) \equiv x_i(\cdot)$ we have $x_i(t_k^r) - \underline{x} < \varepsilon$ for all $i \in \mathcal{D}_r(t_k^r) \cap \mathcal{N}$.

The claim thus trivially follows if $s \in \mathcal{D}_r(t_k^r)$. Suppose $s \notin \mathcal{D}_r(t_k^r)$. Nodes r and s have at least $3F + 1$ neighbors in common by virtue of Assumption 3, so that, because of Assumption 1, they have at least $2F + 1$ neighbors in common belonging to $\mathcal{N}$. At t_k^r , node r can discard at most F normal nodes since there are no normal nodes with value smaller than $x_r(t_k^r)$. Hence, there are at least $F + 1$ normal nodes belonging to $\mathcal{D}_r(t_k^r) \cap \mathcal{Q}_s$. Starting from t_k^r , node s will sample at a time $t_k^s \leq t_k^r + \varepsilon/4$. When s will sample, it cannot discard all these $F + 1$ normal nodes by construction of the control logic and because none of these nodes can take on a value larger than $\bar{x}$ from T' onwards in view of Lemma 2. Following the same reasoning as before, at least one of these nodes, say node i , must satisfy $x_i(t_k^s) > \bar{x} - \varepsilon$ otherwise one would have $\text{ave}_s(t_k^s) \leq -\varepsilon$. Hence,

$$\begin{cases} x_i(t_k^r) < \underline{x} + \varepsilon \\ x_i(t_k^s) > \bar{x} - \varepsilon \\ x_i(t_k^s) \leq x_i(t_k^r) + \frac{\varepsilon}{4} \end{cases} \quad (17)$$

where the last inequality follows since $u_i(t) < 1$ for all $t \in \mathbb{R}_{\geq 0}$ and $i \in \mathcal{N}$, and since $t_k^s \leq t_k^r + \varepsilon/4$. This implies $\bar{x} - \underline{x} < 3\varepsilon$, and letting

$$T := T'$$

the claim follows. ■

IV. A NUMERICAL EXAMPLE

Consider a network system as in (1)-(7), with $n = 7$ nodes interconnected as in Fig. 1 and $F = 1$ misbehaving nodes. State and clock initial values are taken randomly within the intervals $[0, 1]$ and $[0, t_{init}]$, respectively, with $t_{init} = 0.15$ sec. The desired accuracy level for consensus is selected as $\varepsilon = 0.01$ and we set $\underline{\Delta}_i = \varepsilon/(4d_i)$ for every node. We note

Fig. 2. Network state evolution with the resilient consensus protocol under control misbehavior. The evolution of the misbehaving node is depicted in red dashed line.

that the graph satisfies Assumption 3, which is sufficient to guarantee resilience against misbehavior.

Specifically, we assume that the misbehaving node applies the control input $u_i(t) = 10 \sin(10\pi t)$ for all $t \in \mathbb{R}_{\geq 0}$ instead of (3). Fig. 2 illustrates the network state evolution with the proposed resilient consensus protocol. In agreement with the conclusions of Theorem 1, one sees that the normal nodes remain in the convex hull containing their initial values and reach a practical agreement disregarding the behavior of the misbehaving node.

V. CONCLUSIONS

This paper shows the possibility of approaching the resilient consensus problem in a context where the nodes have their own clocks and can make updates at arbitrary time instants. Although the results are preliminary, they indicate that handling misbehaving units can be possible also in network applications involving asynchronous and aperiodic transmissions, as occurs with event-triggered and self-triggered network systems.

We have considered a scenario where the network can support the data flow with reliability and accuracy, thus neglecting issues such as transmission delays, data loss, bandwidth as well as noise. In practice, these issues are also very important but require careful consideration of several technicalities. Nonetheless, we envision that some extensions are indeed possible along the same lines as in [14], [18] where we discuss aspects related to the quality of the transmission medium.

Our approach utilizes a self-triggered update scheme with control saturation. It is of interest to investigate if similar results can be obtained also with event-triggered or other types of aperiodic update schemes [19]. It is also interesting to see if similar results can be obtained with other averaging functions, for example with the classic coupling law for *average* consensus [15]. We envision an application of the present research within the context of distributed optimization [20], with specific reference to self-triggered schemes [21]. Another interesting research venue is in the area of multi-agent systems with cloud access [22]. Also in this context, self-triggered control seems a viable option

for enabling asynchronous coordination without destroying regulation properties.

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