University of Groningen

Absolute stabilization of Lur'e systems via dynamic output feedback

Zhang, Fan ; Trentelman, Harry L.; Feng, Gang; Scherpen, Jacqueliën M.A.

Published in:
European Journal of Control

DOI:
[10.1016/j.ejcon.2018.09.015](https://doi.org/10.1016/j.ejcon.2018.09.015)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2018

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Zhang, F., Trentelman, H. L., Feng, G., & Scherpen, J. M. A. (2018). Absolute stabilization of Lur'e systems via dynamic output feedback. *European Journal of Control*, 44, 15-26.
<https://doi.org/10.1016/j.ejcon.2018.09.015>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Absolute stabilization of Lur'e systems via dynamic output feedback

Fan Zhang^a, Harry L. Trentelman^{b,*}, Gang Feng^c, Jacqueline M.A. Scherpen^b

^a School of Mathematics, Southeast University, Nanjing, China

^b Faculty of Science and Engineering, University of Groningen, Groningen, The Netherlands

^c Department of Mechanical and Biomedical Engineering, City University of Hong Kong, Hong Kong, China

ARTICLE INFO

Article history:

Received 30 January 2018

Revised 2 August 2018

Accepted 20 September 2018

Available online 28 September 2018

Recommended by Dr. A Zemouche

Keywords:

Lur'e system

Absolute stabilization

Dynamic output feedback

Linear matrix inequality

Flexible joint robotic arm

ABSTRACT

In this paper, we study the problem of absolute stabilization under dynamic output feedback for Lur'e systems with sector-bounded unknown nonlinearities. In most of the literature, besides the incremental sector-boundedness condition, the Lur'e-type nonlinearity itself is assumed to be known exactly and used in the dynamic output feedback controller design. In the present paper only the sector-boundedness condition is employed, and exact knowledge of the nonlinearity will not be used in the controller design. More precisely, we will only employ knowledge of the sector in which the unknown nonlinearity lies. Two different approaches will be presented for the dynamic controller design, both using linear matrix inequality techniques. Numerical simulations of a flexible joint robotic arm will illustrate the theoretical results obtained in this paper.

© 2018 European Control Association. Published by Elsevier Ltd. All rights reserved.

1. Introduction

In this paper we study the problem of absolute stabilization by dynamic output feedback for Lur'e systems. A Lur'e system is the interconnection of a linear input-state-output system and an unknown static nonlinearity through a negative feedback loop, commonly represented as

$$\begin{cases} \dot{x} = A_p x + B_p u + E_p d, \\ z = C_p x, \quad d = -\phi(z), \quad y = M_p x, \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$ are the state, control input, loop injection and measurement output, respectively. The matrices A_p , B_p , C_p , E_p and M_p are of compatible dimensions. Without loss of generality, B_p and M_p are assumed to be of full column rank and full row rank, respectively. The equation $d = -\phi(z)$ describes a static negative feedback loop, where $\phi(\cdot)$ is an unknown nonlinear function, see Fig. 1. This nonlinearity is however assumed to be sector-bounded, in the sense that it satisfies the following sector-boundedness condition.

Definition 1. Let $S_1, S_2 \in \mathbb{R}^{p \times p}$ be real symmetric matrices such that S_1 is positive semi-definite and $S_2 - S_1$ is positive definite, i.e. $0 \leq S_1 < S_2$. Then the nonlinearity $\phi(\cdot)$ is called *sector-bounded*

within the sector $[S_1, S_2]$ if it satisfies

$$(\phi(z) - S_1 z)^T (\phi(z) - S_2 z) \leq 0, \quad \forall z \in \mathbb{R}^p. \quad (2)$$

We will be dealing with output feedback controllers of the form

$$\begin{cases} \dot{v} = A_c v + B_c y \\ u = C_c v + D_c y \end{cases}, \quad (3)$$

where $v(t) \in \mathbb{R}^{n_c}$ is the state of the dynamic controller, the positive integer n_c and the matrices $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times q}$, $C_c \in \mathbb{R}^{m \times n_c}$, $D_c \in \mathbb{R}^{m \times q}$ are to be determined. The aim is to design a controller of the form (3) that *absolutely stabilizes* the Lur'e system (1), i.e. the closed-loop system is *absolutely stable*, in the sense defined as follows:

Definition 2. Given the bounds S_1 and S_2 with $0 \leq S_1 < S_2$, the feedback interconnection of the Lur'e system (1) and the controller (3) is called *absolutely stable* if it is globally asymptotically stable for every $\phi(\cdot)$ within the sector $[S_1, S_2]$.

Thus, given the Lur'e system, including the sector bounds, we want to find a single dynamic output feedback controller that makes the closed-loop system globally asymptotically stable for all $\phi(\cdot)$'s satisfying the sector-boundedness condition.

Stabilization of Lur'e systems through output feedback has been studied before. However, in the major part of the literature, the Lur'e-type nonlinearity is assumed to be *known* exactly, and this knowledge is used in the to-be-designed observer-based feedback

* Corresponding author.

E-mail addresses: fan.zhang@seu.edu.cn (F. Zhang), h.l.trentelman@rug.nl (H.L. Trentelman), megfeng@cityu.edu.hk (G. Feng), j.m.a.scherpen@rug.nl (J.M.A. Scherpen).

Fig. 1. Lur'e system.

controllers. In [1,2] this led to a nonlinear observer, called ‘circle-criterion observer’, which used exact knowledge of the nonlinearity. Moreover, a slope-restrictedness condition is required, which is more conservative than the sector-boundedness condition (2). In [9], a circle-criterion observer was adapted in stabilization of Lur’e systems with saturating actuators. It also inspired the output feedback controller design for Lur’e systems with set-valued nonlinearities [3], and the $\mathcal{H}_\infty$ observer design for Lipschitz and monotonic nonlinear systems [11]. However, the above design failed to achieve *absolute stabilization*. In order to remove the requirement on exact knowledge of the Lur’e-type nonlinearity, a nonlinear approximation was proposed in [5], which in fact still needs information on the nonlinearity itself. The same authors studied fixed-order dynamic output feedback control of discrete-time Lur’e systems in [6]. The results therein have been generalized to the continuous-time case in Section 2, where the order of the dynamic controller is not fixed.

The problem formulation in the present paper differs from the above references: the Lur’e-type nonlinearity is not known, and can therefore not be used as part of the controller. The only information on the nonlinearity that we will use are the exact values of the matrices S_1 and S_2 defining the sector. We will present two different approaches to design the controller of the form (3). Numerical simulations for a flexible joint robotic arm will be given to validate the obtained theoretical results.

2. Design approach I

The interconnection of the Lur’e system (1) and the controller (3) is represented by

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} A_p + B_p D_c M_p & B_p C_c \\ B_c M_p & A_c \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} - \begin{bmatrix} E_p \\ \mathbf{0} \end{bmatrix} \phi(C_p x). \quad (4)$$

Collect the matrices A_c , B_c , C_c and D_c describing the controller (3) into the matrix $F \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ defined as

$$F \triangleq \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \quad (5)$$

Henceforth, we will speak about the ‘controller’ (5). The following lemma gives a sufficient condition in terms of solvability of a matrix inequality under which the controller is absolutely stabilizing.

Lemma 1. Consider the Lur’e system (1) with the sector $[S_1, S_2]$. Define $A \triangleq \begin{bmatrix} A_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $B \triangleq \begin{bmatrix} B_p & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$, $C \triangleq [C_p \quad \mathbf{0}]$, $E \triangleq \begin{bmatrix} E_p \\ \mathbf{0} \end{bmatrix}$ and $M \triangleq \begin{bmatrix} M_p & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$. Let a controller F be given by (5). If there exists a positive definite matrix $P \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ such that

$$\begin{bmatrix} P(A + BFM) + (A + BFM)^T P & -PE + \\ -C^T(S_1 S_2 + S_2 S_1)C & C^T(S_1 + S_2) \\ \hline -E^T P + (S_1 + S_2)C & -2I \end{bmatrix} < \mathbf{0}, \quad (6)$$

then the controller (5) is absolutely stabilizing.

Proof. Let $Q > \mathbf{0}$. Define $V(x, v) \triangleq \begin{bmatrix} x \\ v \end{bmatrix}^T Q \begin{bmatrix} x \\ v \end{bmatrix}$. Obviously, $V(x, v)$ is positive definite and radially unbounded. The time derivative of

$V(x, v)$ along trajectories of (4) is given by

$$\begin{aligned} \dot{V}(x, v) &= 2 \begin{bmatrix} x \\ v \end{bmatrix}^T Q \left((A + BFM) \begin{bmatrix} x \\ v \end{bmatrix} - E\phi(C_p x) \right) \\ &= \begin{bmatrix} x \\ v \\ \phi(C_p x) \end{bmatrix}^T \begin{bmatrix} Q(A + BFM) + (A + BFM)^T Q & -QE \\ \hline -E^T Q & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ v \\ \phi(C_p x) \end{bmatrix}. \end{aligned}$$

It is easily verified that the sector-boundedness condition (2) for (1) implies

$$\begin{bmatrix} x \\ \phi(C_p x) \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} C_p^T (S_1 S_2 + S_2 S_1) C_p & -\frac{1}{2} C_p^T (S_1 + S_2) \\ \hline -\frac{1}{2} (S_1 + S_2) C_p & I \end{bmatrix} \begin{bmatrix} x \\ \phi(C_p x) \end{bmatrix} \leq 0,$$

which, in terms of the closed-loop system, yields

$$\begin{bmatrix} x \\ v \\ \phi(C_p x) \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} C^T (S_1 S_2 + S_2 S_1) C & -\frac{1}{2} C^T (S_1 + S_2) \\ \hline -\frac{1}{2} (S_1 + S_2) C & I \end{bmatrix} \begin{bmatrix} x \\ v \\ \phi(C_p x) \end{bmatrix} \leq 0.$$

Thus, we find that $\dot{V}(x, v)$ is negative definite for all $\phi(\cdot)$ ’s within the sector $[S_1, S_2]$ if there exists a positive real number τ together with $Q > \mathbf{0}$ such that

$$\begin{bmatrix} Q(A + BFM) + (A + BFM)^T Q & -QE + \\ \hline -\tau C^T (S_1 S_2 + S_2 S_1) C & \tau C^T (S_1 + S_2) \\ \hline -E^T Q + \tau (S_1 + S_2) C & -2\tau I \end{bmatrix} < \mathbf{0} \quad (7)$$

holds. Obviously, by defining $P \triangleq \frac{1}{\tau} Q$, the matrix inequality (7) is equivalent to (6). This completes the proof. $\square$

Using Lemma 1 one can verify whether a given F is absolutely stabilizing. Of course, the question is now under what conditions an F satisfying the condition (6) actually exists. This will be dealt with in the following. Before doing this, we review the concept of minimal left annihilator of a matrix.

Definition 3 [4]. Let $M \in \mathbb{R}^{n \times m}$ with $\text{rank}(M) = r < n$. We denote by $M^\perp$ any matrix in $\mathbb{R}^{(n-r) \times n}$ of full row rank such that $M^\perp M = \mathbf{0}$. Any such matrix $M^\perp$ is called a *minimal left annihilator* of M .

Note that a minimal left annihilator is only defined for matrices with linearly dependent rows. The set of all such matrices is given by $M^\perp = WU_2^T$, where W is an arbitrary nonsingular matrix and U_2 is obtained from the singular value decomposition $M = [U_1 \quad U_2] \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} [V_1 \quad V_2]^T$. Thus, for a given M , $M^\perp$ is not unique. Throughout this paper, $M^\perp$ will denote any choice from this set of matrices.

The following lemma now gives necessary and sufficient conditions under which, for a given dimension n_c , there exists a controller $F \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ for which the matrix inequality (6) has a solution $P > \mathbf{0}$:

Lemma 2. Let n_c be a given nonnegative integer. There exist F and $P > \mathbf{0}$ such that (6) holds if and only if there exist positive definite matrices $X \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ and $Y \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ such that $XY = I$,

$$\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^\perp \begin{bmatrix} (A - \frac{1}{2}E(S_1 + S_2)C)X + \\ X(A - \frac{1}{2}E(S_1 + S_2)C)^T \\ \hline + \frac{1}{2}EE^T \\ \hline CX & -2(S_2 - S_1)^{-2} \end{bmatrix} \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \quad (8)$$

$$\begin{bmatrix} M^T \\ \mathbf{0} \end{bmatrix}^\perp \begin{bmatrix} Y(A - \frac{1}{2}E(S_1 + S_2)C) + (A - \frac{1}{2}E(S_1 + S_2)C)^T Y & YE \\ + \frac{1}{2}C^T(S_2 - S_1)^2 C & \\ \hline E^T Y & -2I \end{bmatrix} \begin{bmatrix} M^T \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}. \quad (9)$$

Furthermore, in terms of X and Y , the following P and F satisfy (6): $P \triangleq X^{-1}$ and

$$F \triangleq -rB^T \Theta_x^{-1} X M^T (M X \Theta_x^{-1} X M^T)^{-1}, \quad (10)$$

where r and Θ_x are determined as follows: choose a real number r such that

$$\Theta_x \triangleq rBB^T - Q_x - \frac{1}{2}EE^T - \frac{1}{2}XC^T(S_2 - S_1)^2 CX > \mathbf{0} \quad (11)$$

with $Q_x \triangleq (A - \frac{1}{2}E(S_1 + S_2)C)X + X(A - \frac{1}{2}E(S_1 + S_2)C)^T$.

Remark 1. In order to enhance readability of this paper, we defer the proof of Lemma 2 to the Appendix. Several ideas in our proof are borrowed from [10], Section 2.3. For instance, the existence of $r \in \mathbb{R}$ such that Θ_x in (11) is positive definite follows by applying Finsler's lemma (see e.g. [4]) to the LMI (8).

Remark 2. We note that Lemma 2 also applies to the situation that $n_c = 0$, in which we want to obtain a static output feedback controller. We also refer to [6] for the design of static output feedback in the context of Lur'e systems. In general, asking for solvability with $n_c = 0$ is too restrictive, and it will turn out that we need $n_c > 0$ for the LMI's (8) and (9) to be solvable.

Note that in Lemma 2, the controller dimension n_c is assumed to be given, and the lemma gives necessary and sufficient conditions for solvability of (6) with that given dimension n_c . But, in general finding the dimension of the controller is part of the design problem. Our following result addresses the problem of how to find an n_c together with a suitable $F \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$:

Lemma 3. There exists a nonnegative integer n_c , matrices $X > \mathbf{0}$ and $Y > \mathbf{0}$ of size $(n + n_c) \times (n + n_c)$ such that $XY = I$ and the LMI's (8), (9) hold if and only if there exist matrices $X_p > \mathbf{0}$, $Y_p > \mathbf{0}$ of size $n \times n$ such that

$$\begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix}^\perp \begin{bmatrix} Q_x + \frac{1}{2}E_p E_p^T & X_p C_p^T \\ C_p X_p & -2(S_2 - S_1)^{-2} \end{bmatrix} \begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \quad (12)$$

$$\begin{bmatrix} M_p^T \\ \mathbf{0} \end{bmatrix}^\perp \begin{bmatrix} Q_y + \frac{1}{2}C_p^T(S_2 - S_1)^2 C_p & Y_p E_p \\ E_p^T Y_p & -2I \end{bmatrix} \begin{bmatrix} M_p^T \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \quad (13)$$

$$\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \geq \mathbf{0}, \quad (14)$$

$$\text{rank} \left(\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \right) \leq n + n_c, \quad (15)$$

where $Q_x \triangleq (A_p - \frac{1}{2}E_p(S_1 + S_2)C_p)X_p + X_p(A_p - \frac{1}{2}E_p(S_1 + S_2)C_p)^T$,

$$Q_y \triangleq Y_p(A_p - \frac{1}{2}E_p(S_1 + S_2)C_p) + (A_p - \frac{1}{2}E_p(S_1 + S_2)C_p)^T Y_p.$$

Proof. (only if) Assume there exists nonnegative integer n_c , $X > \mathbf{0}$ and $Y > \mathbf{0}$ of size $(n + n_c) \times (n + n_c)$ such that $XY = I$, (8) and (9) hold. Partition $X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}$ and $Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}$ appropriately. Note that $B^\perp = [B_p^\perp \quad \mathbf{0}]$, $M^{\perp T} = [M_p^{\perp T} \quad \mathbf{0}]$, $\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^\perp = \begin{bmatrix} B_p^\perp & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$,

$\begin{bmatrix} M^T \\ \mathbf{0} \end{bmatrix}^\perp = \begin{bmatrix} M^{\perp T} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$. In this way we obtain (12) and (13). $XY = I$ implies that $X_p Y_p + X_{pc} Y_{pc}^T = I$ and $X_p Y_{pc} + X_{pc} Y_c = \mathbf{0}$. Thus

$$Y_p - X_p^{-1} = Y_{pc} Y_c^{-1} Y_{pc}^T \geq \mathbf{0}. \quad (16)$$

Using the Schur complement lemma, (16) is equivalent to (14). In addition,

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \right) &= \text{rank}(X_p) + \text{rank}(Y_p - X_p^{-1}) \\ &= n + \text{rank}(Y_{pc} Y_c^{-1} Y_{pc}^T) \leq n + n_c. \end{aligned}$$

So (15) holds.

(if) Let Y_{pc} and $Y_c > \mathbf{0}$ be any matrices satisfying (16) while $X_p > \mathbf{0}$ and $Y_p > \mathbf{0}$ satisfy (12)–(14), and n_c is chosen to satisfy (15).

It can be verified that a matrix pair (X, Y) such that $Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}$

and $X = Y^{-1}$ satisfy the conditions in Lemma 2. This completes the proof. $\square$

Lemmas 1, 2 and 3 together give a design method for an absolutely stabilizing controller (3), summarized in the following theorem.

Theorem 1. Consider the Lur'e system (1) with the sector $[S_1, S_2]$. If there exist positive definite matrices $X_p \in \mathbb{R}^{n \times n}$ and $Y_p \in \mathbb{R}^{n \times n}$ such that (12), (13) and (14) holds, then there exists an absolutely stabilizing controller (3). Its design can be performed as follows:

- 1) Compute $X_p > \mathbf{0}$ and $Y_p > \mathbf{0}$ such that (12), (13) and (14).
- 2) Choose the controller dimension as $n_c \triangleq \text{rank} \left(\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \right) - n$, then define A, B, C, E and M as introduced in Lemma 1.
- 3) Choose $Y_c > \mathbf{0}$ and Y_{pc} satisfying (16) to obtain $Y > \mathbf{0}$ and define $X \triangleq Y^{-1}$.
- 4) Compute $r > 0$ in (11) to get $\Theta_x > \mathbf{0}$.
- 5) Compute F defined by (10) and partition it as $\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$.

For Case 1 of the simulation example in the subsequent Section 4 we have applied the design approach I to stabilize a flexible joint robotic arm modelled by the Lur'e system (1). It turns out that, although absolute stabilization is achieved, the transient performance of the closed-loop system is not satisfactory and a rather large control input is required. Therefore, in order to overcome the above drawbacks, in the next section we will present an alternative design approach.

3. Design approach II

In this section we will take an alternative approach to designing an absolutely stabilizing dynamic controller for the Lur'e system (1). A main difference is that we will fix the controller state dimension n_c to be equal to n , the state space dimension of the Lur'e system. We set the direct feedthrough matrix of the controller (3) equal to $D_c = \mathbf{0}$, and look at controllers of the form

$$\begin{cases} \dot{\hat{x}} = A_c \hat{x} + B_c y \\ u = C_c \hat{x} \end{cases}. \quad (17)$$

Then the interconnection of (1) and (17) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A_p & B_p C_c \\ B_c M_p & A_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} - \begin{bmatrix} E_p \\ \mathbf{0} \end{bmatrix} \phi(C_p x). \quad (18)$$

The controller state $\hat{x}$ can be viewed as an estimate of x . By defining the error $e(t) \triangleq x(t) - \hat{x}(t)$, we also have

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_p + B_p C_c & -B_p C_c \\ A_p - B_c M_p + B_p C_c - A_c & -B_p C_c + A_c \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} - \begin{bmatrix} E_p \\ E_p \end{bmatrix} \phi(C_p x).$$

(19)

Clearly, for a given sector $[S_1, S_2]$, the closed-loop system (18) is absolutely stable if and only if the same holds for (19).

The following theorem provides a solution to the design problem for A_c , B_c and C_c in (17).

Theorem 2. Consider the Lur'e system (1) with the sector $[S_1, S_2]$. Denote $\bar{A}_p \triangleq A_p - \frac{1}{2}E_p(S_1 + S_2)C_p$ and $\bar{C}_p \triangleq (S_2 - S_1)C_p$. If there exist positive definite matrices $X_p \in \mathbb{R}^{n \times n}$ and $Y_p \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix}^\perp \begin{bmatrix} \bar{A}_p X_p + X_p \bar{A}_p^T + \frac{1}{4}E_p E_p^T & X_p \bar{C}_p^T \\ \bar{C}_p X_p & -I \end{bmatrix} \begin{bmatrix} B_p \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \quad (20)$$

$$\begin{bmatrix} M_p^T \\ \mathbf{0} \end{bmatrix}^\perp \begin{bmatrix} Y_p \bar{A}_p + \bar{A}_p^T Y_p + \frac{1}{4}\bar{C}_p^T \bar{C}_p & Y_p E_p \\ E_p^T Y_p & -I \end{bmatrix} \begin{bmatrix} M_p^T \\ \mathbf{0} \end{bmatrix}^{\perp T} < \mathbf{0}, \quad (21)$$

$$Y_p - \frac{1}{4}X_p^{-1} > \mathbf{0}, \quad (22)$$

then there exists an absolutely stabilizing controller of the form (17).

Moreover, if the positive definite matrices X_p and Y_p satisfy the LMI's (20), (21) and the matrix inequality (22), then suitable A_c , B_c and C_c are obtained as follows:

1) Choose a real number r_1 such that

$$\begin{bmatrix} \bar{A}_p X_p + X_p \bar{A}_p^T + \frac{1}{4}E_p E_p^T - 2r_1 B_p B_p^T & X_p \bar{C}_p^T \\ \bar{C}_p X_p & -I \end{bmatrix} < \mathbf{0}, \quad (23)$$

2) Choose a real number r_2 such that

$$\begin{bmatrix} Y_p \bar{A}_p + \bar{A}_p^T Y_p + \frac{1}{4}\bar{C}_p^T \bar{C}_p - 2r_2 M_p^T M_p & Y_p E_p \\ E_p^T Y_p & -I \end{bmatrix} < \mathbf{0}, \quad (24)$$

3) Define $H \triangleq -r_1 B_p^T X_p^{-1}$, $G \triangleq -r_2 Y_p^{-1} M_p^T$,

$$R_H \triangleq X_p^{-1}(\bar{A}_p + B_p H) + (\bar{A}_p + B_p H)^T X_p^{-1} + \frac{1}{4}X_p^{-1}E_p E_p^T X_p^{-1} + \bar{C}_p^T \bar{C}_p,$$

$$R_G \triangleq (\bar{A}_p + G M_p) Y_p^{-1} + Y_p^{-1}(\bar{A}_p + G M_p)^T + \frac{1}{4}Y_p^{-1} \bar{C}_p^T \bar{C}_p Y_p^{-1} + E_p E_p^T,$$

4) Choose a real number $k \in (0, 1)$ such that

$$Y_p R_G Y_p < \frac{1}{4}(1 - k)R_H, \quad (25)$$

5) Define $Z_p \triangleq Y_p - \frac{1}{4}X_p^{-1}$, $\bar{G} \triangleq Z_p^{-1}Y_p G$,

$$\Delta_1 \triangleq \frac{1}{4}k Z_p^{-1} \left(X_p^{-1} \bar{A}_p + \bar{A}_p^T X_p^{-1} + \frac{1}{4}X_p^{-1} E_p E_p^T X_p^{-1} + \bar{C}_p^T \bar{C}_p \right),$$

$$\Delta_2 \triangleq -\frac{1}{4}Z_p^{-1} \left((1 - k)H^T B_p^T X_p^{-1} - kX_p^{-1} B_p H \right),$$

6) Choose

$$A_c \triangleq \bar{A}_p + \frac{1}{4}E_p E_p^T X_p^{-1} + \bar{G} M_p + \Delta_1 + B_p H + \Delta_2,$$

$$B_c \triangleq -\bar{G},$$

and

$$C_c \triangleq H.$$

Proof. In this proof, we denote $C \triangleq [C_p \quad \mathbf{0}]$, $\bar{E} \triangleq \begin{bmatrix} E_p \\ \bar{C}_p \end{bmatrix}$ and

$$A \triangleq \begin{bmatrix} A_p + B_p C_c & -B_p C_c \\ A_p - B_c M_p + B_p C_c - A_c & -B_p C_c + A_c \end{bmatrix}.$$

Fig. 2. Two nonlinearities within sector $[0, 2]$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Similarly as in the proof of Lemma 1, it can be shown that if there exists a positive definite matrix $P \in \mathbb{R}^{2n \times 2n}$ and a positive real number τ such that

$$\begin{bmatrix} PA + A^T P - \tau C^T (S_1 S_2 + S_2 S_1) C & -P \bar{E} + \tau C^T (S_1 + S_2) \\ -\bar{E}^T P + \tau (S_1 + S_2) C & -2\tau I \end{bmatrix} < \mathbf{0}, \quad (26)$$

then (19) is absolutely stable. Note that the unknown $\tau > 0$ can be removed from (26) through dividing (26) by 2τ and defining $\bar{P} \triangleq \frac{1}{2\tau}P$. We see that (26) has a solution pair (P, τ) if and only if there is a positive definite solution $\bar{P}$ to

$$\bar{P} \bar{A} + \bar{A}^T \bar{P} + \frac{1}{4}\bar{C}^T \bar{C} + \bar{P} \bar{E} \bar{E}^T \bar{P} < \mathbf{0}, \quad (27)$$

where $\bar{C} \triangleq [\bar{C}_p \quad \mathbf{0}]$ and

$$\bar{A} \triangleq \begin{bmatrix} \bar{A}_p + B_p C_c & -B_p C_c \\ \bar{A}_p - B_c M_p + B_p C_c - A_c & -B_p C_c + A_c \end{bmatrix}.$$

Before moving on, by Finsler's lemma, (20) and (21) imply that there exist real numbers r_1 and r_2 such that (23) and (24) hold, respectively. By taking Schur complements, (23) is equivalent to $R_H < \mathbf{0}$, and similarly we have $R_G < \mathbf{0}$. In addition, we choose $\bar{P}$ to be

$$\bar{P} \triangleq \begin{bmatrix} \frac{1}{4}X_p^{-1} & \mathbf{0} \\ \mathbf{0} & Z_p \end{bmatrix}.$$

By straightforward computation, the block (1,1) of the left hand side of (27) turns out to be $\frac{1}{4}R_H$. The blocks (2,1) and (2,2) can be computed to be equal to $-\frac{1}{4}kR_H$ and $Y_p R_G Y_p - \frac{1}{4}R_H + \frac{1}{2}kR_H$, respectively. Thus the left hand side of (27) equals

$$\begin{bmatrix} \frac{1}{4}R_H & -\frac{1}{4}kR_H \\ -\frac{1}{4}kR_H & Y_p R_G Y_p - \frac{1}{4}R_H + \frac{1}{2}kR_H \end{bmatrix},$$

i.e.

$$\begin{bmatrix} \frac{1}{4}(1 - k)R_H & \mathbf{0} \\ \mathbf{0} & Y_p R_G Y_p - \frac{1}{4}(1 - k)R_H \end{bmatrix} + \frac{1}{4}k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes R_H.$$

Obviously, the first term in the above is negative definite, and the second term is negative semi-definite since $k > 0$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \geq \mathbf{0}$ and $R_H < \mathbf{0}$. Therefore, (26) as well as (27) holds. This completes the proof. $\square$

Fig. 3. The state trajectories regarding $\phi(\cdot)$ in Case 1. Fig. 4. The control input regarding $\phi(\cdot)$ in Case 1.

4. A simulation example

In this section we will consider a flexible joint robotic arm model as an application example of Lur'e systems and show how to compute absolutely stabilizing dynamic output feedback controllers. The corresponding simulation results of the closed-loop control systems will be also given, along with some discussions.

A type of flexible joint robotic arms can be modeled using the Lur'e system (1) [7], where $x = [x_1 \ x_2 \ x_3 \ x_4]^T$, $A_p = [0 \ 1 \ 0 \ 0; -48.6 \ -1.25 \ 48.6 \ 0; 0 \ 0 \ 0 \ 1; 19.5 \ 0 \ -16.17 \ 0]$, $B_p = [0 \ 21.6 \ 0 \ 0]^T$, $C_p = [0 \ 0 \ 1 \ 0]$, $E_p = [0 \ 0 \ 0 \ 3.33]^T$ and $M_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Its nonlinearity is represented by $\phi(x_3) = x_3 + \sin x_3$, which satisfies the sector-boundedness condi-

tion $\phi(x_3)(\phi(x_3) - 2x_3) \leq 0$, see the red curve in Fig. 2. Note that different from [7] (see Page 727 therein), here the dynamics of x_4 has been rewritten in order to render the new nonlinear part to lie inside the sector $[0, 2]$.

In the following two cases, we will apply our first and second approaches, respectively. In the first case, we will compute a suitable controller of the form (3) by using Theorem 1.

Case 1. From (12), we can compute $X_p > 0$ to be

$$X_p = \begin{bmatrix} 0.4305 & -0.6514 & 0.2940 & -1.0916 \\ -0.6514 & 10.2188 & 0.3089 & -1.5696 \\ 0.2940 & 0.3089 & 0.6282 & -0.8073 \\ -1.0916 & -1.5696 & -0.8073 & 7.9034 \end{bmatrix}.$$

Fig. 5. The state trajectories regarding $\psi(\cdot)$ in Case 1.

Fig. 6. The control input regarding $\psi(\cdot)$ in Case 1.

Here we enforce the left hand side of (14) to be strictly positive definite. Thus $Y_p > \mathbf{0}$ determined by LMI (13) and the strict version of (14) can be computed to be

$$Y_p = \begin{bmatrix} 32.2408 & 0.0115 & -0.0063 & 0.0089 \\ 0.0115 & 32.0406 & -1.2093 & -0.3112 \\ -0.0063 & -1.2093 & 34.7891 & -1.7252 \\ 0.0089 & -0.3112 & -1.7252 & 0.6567 \end{bmatrix}.$$

Hence the state dimension n_c of a possible controller can be chosen as $n_c = 4$ by (15). It is equal to the state space dimension of the flexible joint robotic arm. Without loss of generality, we choose

Y_c as $Y_c = I_4$. It follows that

$$Y_{pc} = \begin{bmatrix} 4.9737 & 0 & 0 & 0 \\ -0.1380 & 5.6440 & 0 & 0 \\ 0.5414 & -0.1516 & 5.6269 & 0 \\ -0.1790 & -0.0774 & -0.2664 & 0.5398 \end{bmatrix}$$

from (16) by using Cholesky decomposition. Then, a suitable $Y > \mathbf{0}$ is obtained. Consequently, a suitable $X > \mathbf{0}$ is derived by $X = Y^{-1}$ which is given in (28).

Fig. 7. The state trajectories regarding $\phi(\cdot)$ in Case 2. Fig. 8. The control input regarding $\phi(\cdot)$ in Case 2.

$$X = \begin{bmatrix} 0.4305 & -0.6514 & 0.2940 & -1.0916 & -2.5854 & 3.6364 & -1.9452 & 0.5892 \\ -0.6514 & 10.2188 & 0.3089 & -1.5696 & 4.2019 & -57.7501 & -2.1563 & 0.8472 \\ 0.2940 & 0.3089 & 0.6282 & -0.8073 & -1.9043 & -1.7107 & -3.7499 & 0.4358 \\ -1.0916 & -1.5696 & -0.8073 & 7.9034 & 7.0643 & 9.3485 & 6.6483 & -4.2659 \\ -2.5854 & 4.2019 & -1.9043 & 7.0643 & 16.7348 & -23.4576 & 12.5974 & -3.8130 \\ 3.6364 & -57.7501 & -1.7107 & 9.3485 & -23.4576 & 327.4086 & 12.1161 & -5.0458 \\ -1.9452 & -2.1563 & -3.7499 & 6.6483 & 12.5974 & 12.1161 & 23.8713 & -3.5884 \\ 0.5892 & 0.8472 & 0.4358 & -4.2659 & -3.8130 & -5.0458 & -3.5884 & 3.3025 \end{bmatrix} \quad (28)$$

Now a suitable real number r in (11) can be computed as $r = 10,866$ and $\Theta_x > \mathbf{0}$ is also known. So (10) provides a suitable F which can be appropriately partitioned to get

Fig. 9. The state trajectories regarding $\psi(\cdot)$ in Case 2.

Fig. 10. The control input regarding $\psi(\cdot)$ in Case 2.

$$A_c = 10^4 \begin{bmatrix} -1.1022 & 2.0013 & 0.1776 & -0.0569 \\ -0.0285 & 0.2499 & 0.0212 & -0.0074 \\ 0.2491 & -2.9095 & -0.1279 & -0.2999 \\ 0.1392 & -8.0287 & -0.3002 & -0.9877 \end{bmatrix},$$

$$B_c = 10^5 \begin{bmatrix} -0.5235 & 1.1507 \\ 0.0023 & 0.1454 \\ 0.0878 & -1.6542 \\ -0.0300 & -4.5570 \end{bmatrix},$$

$$C_c = 10^3 [0.3098 \quad -5.8578 \quad -0.2395 \quad -0.6580],$$

and

$$D_c = 10^4 [0.0806 \quad -3.3280].$$

Ignoring the physical quantities and units we take the initial state $x(0) = [1 \ 2 \ 3 \ 4]^T$. Setting the initial state of the dynamic controller (3) to be zero, the state trajectories of the closed-loop system (4) are plotted in Fig. 3. Clearly, our designed dynamic output feedback controller works. However, $v(t)$ varies greatly compared to $x(t)$ in transient state. This can be also seen from the plot of the control input given in Fig. 4, which is extremely large in transient state. Moreover, due to the very large controller matrices, it takes too much time to compute the state trajectories using Matlab and sometimes the computer will run out of memory.

In order to illustrate the feature of absolute stabilization for the dynamic output feedback controller (3), we replace $\phi(x_3)$ with the dead-zone nonlinearity

Fig. 11. The state trajectories regarding $\phi(\cdot)$ in Case 1 with measurement noise.

Fig. 12. The control input regarding $\phi(\cdot)$ in Case 1 with measurement noise.

$$\psi(x_3) = \begin{cases} \frac{3}{2}x_3 + 3 & x_3 < -2 \\ 0 & -2 \leq x_3 < 2 \\ \frac{3}{2}x_3 - 3 & x_3 \geq 2 \end{cases},$$

which is also sector-bounded within $[0, 2]$, see the orange curve in Fig. 2. Using the same initial states and the same $F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$, the corresponding state trajectories are plotted in Fig. 5 along with the plot of the control input in Fig. 6. Obviously, the same F works against $\psi(\cdot)$ within the same sector while there exists the same issue discussed above.

Case 2. In this case, we will compute a suitable dynamic output feedback controller of the form (17) by using Theorem 2. From (20), we can compute $X_p > \mathbf{0}$ to be

$$X_p = \begin{bmatrix} 0.3030 & -0.7051 & 0.1766 & -0.9492 \\ -0.7051 & 11.0552 & 0.3412 & -1.2957 \\ 0.1766 & 0.3412 & 0.4949 & -0.7042 \\ -0.9492 & -1.2957 & -0.7042 & 7.6735 \end{bmatrix}.$$

Then (22) becomes linear and together with LMI (21), a suitable $Y_p > \mathbf{0}$ is computed as

Fig. 13. The state trajectories regarding $\phi(\cdot)$ in Case 2 with measurement noise.

Fig. 14. The control input regarding $\phi(\cdot)$ in Case 2 with measurement noise.

$$Y_p = \begin{bmatrix} 3.7727 & 0.2374 & -0.7035 & 0.3210 \\ 0.2374 & 1.0201 & -0.0941 & 0.0119 \\ -0.7035 & -0.0941 & 1.7614 & -0.1749 \\ 0.3210 & 0.0119 & -0.1749 & 0.1095 \end{bmatrix}.$$

$$B_c = 10^3 \begin{bmatrix} 4.7265 & 0.0000 \\ 0.0000 & 4.9183 \\ 0.0000 & 1.2617 \\ 0.0000 & 7.8693 \end{bmatrix},$$

Similarly, by LMI's (23), (24) and (25), suitable real numbers r_1 , r_2 and $k \in (0, 1)$ are computed to be $r_1 = 0.9299$, $r_2 = 4632$ and $k = 0.9794$, respectively. Now it is straightforward to get

and

$$C_c = [-19.0775 \quad -3.4955 \quad 5.7736 \quad -2.4202].$$

Using the same initial states as in Case 1, the state trajectories of the closed-loop system (18) are plotted in Fig. 7. Clearly, this dynamic output feedback controller works well. The required control input is not large either, see Fig. 8.

Finally, we replace $\phi(\cdot)$ with $\psi(\cdot)$ in (18). The corresponding state trajectories are plotted in Fig. 9 along with the plot of the

$$A_c = 10^3 \begin{bmatrix} -4.8316 & -0.0155 & 0.0371 & -0.0126 \\ -0.4889 & -4.9996 & 0.1835 & -0.0557 \\ -0.0475 & -1.2683 & 0.0153 & -0.0051 \\ -0.5459 & -7.9553 & 0.1742 & -0.0697 \end{bmatrix},$$

control input in Fig. 10. Certainly, the feature of absolute stabilization holds.

Remark 3. If we compare the two design approaches I and II, from the simulation example we see that they perform quite differently. In particular, the obtained controllers using the two design approaches perform quite differently in transient state. More precisely, in order to drive the flexible joint robotic arm from the same initial states to the origin, the controller based on design approach I uses much more energy and its internal states vary greatly as compared to the states of the flexible joint robotic arm. A very powerful actuator would be needed, which could be expensive or does not even exist in practice. In the design approach I, all (linear) matrix inequalities are handled by algebraic techniques, while the choice of the dimension of the dynamic controller is flexible. In contrast, the design approach II is inspired by an $\mathcal{H}_\infty$ optimal control algorithm for linear systems against external and unknown disturbance [8]. The Lur'e-type nonlinearity is viewed as a kind of state-dependent disturbance acting on the linear part of the Lur'e system. Thus the design approach II is based on $\mathcal{H}_\infty$ -optimization. It is not unreasonable to believe that, in general, it achieves better transient performance than the first approach. Unfortunately, we have no rigorous analysis at the moment to provide comparisons between the two design approaches.

We note that in dynamic output feedback control, often measurement noise is involved. In Case 1, measurement noise is amplified by the very large direct feedthrough matrix D_c and then enter the channels of the system states directly. Thus, even if the dynamic controller itself is not sensitive to measurement noise, the performance of the closed-loop system can be destroyed. In Case 2, we have no such issue thanks to the fact that $D_c = \mathbf{0}$.

A simulation has also been performed considering an additive measurement noise, which is a Gaussian distributed random signal with mean zero and standard deviation 0.01. The simulation results are given below. Clearly, the dynamic controller obtained in Case 1 does not work at all in the presence of measurement noise, see Figs. 11 and 12. Both the state trajectories and the control input are expanding. In contrast, with the same measurement noise, the controller obtained in Case 2 still works and the effect of the measurement noise is not that prominent, see Fig. 13. Apart from smaller overshoot, the dynamic controller based on design approach II is not sensitive to measurement noise, see Fig. 14.

5. Conclusions

We have studied absolute stabilization of Lur'e systems under dynamic output feedback in the sense that the Lur'e-type nonlinearities are unknown. Only their sector bounds are assumed to be known. This setup is different from existing work on dynamic output feedback control for Lur'e systems, where the nonlinearities are always assumed to be known precisely and used to design the controllers. Two approaches are presented for the controller design. Our designed controllers can be computed using the Matlab LMI Control Toolbox. Their effectiveness has been validated through numerical simulation results on a Lur'e-type model of flexible joint robotic arms.

For future research it is appealing to design a separate observer for Lur'e systems without using the Lur'e-type nonlinearities themselves. Such observer is indeed stronger than the circle-criterion observer, and would be applied to, for example, output feedback control and fault diagnosis of Lur'e systems. The robustness will be also considered against incomplete feedback information and model uncertainties in the future.

Acknowledgments

This work was supported by the State Key Laboratory of Intelligent Control and Decision of Complex Systems, the Young Scientist Fund of the National Natural Science Foundation of China (Grant No. 61703099), the China Postdoctoral Science Foundation Funded Project (Grant No. 2017M621589), and grants from the Research Grants Council of Hong Kong (Grant No. CityU-11261516).

Appendix

Proof of Lemma 2. The existence of F and $P > \mathbf{0}$ satisfying (6) is equivalent to the existence of solutions F and $X > \mathbf{0}$ to

$$\begin{bmatrix} BFMX + (BFMX)^T + Q_X + \frac{1}{2}EE^T & XC^T \\ CX & -2(S_2 - S_1)^{-2} \end{bmatrix} < \mathbf{0}. \quad (29)$$

This can be seen by taking $X = P^{-1}$ and using appropriate Schur complements.

(Only if) Let X be a positive definite solution to (29). Define $Y = X^{-1}$. Then $Y > \mathbf{0}$ as well, and $XY = I$. We have $\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}^\perp = \begin{bmatrix} B^\perp & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$. Obviously, (8) holds. (29) implies that

$$\begin{bmatrix} YBFM + (YBFM)^T + Q_Y + \frac{1}{2}C^T(S_2 - S_1)^2C & YE \\ E^TY & -2I \end{bmatrix} < \mathbf{0},$$

where $Q_Y \triangleq Y(A - \frac{1}{2}E(S_1 + S_2)C) + (A - \frac{1}{2}E(S_1 + S_2)C)^T Y$, which then implies (9).

(if) By Finsler's lemma [4], (8) implies that there exists a real number r such that

$$\begin{bmatrix} rBB^T - Q_X - \frac{1}{2}EE^T & -XC^T \\ -CX & 2(S_2 - S_1)^{-2} \end{bmatrix} > \mathbf{0},$$

equivalently, $\Theta_X > \mathbf{0}$. Similarly, (9) implies that there exists a positive definite matrix $S > \mathbf{0}$ such that

$$\begin{bmatrix} M^T S^{-1} M - Q_Y - \frac{1}{2}C^T(S_2 - S_1)^2C & -YE \\ -E^TY & 2I \end{bmatrix} > \mathbf{0},$$

equivalently, $\Theta_Y \triangleq XM^T S^{-1} MX - Q_X - \frac{1}{2}XC^T(S_2 - S_1)^2CX - \frac{1}{2}EE^T > \mathbf{0}$.

Define

$$\Xi \triangleq rI - r^2 B^T \Theta_X^{-1} B + r^2 B^T \Theta_X^{-1} X M^T (M X \Theta_X^{-1} X M^T)^{-1} M X \Theta_X^{-1} B,$$

where $M X \Theta_X^{-1} X M^T$ is positive definite since $\Theta_X > \mathbf{0}$, $X > \mathbf{0}$ and M has full row rank. Obviously, $\Xi > \mathbf{0}$ if and only if there exists a positive definite matrix $Z > \mathbf{0}$ such that

$$rI - r^2 B^T \Theta_X^{-1} B + r^2 B^T \Theta_X^{-1} X M^T (M X \Theta_X^{-1} X M^T + Z)^{-1} M X \Theta_X^{-1} B > \mathbf{0},$$

or equivalently, using the matrix inversion lemma [10],

$$rI - r^2 B^T (\Theta_X + X M^T Z^{-1} M X)^{-1} B > \mathbf{0},$$

equivalently, using the Schur complement lemma,

$$\begin{bmatrix} rI & rB^T \\ rB & \Theta_X + X M^T Z^{-1} M X \end{bmatrix} > \mathbf{0},$$

which is equivalent to

$$\begin{aligned} & \Theta_X + X M^T Z^{-1} M X - rBB^T \\ &= -Q_X - \frac{1}{2}EE^T - \frac{1}{2}XC^T(S_2 - S_1)^2CX + X M^T Z^{-1} M X > \mathbf{0}. \end{aligned}$$

If we take $Z = S$, the above inequality is exactly $\Theta_y > \mathbf{0}$. This thus shows that $\Xi > \mathbf{0}$.

Now, clearly,

$$\begin{bmatrix} F + rB^T \Theta_x^{-1} X M^T (M X \Theta_x^{-1} X M^T)^{-1} \\ M^T [F + rB^T \Theta_x^{-1} X M^T (M X \Theta_x^{-1} X M^T)^{-1}]^T \end{bmatrix} M X \Theta_x^{-1} X$$

$$M^T [F + rB^T \Theta_x^{-1} X M^T (M X \Theta_x^{-1} X M^T)^{-1}]^T = \mathbf{0} < \Xi$$

for the particular choice $F \triangleq -rB^T \Theta_x^{-1} X M^T (M X \Theta_x^{-1} X M^T)^{-1}$. The latter inequality holds if and only if

$$(rB^T + FMX) \Theta_x^{-1} (rB^T + FMX)^T < rI,$$

which in turn is equivalent to

$$\begin{bmatrix} \Theta_x & (rB^T + FMX)^T \\ rB^T + FMX & rI \end{bmatrix} > \mathbf{0},$$

and to

$$\begin{aligned} \frac{1}{r} (rB^T + FMX)^T (rB^T + FMX) &< \Theta_x \\ &= rBB^T - Q_X - \frac{1}{2} EE^T - \frac{1}{2} XC^T (S_2 - S_1)^2 CX. \end{aligned}$$

It then follows that

$$\begin{aligned} BFMX + (BFMX)^T + \frac{1}{r} XM^T F^T FMX + Q_X + \frac{1}{2} EE^T \\ + \frac{1}{2} XC^T (S_2 - S_1)^2 CX < \mathbf{0}, \end{aligned}$$

which yields (29) since $\frac{1}{r} XM^T F^T FMX$ is always positive semi-definite. This completes the proof. $\square$

References

- [1] M. Arcak, P. Kokotović, Nonlinear observers: a circle criterion design and robustness analysis, *Automatica* 37 (12) (2001) 1923–1930.
- [2] M. Arcak, P. Kokotović, Observer-based control of systems with slope-restricted nonlinearities, *IEEE Trans. Autom. Control* 46 (7) (2001) 1146–1150.
- [3] J. de Bruin, A. Doris, N. van de Wouw, W. Heemels, H. Nijmeijer, Control of mechanical motion systems with non-collocation of actuation and friction: a Popov criterion approach for input-to-state stability and set-valued nonlinearities, *Automatica* 45 (2) (2009) 405–415.
- [4] T. Iwasaki, R. Skelton, All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas, *Automatica* 30 (8) (1994) 1307–1317.
- [5] K.-K. Kim, R. Braatz, Observer-based output feedback control of discrete-time Lur'e systems with sector-bounded slope-restricted nonlinearities, *Int. J. Robust Nonlinear Control* 24 (16) (2014) 2458–2472.
- [6] K.-K. Kim, R. Braatz, Robust static and fixed-order dynamic output feedback control of discrete-time parametric uncertain Lur'e systems: sequential SDP relaxation approaches, *Optim. Control Appl. Methods* 38 (1) (2017) 36–58.
- [7] R. Rajamani, Y. Cho, Existence and design of observers for nonlinear systems: relation to distance to unobservability, *Int. J. Control* 69 (5) (1998) 717–731.
- [8] C. Scherer, *The Riccati Inequality and State-Space H_∞ -Optimal Control*, University of Wuerzburg, 1991.
- [9] J. da Silva Jr., E. Castelan, J. Corso, D. Eckhard, Dynamic output feedback stabilization for systems with sector-bounded nonlinearities and saturating actuators, *J. Frankl. Inst.* 350 (3) (2013) 464–484.
- [10] R. Skelton, T. Iwasaki, K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, CRC Press, 1997.
- [11] A. Zemouche, R. Rajamani, G. Phanomchoeng, B. Boukroune, H. Rafaralahy, Circle criterion-based H_∞ observer design for Lipschitz and monotonic nonlinear systems - enhanced LMI conditions and constructive discussions, *Automatica* 85 (2017) 412–425.