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Rent-seeking with uncertain discriminatory power[☆]

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ABSTRACT

We consider a two-player Tullock rent-seeking contest with uncertain discriminatory power in the contest success function. We examine the cases where both players are either informed or uninformed about the size of the discriminatory power, as well as the case where only one player has private information about it. We show that in all three cases the contest has a unique (Bayesian) Nash equilibrium. In each case we characterize key properties of the equilibrium.

1. Introduction

Consider a situation where, say two, firms lobby a foreign politician to obtain a license for a monopoly position on a market in the politician's country. It might happen that both firms are uncertain about the local political culture, i.e. the extent to which the politician rewards their lobbying activities (Che and Gale, 1997). It can also occur that one of the firms is local while the other one is foreign. The local firm might be able to acquire information about the politician's responsiveness to lobbying efforts, for instance if it has direct contact with people close to the politician, whereas the foreign firm might not have access to these sources of information and remains uncertain about it. We investigate how firms' lobbying efforts and payoffs depend on this kind of uncertainty.

In order to do this, we consider a rent-seeking contest between two players competing for a single prize that will be awarded to one of them (Tullock, 1980). Each player can exert effort in order to increase the probability that he wins the contest. The winning probability of a player is given by a contest success function (CSF), which includes a discriminatory power parameter that represents the effectiveness of the players' efforts in determining their winning chances. For larger values of the parameter, winning becomes more sensitive to efforts. Skaperdas (1996) shows that this CSF is the only one that satisfies a set of reasonable axioms. In the standard contest the players have complete information. The contest then has a (unique) pure-strategy Nash equilibrium if and only if the (positive) discriminatory power parameter is smaller than or equal to two (Nti, 1999). The contest can be used to analyze lobbying behavior, as well as a variety of other settings, like patent races, litigation, political campaigns, etc. See Konrad (2009) for a comprehensive survey of the literature.

We examine the case where the value of the discriminatory power parameter is uncertain. For simplicity, it is either high or low with given probabilities, where both values are smaller than or equal to two. We call a player informed if he learns the true value of the parameter, and uninformed if he only knows the distribution. We investigate three cases: both players are informed, both players are uninformed, and one player is informed while the other is uninformed. Information is symmetric in the first two cases and asymmetric in the last case. We demonstrate that in each case there exists a unique pure-strategy (Bayesian) Nash equilibrium.

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We give the equilibria for the two cases with symmetric information in closed form; they closely correspond to the equilibrium of the case with complete information. It is not possible to derive the equilibrium of the case with asymmetric information in closed form. Instead we characterize the properties of that equilibrium. First, the equilibrium effort of the informed player is largest if the value of the discriminatory power parameter is high. This is intuitive since winning is more sensitive to efforts in that case, which makes the informed player compete more fiercely. The equilibrium effort of the uninformed player is of intermediate size, since this player takes into account the realization of both values of the discriminatory power parameter. Second, a priori it is not clear how the ex ante equilibrium efforts, winning probabilities and payoffs of the informed and uninformed player are related to each other. We find that the ex ante efforts of the two players are the same, but the informed player wins the contest with a larger probability than the uninformed player, and hence also has a larger payoff.

We also compare the equilibria of the three cases. We find that the ex ante aggregate efforts are smallest in the case with asymmetric information. The payoff of the informed player in the case with asymmetric information is larger than his payoff in the two cases with symmetric information. Interestingly, numerical analysis shows that the payoff of the uninformed player in the case with asymmetric information can be larger than his payoff in the two cases with symmetric information. Hence, both players can be better off in the equilibrium under asymmetric information.

A number of papers have considered rent-seeking Tullock contests with uncertainty about parameters of the model. The case with an uncertain prize is investigated by Hurley and Shogren (1998a, 1998b), Malueg and Yates (2004), Schoonbeek and Winkel (2006), Wärneryd (2003, 2013) and Wasser (2013); uncertain costs of effort by Fey (2008) and Ryvkin (2010); and an uncertain number of contestants by Münster (2006) and Myerson and Wärneryd (2006). Einy et al. (2015) and Ewerhart and Quartieri (2016) demonstrate the existence, respectively the existence and uniqueness, of a pure strategy Bayesian Nash equilibrium for general contests that allow for uncertainty in the prize, costs of efforts and CSF. Their models cover our contest as a special case, but only for the situation where the discriminatory power parameter is smaller than or equal to unity, which is smaller than the range of values considered by us. Moreover, Einy et al. and Ewerhart and Quartieri do not derive additional properties of their equilibria, like we do in our paper (see above). Epstein and Meale (2013) consider a two-player contest in which the CSF has a discriminatory power parameter equal to unity, while one player is uncertain about the ability of his opponent (a player has a larger ability than his opponent if his effort has a larger effect on the winning probability than the same effort would have if exerted by the opponent).¹

Grossmann (2014) investigates a model where two players are uncertain about which contest they will play, similar to our case with two uninformed players. In his model there are also two possible contests which differ with regard to the sensitivity of the winning probabilities to the players' efforts. Grossmann's model employs an alternative CSF in which a given noise parameter is added linearly to each other player's effort, see also Amegashie (2006). This CSF does not satisfy all axioms of Skaperdas (1996). We focus here on the CSF of the standard Tullock contest.²

The paper is organized as follows. Section 2 gives the model and Section 3 presents the equilibrium results. We conclude in Section 4. The Appendix presents proofs.

2. The model

Consider the standard Tullock contest where risk-neutral players 1 and 2 compete for a prize $V > 0$ that will be awarded to one of them. Without loss of generality, we may assume that $V = 1$. Let $x \geq 0$ and $y \geq 0$ be the nonrefundable efforts exerted by players 1 and 2, respectively. The probability that player 1 wins the contest is given by the CSF

$$p \equiv p(x, y; r) = \frac{x^r}{x^r + y^r} \quad (1)$$

if $x + y > 0$, where the discriminatory power parameter $0 < r \leq 2$ is given. We take $p = \frac{1}{2}$ if $x = y = 0$. Player 2 wins the contest with probability $1 - p$. We call x^r the impact of player 1's effort in the CSF. The impact is increasing in x ; it is strictly concave, proportional or strictly convex in x if, respectively, $0 < r < 1$, $r = 1$ or $1 < r \leq 2$. The same applies to player 2's impact of effort, i.e. y^r .

It can be verified that the discriminatory power parameter r denotes the elasticity of the ratio of the winning probabilities of players 1 and 2 with respect to their relative efforts (Wang, 2010; Yildirim, 2015).³ If r increases, then winning becomes more sensitive to efforts and less to exogenous uncertainty. The prize is allocated randomly to one of the players regardless of the efforts, if r converges to zero.

Notice also that the relative winning probability $p/(1 - p)$ is increasing in x . It is increasing in x at a decreasing, constant or increasing rate if, respectively, $0 < r < 1$, $r = 1$ or $1 < r \leq 2$. Hence, the size of r also indicates whether we have, respectively, decreasing, constant or increasing returns to scale of player 1's effort in determining the winning odds between players 1 and 2. Analogously, the same holds for the effort of player 2.⁴

Players maximize their own expected payoff, i.e. player 1 solves

$$\max_{x \geq 0} \frac{x^r}{x^r + y^r} - x, \quad (2)$$

¹ See also Clark (1997).

² Below we will further discuss the papers by Grossmann (2014) and Amegashie (2006).

³ Wang (2010) and Yildirim (2015) refer to r as the accuracy of the contest, Che and Gale (1997) call it the political culture associated with the contest.

⁴ See further Dari-Mattiacci and Parisi (2014).

and player 2 solves

$$\max_{y \geq 0} \frac{y^r}{x^r + y^r} - y. \quad (3)$$

For later use, we make the following remarks. First, each player will be active (i.e. his effort will be positive) in a pure-strategy Nash equilibrium. Second, the (expected) revenues of the two players are, respectively, $R_1(x, y; r) = p(x, y; r)$ and $R_2(x, y; r) = 1 - p(x, y; r)$, and the corresponding marginal revenues are $MR_1(x, y; r) = \partial p / \partial x$ and $MR_2(x, y; r) = -\partial p / \partial y$. The marginal costs (of effort) of both players are equal to unity. Hence, the first-order conditions (FOCs) of a pure-strategy Nash equilibrium are $MR_1(x, y; r) = 1$ and $MR_2(x, y; r) = 1$. From this we obtain that $x = y$, i.e. the players' efforts must be the same in equilibrium. Third, the FOC of player 1 can be written as

$$\frac{rx^{r-1}y^r}{(x^r + y^r)^2} = 1. \quad (4)$$

Note that (4) reduces to $\frac{r}{4x} = 1$ if $x = y > 0$. Denoting equilibrium values with a star, we find that $x^* = y^* = \frac{r}{4}$ and $p^* = \frac{1}{2}$. Importantly, efforts x^* and y^* rise linearly in r . The corresponding payoff of player i is $\pi_i^* = \frac{2-r}{4}$, $i = 1, 2$. Nti (1999) shows that the model has a (unique) pure-strategy Nash equilibrium if and only if $0 < r \leq 2$.⁵ Finally, given r , let $x = b(y; r)$ be player 1's best-response function (player 2's best-response function, $y = b(x; r)$, can be discussed similarly). It has a maximum value at $y = \frac{r}{4}$, with $\frac{r}{4} = b(\frac{r}{4}; r)$, while $b(y; r)$ increases if $y < \frac{r}{4}$ and decreases if $y > \frac{r}{4}$.⁶

Now suppose that $r = r_t$ with probability α_t , where $t = H, L$, $0 < r_L < r_H \leq 2$ and $0 < \alpha_t < 1$ are given (with $\alpha_L = 1 - \alpha_H$). The expected value of r is $\mathbb{E}(r) = \alpha_H r_H + \alpha_L r_L$. Let p_t denote the probability that player 1 wins the contest if $r = r_t$. The timing of the game is as follows: Nature draws the true value of r in stage 1; the actual contest takes place in stage 2. We consider three cases regarding the information received by the players in stage 1 about the realized value of r : (i) both players learn the true value of r (both are informed), (ii) both players do not learn the true value of r , they only know the distribution (both are uninformed); (iii) one player learns the true value of r whereas the other player remains uninformed and only knows the distribution (asymmetric information). Without loss of generality, we may assume that player 1 is informed and player 2 is uninformed in case (iii). Note that information is symmetric between the two players in cases (i) and (ii). We will characterize the pure-strategy (Bayesian) Nash equilibria of the three cases. For brevity we refer to them as equilibria.

3. The equilibrium results

We first consider the cases where information is symmetric between the two players. Take case (i), where both players are informed. In stage 2, the players then play a contest with payoffs similar to (2) and (3) contingent on the realized value of r . Using obvious notation, we denote an equilibrium in this case by $(\bar{x}_H, \bar{x}_L, \bar{y}_H, \bar{y}_L)$ and the corresponding probabilities that player 1 wins the contest by $\bar{p}_H$ and $\bar{p}_L$, respectively. The ex ante equilibrium effort of player 1 is $\mathbb{E}(\bar{x}) = \alpha_H \bar{x}_H + \alpha_L \bar{x}_L$, and the ex ante equilibrium probability that player 1 wins the contest is $\mathbb{E}(\bar{p}) = \alpha_H \bar{p}_H + \alpha_L \bar{p}_L$. In a similar way, we denote the ex ante equilibrium effort of player 2 by $\mathbb{E}(\bar{y})$, and the ex ante equilibrium payoffs of the two players by $\mathbb{E}(\bar{\pi}_1) = \mathbb{E}(\bar{p}) - \mathbb{E}(\bar{x})$ and $\mathbb{E}(\bar{\pi}_2) = 1 - \mathbb{E}(\bar{p}) - \mathbb{E}(\bar{y})$.

Next, examine case (ii) where both players are uninformed. Player 1 then solves

$$\max_{x \geq 0} \sum_t \alpha_t \left(\frac{x^{r_t}}{x^{r_t} + y^{r_t}} \right) - x, \quad (5)$$

while player 2 considers

$$\max_{y \geq 0} \sum_t \alpha_t \left(\frac{y^{r_t}}{x^{r_t} + y^{r_t}} \right) - y. \quad (6)$$

We denote an equilibrium in this case by $(\hat{x}, \hat{y})$, the corresponding ex ante probability that player 1 wins the contest by $\mathbb{E}(\hat{p})$, and the ex ante payoffs of the two players by $\mathbb{E}(\hat{\pi}_1) = \mathbb{E}(\hat{p}) - \hat{x}$ and $\mathbb{E}(\hat{\pi}_2) = 1 - \mathbb{E}(\hat{p}) - \hat{y}$.

We obtain the following result. The Proof is in the [Appendix](#).

Proposition 1. *We have the following:*

- (i) *The model where both players are informed has a unique equilibrium $(\bar{x}_H, \bar{x}_L, \bar{y}_H, \bar{y}_L)$, with $\bar{x}_t = \bar{y}_t = \frac{r_t}{4}$, $t = H, L$. Further, $\mathbb{E}(\bar{x}) = \mathbb{E}(\bar{y}) = \frac{\mathbb{E}(r)}{4}$, $\mathbb{E}(\bar{p}) = \frac{1}{2}$, and $\mathbb{E}(\bar{\pi}_1) = \mathbb{E}(\bar{\pi}_2) = \frac{2 - \mathbb{E}(r)}{4}$.*
- (ii) *The model where both players are uninformed has a unique equilibrium $(\hat{x}, \hat{y})$, with $\hat{x} = \hat{y} = \frac{\mathbb{E}(r)}{4}$. Further, $\mathbb{E}(\hat{p}) = \frac{1}{2}$ and $\mathbb{E}(\hat{\pi}_1) = \mathbb{E}(\hat{\pi}_2) = \frac{2 - \mathbb{E}(r)}{4}$.*

⁵ For a discussion of the mixed equilibria of the case with $2 < r < \infty$, see e.g. Ewerhart (2015). We obtain an all-pay auction if $r = \infty$; the prize is then awarded to the player exerting the largest effort.

⁶ Pérez-Castrillo and Verdier (1992) show that (i) $b(y; r) > 0$ for all $y > 0$, given $0 < r \leq 1$; (ii) $b(y; r) > 0$ for $0 < y < m(r)$ and $b(y; r) = 0$ for $y \geq m(r)$, given $1 < r \leq 2$, where $m(r) \equiv (r - 1)^{1-\frac{1}{r}}/r$; (iii) $db(y; r)/dy > 0$ if $0 < y < r/4$, and $db(y; r)/dy < 0$ if $y > r/4$ as long as $b(y; r) > 0$.

Fig. 1. (I) and (II) denote the best-response of player 1 for, respectively, $r_L = 1.2$ and $r_H = 1.8$; (III) and (IV) denote the best-response of player 2 for, respectively, $r_L = 1.2$ and $r_H = 1.8$.

We see that in equilibrium the players' ex ante efforts are identical and the same in case (i) and case (ii). Moreover, the ex ante equilibrium probability that player 1 wins the contest is the same in both cases and equal to one half. As a result, the expected equilibrium payoffs of the two players are identical and the same in both cases.

The interpretation of case (i) is immediate since it is composed of two standard Tullock contests with, respectively, $r = r_H$ and $r = r_L$, which occur with probabilities α_H and α_L . Fig. 1 illustrates the players' best-response functions for $r_H = 1.8$ and $r_L = 1.2$. The equilibrium efforts corresponding to the two possible values of r are located on the 45-degree line. The ex ante equilibrium efforts are given by a convex combination of these two pairs of efforts, with weights α_H and α_L . Case (ii) can be understood with the same kind of arguments as used for the standard Tullock contest in Section 2, see the remarks below (2) and (3). In particular, it follows easily that each player must be active in equilibrium. The players' (expected) revenues in case (ii) are $R_1(x, y; r_H, r_L) = \alpha_H p(x, y; r_H) + \alpha_L p(x, y; r_L)$ and $R_2(x, y; r_H, r_L) = \alpha_H (1 - p(x, y; r_H)) + \alpha_L (1 - p(x, y; r_L))$. Further, marginal revenues equal marginal costs for each player in equilibrium. From this, we obtain that players' equilibrium efforts must be the same. In turn, it also follows directly that equilibrium efforts are $\mathbb{E}(r)/4$, i.e. they are linear in $\mathbb{E}(r)$.

Remark: Amegashie (2006) examines an alternative CSF. Taking the case with two players, the probability that player 1 wins the contest is then given by

$$q \equiv q(x, y; \epsilon) = \frac{x + \epsilon}{x + \epsilon + y + \epsilon}, \quad (7)$$

where, for notational simplicity, the players' efforts are denoted again by x and y , and $\epsilon > 0$ is given. Player 2 wins with probability $1 - q$. Parameter ϵ measures how sensitive q is to a player's effort. A smaller ϵ means that the sensitivity to effort is higher. Contrary to (1), the CSF in (7) does not fulfill all axioms of Skaperdas (1996), i.e. it is not homogeneous of degree zero in the efforts and does not fulfill the axiom of independence of irrelevant alternatives.⁷ For details and further discussion, see Amegashie (2006). Grossmann (2014) considers a model where two players play with given probability $\lambda \in (0, 1)$ a contest with CSF (7) (contest A), and with probability $1 - \lambda$ a contest with a CSF given by $q(x, y; -\epsilon)$ (contest B). A smaller ϵ means that the sensitivity to effort is higher in contest A, but smaller in contest B. Grossmann's model is related to our case (ii) in the sense that in both instances the players are playing with given positive probabilities one of two possible contests which differ in their sensitivity to effort in the CSF. A change in ϵ simultaneously and 'symmetrically' affects both contest A and contest B in the model of Grossmann. On the other hand, a change in say r_H affects only the contest corresponding to this parameter in our model and not the contest associated with r_L . As a result, the equilibrium results are not the same in general. For example, in Grossmann's model there exists a unique (pure-strategy) equilibrium if and only if $0 < \epsilon < \frac{1}{4}$, i.e. if the sensitivity to effort is small enough in contest A and high enough in contest B (cf. Proposition 1 of Grossmann, which considers the case where the valuation of the prize is identical for both players similar to our model). Our model in case (ii) has a unique equilibrium if and only if $0 < r_H \leq 2$, i.e. if the sensitivity to effort is small enough in the contest related to r_H . Since $0 < r_L < r_H$ by assumption, the sensitivity to effort in the contest associated with r_L is then small enough as well. $\triangle$

Proceeding, consider case (iii) with asymmetric information, where player 1 is informed and player 2 is uninformed. We say that player 1 is of type t if r_t is realized, with $t = H, L$. Player 1 of type $t = H, L$ solves

$$\max_{x_t \geq 0} \frac{x_t^{r_t}}{x_t^{r_t} + y^{r_t}} - x_t, \quad (8)$$

⁷ Rai and Sarin (2009) presents alternative axioms characterizing (7).

and player 2 solves

$$\max_{y \geq 0} \sum_t \alpha_t \left(\frac{y^{r_t}}{x_t^{r_t} + y^{r_t}} \right) - y. \quad (9)$$

We denote an equilibrium in case (iii) by $(\tilde{x}_H, \tilde{x}_L, \tilde{y})$. The corresponding ex ante effort and probability of winning of player 1 are $\mathbb{E}(\tilde{x})$ and $\mathbb{E}(\tilde{p})$, respectively. The corresponding equilibrium payoffs of the two players are $\mathbb{E}(\tilde{\pi}_1) = \mathbb{E}(\tilde{p}) - \mathbb{E}(\tilde{x})$ and $\mathbb{E}(\tilde{\pi}_2) = 1 - \mathbb{E}(\tilde{p}) - \tilde{y}$.

There is no closed form equilibrium in the case with asymmetric information. However, we are able to present the following Proposition, which is proved in the [Appendix](#).

Proposition 2. *The model with asymmetric information has a unique equilibrium $(\tilde{x}_H, \tilde{x}_L, \tilde{y})$. In the equilibrium we have $0 < \tilde{x}_L < \frac{r_L}{4} < \tilde{y} < \tilde{x}_H < \frac{r_H}{4}$. Further, $\mathbb{E}(\tilde{x}) = \tilde{y}$, $\mathbb{E}(\tilde{p}) > \frac{1}{2}$, and $\mathbb{E}(\tilde{\pi}_1) > \mathbb{E}(\tilde{\pi}_2)$.*

We make five observations regarding the results of the Proposition. First, we see that in equilibrium the ex ante efforts of the two players are the same. In order to understand this, we use again arguments similar to those employed in Section 2 for the standard Tullock contest. In particular, it can be verified that both types of player 1 and player 2 must be active in equilibrium. Next, let $R_1(x_H, y; r_H) = p(x_H, y; r_H)$ and $R_1(x_L, y; r_L) = p(x_L, y; r_L)$ be the (expected) revenues of, respectively, the high-type and low-type of player 1, and $R_2(x_H, x_L, y; r_H, r_L) = \alpha_H(1 - p(x_H, y; r_H)) + \alpha_L(1 - p(x_L, y; r_L))$ the revenues of player 2. Using that in equilibrium marginal revenues equal marginal costs for both types of player 1 and player 2, it follows easily that players' ex ante efforts are the same. Second, the equilibrium effort of the high-type player 1 is larger than the effort of the low-type. This is intuitive since the relative winning probabilities are more sensitive to the relative efforts of the players if the discriminatory power parameter is high.

Fig. 2. Comparison of the equilibrium payoffs of player 2 in the cases (i), (ii) and (iii) for different values of α_H and r_H , given $r_L = 1.2$ (upper panel) and $r_L = 1.6$ (bottom panel).

Fig. 3. The benefit B (dotted line) and cost C (solid line) for player 2 in the case with asymmetric information versus the two cases with symmetric information for different values of α_H , given $r_H = 1.6, r_L = 1.2$.

Third, player 2's equilibrium effort is smaller than the effort of the high-type of player 1 and larger than the effort of the low-type. This is also intuitive since the uninformed player takes into account the realization of both r_H and r_L . It also implies that $\tilde{p}_L < \frac{1}{2} < \tilde{p}_H$, i.e. the high-type player 1 has a larger probability to win the contest than the uninformed player 2, whereas the opposite holds for the low-type player 1. Fourth, in case (i) each player exerts effort $\frac{r_L}{4}$ if r_t is realized ($t = H, L$). Starting from case (i), let us move to case (iii), i.e. player 2 now becomes uninformed and exerts effort $\tilde{y} \in (\frac{r_L}{4}, \frac{r_H}{4})$. The shape of the best-response curve of player 1 then implies that he exerts less effort in case (iii) than in case (i) for both possible values of r_t , i.e. $\tilde{x}_t < \frac{r_t}{4} = \bar{x}_t$ ($t = H, L$).⁸ Fifth, in equilibrium the ex ante payoff of player 1 is higher than the ex ante payoff of player 2, i.e. the informed player has an informational advantage. This can be understood as follows. If player 2 exerts effort $\tilde{y}$, both types of player 1 might (passively) set $\tilde{y}$ as well. Each type would then win the contest with probability one half and have payoff $\frac{1}{2} - \tilde{y}$, and thus player 1's ex ante payoff would be $\frac{1}{2} - \tilde{y}$. However, for each type of player 1 the (unique) best-response is different from $\tilde{y}$, and consequently player 1's ex ante payoff is larger than $\frac{1}{2} - \tilde{y}$. From this and $\mathbb{E}(\tilde{x}) = \tilde{y}$, it then follows that player 1's ex ante payoff is larger than that of player 2, while player 1 must have a higher ex ante probability to win the contest than player 2. We remark here that Wärneryd (2003) considers a two-player Tullock contest with asymmetric information about the value of the contested prize. Contrary to our case, in his model the uninformed player has the largest ex ante probability to win the prize.

Next, we compare the players' ex ante efforts and payoffs in the equilibria of the cases (i) - (iii). Using Propositions 1 and 2, we directly obtain the following.

Corollary 1. Consider the equilibria of the models where both players are either informed or uninformed and the equilibrium of the model with asymmetric information, given in Proposition 1 and Proposition 2, respectively. We have:

- (i) $\mathbb{E}(\tilde{x}) = \hat{x} > \mathbb{E}(\tilde{x})$ and $\mathbb{E}(\tilde{y}) = \hat{y} > \tilde{y}$.
- (ii) $\mathbb{E}(\tilde{\pi}_1) = \mathbb{E}(\hat{\pi}_1) < \mathbb{E}(\tilde{\pi}_1)$.

The interpretation of the corollary is straightforward. Part (i) shows that in equilibrium the ex ante effort of each player is smallest in the case with asymmetric information. Hence, if aggregate efforts are seen as social costs (e.g. if efforts denote bribing), then a social planner minimizing social costs (rent dissipation) prefers the situation with asymmetric information. On the other hand, if a contest organizer wants to maximize ex ante aggregate efforts (e.g. if they are expenditures in a patent race), then he prefers the cases with symmetric information.⁹

Part (ii) of Corollary 1 shows that player 1's equilibrium payoff is higher in the case with asymmetric information (where only he is informed) than in the two cases with symmetric information between the two players (under asymmetric information not only player 1's ex ante effort is smaller but also his ex ante winning probability is larger than under symmetric information). Thus, also in this sense, player 1 benefits from his private information. We are not able to find a related analytical result for player 2. Beforehand one might think that player 2 is worse off in the case with asymmetric information (where only he is uninformed) than in the two

⁸ In case (iii) player 1's best-response if r_t is realized ($t = H, L$) has a maximum value at $y = \frac{r_t}{4}$. Using $\tilde{y} \in (\frac{r_L}{4}, \frac{r_H}{4})$, we thus obtain $\tilde{x}_H = b(\tilde{y}; r_H) < \frac{r_H}{4}$ and $\tilde{x}_L = b(\tilde{y}; r_L) < \frac{r_L}{4}$.

⁹ The result that aggregate efforts are higher in cases (i) and (ii) than in case (iii) corresponds to the common wisdom in contest theory that, usually, aggregate efforts are maximized when the contest is symmetric, see e.g. Drugov and Ryvkin (2017) and Mealem and Nitzan (2016).

cases with symmetric information. However, numerical analysis reveals that this is not always true. The computations show that (a) $\mathbb{E}(\tilde{\pi}_2) < \mathbb{E}(\bar{\pi}_2)$ if both r_L and r_H are ‘small’ (regardless of α_H), (b) $\mathbb{E}(\tilde{\pi}_2) > \mathbb{E}(\bar{\pi}_2)$ if, given ‘small’ or ‘medium-sized’ r_L , both r_H is ‘large’ and α_H is ‘small’, and (c) $\mathbb{E}(\tilde{\pi}_2) > \mathbb{E}(\bar{\pi}_2)$ if both r_L and r_H are ‘large’ (regardless of α_H). Hence, there are situations where both players are better off under asymmetric information.¹⁰ Fig. 2 illustrates case (b) for $r_L = 1.2$ and case (c) for $r_L = 1.6$.

Remark: Note that $\mathbb{E}(\tilde{\pi}_2) > \mathbb{E}(\bar{\pi}_2)$ if and only if $B > C$, where $B \equiv B(\alpha_H, r_H, r_L) = \frac{\mathbb{E}(r)}{4} - \tilde{y} (> 0)$ denotes player 2’s benefit of exerting less effort in the case with asymmetric information and $C \equiv C(\alpha_H, r_H, r_L) = \frac{1}{2} - (1 - \mathbb{E}(\tilde{p})) = \mathbb{E}(\tilde{p}) - \frac{1}{2} (> 0)$ the cost of the lower probability of winning for player 2 in that case (versus the cases with symmetric information). Fig. 3 illustrates B and C for different values of α_H , given $r_L = 1.2$ and $r_H = 1.6$. The reactions of B and C to a change in α_H are quite similar. The same holds if we consider one-sided changes of either r_L or r_H . This hinders the analytical comparison of $\mathbb{E}(\tilde{\pi}_2)$ versus $\mathbb{E}(\bar{\pi}_2)$. $\triangle$

4. Conclusion

One of the basic elements of the standard two-player Tullock rent-seeking contest is the positive discriminatory power parameter. It measures the elasticity of the relative winning probabilities between the players with respect to their relative efforts. We have investigated the situation in which ex ante there is uncertainty about the size of this parameter. We have considered the cases where both players are either informed or uninformed about the realized value of the parameter, as well as the case with asymmetric information where only one player is informed about it. We have shown that in all three cases the contest has a unique equilibrium and derived key properties of these equilibria. As an example, mentioned in Section 1, our model can be applied to a situation where two firms lobby a politician who will award to one of them a license for a monopoly position on a market. The discriminatory power parameter then measures the responsiveness of the politician’s decision to the lobbying efforts. Our analysis implies that the aggregate lobbying expenditures will be smallest if one firm knows the value of the parameter whereas the other firm is uncertain about it. More generally, our findings thus suggest that asymmetric information about the size of the discriminatory power parameter might offer a new explanation for the results of empirical studies which indicate that rent-seeking expenditures are normally much lower than the value of the contested prize, cf. Hillman and Van Long (2017) and Wärneryd (2003).

Appendix. Proof of Propositions 1 and 2

Proof of Proposition 1. Part (i) follows directly. Next, the FOCs (the equilibrium is interior) associated with (5) and (6) are

$$\sum_t \frac{\alpha_t r_t \hat{x}^{t-1} \hat{y}^t}{(\hat{x}^t + \hat{y}^t)^2} = 1, \quad (\text{A.1})$$

$$\sum_t \frac{\alpha_t r_t \hat{x}^t \hat{y}^{t-1}}{(\hat{x}^t + \hat{y}^t)^2} = 1. \quad (\text{A.2})$$

It follows that $\hat{x} = \hat{y} = \frac{\mathbb{E}(r)}{4}$. We obtain part (ii) by noting that the SOC’s are fulfilled. $\blacksquare$

Next, we provide the Proof of Proposition 2 by presenting a number of intermediate steps. First, we give the following lemma.

Lemma A.1. *In any equilibrium of the model with asymmetric information we have $\tilde{x}_H > 0$, $\tilde{x}_L > 0$ and $\tilde{y} > 0$.*

Proof. Let $(\tilde{x}_H, \tilde{x}_L, \tilde{y})$ be an equilibrium. It is easy to see that there is no equilibrium where both types of player 1 exert zero effort, or where player 2 exerts zero effort. Hence, the FOC’s associated with the equilibrium are

$$\frac{r_H \tilde{x}_H^{r_H-1} \tilde{y}^{r_H}}{(\tilde{x}_H + \tilde{y})^2} \leq 1, \quad (\text{A.3})$$

$$\frac{r_L \tilde{x}_L^{r_L-1} \tilde{y}^{r_L}}{(\tilde{x}_L + \tilde{y})^2} \leq 1, \quad (\text{A.4})$$

$$\frac{\alpha_H r_H \tilde{x}_H^{r_H-1} \tilde{y}^{r_H-1}}{(\tilde{x}_H + \tilde{y})^2} + \frac{\alpha_L r_L \tilde{x}_L^{r_L-1} \tilde{y}^{r_L-1}}{(\tilde{x}_L + \tilde{y})^2} = 1, \quad (\text{A.5})$$

and at least one of the inequality signs in (A.3) and (A.4) can be replaced by an equality sign.

We show that $\tilde{x}_L > 0$. Suppose, on the contrary, that $\tilde{x}_L = 0$. Then (A.3) holds with an equality sign. Hence, (A.3) and (A.5) imply that $\tilde{y} = \alpha_H \tilde{x}_H$. Substituting this in (A.3), we find that

$$\tilde{x}_H = \frac{r_H \alpha_H^{r_H}}{(1 + \alpha_H^{r_H})^2}, \quad (\text{A.6})$$

¹⁰ The fact that both players might be better off in our case with asymmetric information corresponds to other results that show that, given that uncertainty decreases effort, it can increase a player’s expected payoff. For a recent example, see Kahana and Klunover (2016).

and thus

$$\tilde{y} = \frac{r_H \alpha_H^{r_H+1}}{(1 + \alpha_H^{r_H})^2}. \quad (\text{A.7})$$

If player 1 of type L plays $\tilde{x}_L = 0$, his payoff is zero. If, instead, he would play $x_L = \tilde{y}$, with $\tilde{y}$ given by (A.7), his payoff is

$$\frac{1}{2} - \frac{r_H \alpha_H^{r_H+1}}{(1 + \alpha_H^{r_H})^2} = \frac{1}{2} - r_H \alpha_H \times \frac{\alpha_H^{r_H}}{(1 + \alpha_H^{r_H})^2} > 0. \quad (\text{A.8})$$

The latter inequality follows since $0 < r_H \alpha_H < 2$ and the fraction behind this term in (A.8) is smaller than $\frac{1}{4}$. Hence, $\tilde{x}_L = 0$ is not a best-response for player 1 of type L to $\tilde{y}$ of (A.7). We have a contradiction, and it follows that $\tilde{x}_L > 0$. In the same way we can show that $\tilde{x}_H > 0$. ■

Lemma A.1 implies that the FOCs can be written as

$$\frac{r_H \tilde{x}_H^{r_H-1} \tilde{y}^{r_H}}{(\tilde{x}_H^{r_H} + \tilde{y}^{r_H})^2} = 1, \quad (\text{A.9})$$

$$\frac{r_L \tilde{x}_L^{r_L-1} \tilde{y}^{r_L}}{(\tilde{x}_L^{r_L} + \tilde{y}^{r_L})^2} = 1, \quad (\text{A.10})$$

$$\alpha_H \tilde{x}_H + \alpha_L \tilde{x}_L = \tilde{y}. \quad (\text{A.11})$$

Note that (A.11) means that the ex ante equilibrium efforts of players 1 and 2 are the same, i.e. $\mathbb{E}(\tilde{x}) = \tilde{y}$. Suppose now that $\tilde{x}_L = \tilde{y}$. Then (A.11) implies that $\tilde{x}_H = \tilde{y}$. Substituting this in (A.9) and (A.10), we find that $r_H = 4\tilde{y}$ and $r_L = 4\tilde{y}$, which is inconsistent with $r_H \neq r_L$. So, $\tilde{x}_L \neq \tilde{y}$. In a similar way, we can show that $\tilde{x}_H \neq \tilde{y}$. Next, suppose that $\tilde{x}_L = \tilde{x}_H$. Then (A.11) implies that $\tilde{x}_L = \tilde{x}_H = \tilde{y}$, which gives a contradiction. Hence, $\tilde{x}_L \neq \tilde{x}_H$. As a result, the following two possible cases remain: $0 < \tilde{x}_H < \tilde{y} < \tilde{x}_L$ and $0 < \tilde{x}_L < \tilde{y} < \tilde{x}_H$.

Proceeding, write $\tilde{x}_H = \tilde{\lambda}_H \tilde{y}$ and $\tilde{x}_L = \tilde{\lambda}_L \tilde{y}$. We know that we have two possible cases: $0 < \tilde{\lambda}_H < 1 < \tilde{\lambda}_L$ and $0 < \tilde{\lambda}_L < 1 < \tilde{\lambda}_H$. We will show that only the latter case can be relevant. The equilibrium $(\tilde{x}_H, \tilde{x}_L, \tilde{y})$ can equivalently be denoted as $(\tilde{\lambda}_H, \tilde{\lambda}_L, \tilde{y})$. The FOCs characterizing the equilibrium can be rewritten as

$$\frac{r_H}{\tilde{y}} \times \frac{\tilde{\lambda}_H^{r_H-1}}{(\tilde{\lambda}_H^{r_H} + 1)^2} = 1, \quad (\text{A.12})$$

$$\frac{r_L}{\tilde{y}} \times \frac{\tilde{\lambda}_L^{r_L-1}}{(\tilde{\lambda}_L^{r_L} + 1)^2} = 1, \quad (\text{A.13})$$

$$\alpha_H \tilde{\lambda}_H + \alpha_L \tilde{\lambda}_L = 1. \quad (\text{A.14})$$

In order to analyze these FOCs, we first introduce two auxiliary functions. Given $0 < r \leq 2$ and $0 < y < 1$, we define

$$f(\lambda) \equiv f(\lambda; r, y) = \frac{r \lambda^{r-1}}{y}, \quad (\text{A.15})$$

and

$$g(\lambda) \equiv g(\lambda; r) = (\lambda^r + 1)^2, \quad (\text{A.16})$$

for $\lambda > 0$. Considering the graph of $f(\lambda)$ and $g(\lambda)$ with λ along the horizontal axis, we present the following two lemmas.

Lemma A.2. *We have the following:*

- (i) $f(\lambda)$ and $g(\lambda)$ are twice continuously differentiable in λ .
- (ii) $f(\lambda)$ is strictly decreasing in λ if $0 < r < 1$, constant if $r = 1$, and strictly increasing in λ if $1 < r \leq 2$. Moreover, $f(\lambda)$ is strictly convex in λ if $0 < r < 1$, strictly concave in λ if $1 < r < 2$, and linear in λ if $r = 2$.
- (iii) $g(\lambda)$ is strictly increasing in λ . Moreover, $g(\lambda)$ is strictly convex in λ if $1 \leq r \leq 2$.
- (iv) $f(1) > 4$ if $0 < y < \frac{r}{4}$, $f(1) = 4$ if $y = \frac{r}{4}$, and $f(1) < 4$ if $\frac{r}{4} < y < 1$.
- (v) $\lim_{\lambda \rightarrow 0} f(\lambda) = \infty$ if $0 < r < 1$, $\lim_{\lambda \rightarrow 0} f(\lambda) = 0$ if $1 < r \leq 2$, $\lim_{\lambda \rightarrow \infty} f(\lambda) = 0$ if $0 < r < 1$, and $\lim_{\lambda \rightarrow \infty} f(\lambda) = \infty$ if $1 < r \leq 2$.
- (vi) $\lim_{\lambda \rightarrow 0} g(\lambda) = 1$, $g(1) = 4$, and $\lim_{\lambda \rightarrow \infty} g(\lambda) = \infty$.

Proof. Part (i) is obvious. Next, note that

$$f'(\lambda) = \frac{r(r-1)\lambda^{r-2}}{y}. \quad (\text{A.17})$$

Hence, $f'(\lambda) < 0$ if $0 < r < 1$ and $f'(\lambda) > 0$ if $1 < r \leq 2$. Further,

$$f''(\lambda) = \frac{r(r-1)(r-2)\lambda^{r-3}}{y}. \quad (\text{A.18})$$

So, $f''(\lambda) > 0$ if $0 < r < 1$ and $f''(\lambda) < 0$ if $1 < r < 2$. Part (ii) follows directly. Next,

$$g'(\lambda) = 2r\lambda^{r-1}(\lambda^r + 1) > 0, \quad (\text{A.19})$$

and

$$g''(\lambda) = 2r\lambda^{r-2}[(2r-1)\lambda^r + r-1]. \quad (\text{A.20})$$

So, $g''(\lambda) > 0$ if $1 \leq r \leq 2$. Part (iii) follows directly. Part (iv) follows from $f(1) = \frac{r}{y}$. Parts (v) and (vi) follow easily. ■

Lemma A.3. *We have the following:*

- (i) Let $0 < r \leq 1$. Then $f(\lambda)$ and $g(\lambda)$ intersect once, say at $\lambda = \lambda_1 > 0$. We have $\lambda_1 > 1$ if $0 < y < \frac{r}{4}$, $\lambda_1 = 1$ if $y = \frac{r}{4}$, and $\lambda_1 < 1$ if $\frac{r}{4} < y < 1$. Further, $f(\lambda) - g(\lambda) > 0$ if $0 < \lambda < \lambda_1$, whereas $f(\lambda) - g(\lambda) < 0$ if $\lambda > \lambda_1$.
- (ii) Let $1 < r \leq 2$ and $0 < y \leq \frac{r}{4}$. Then $f(\lambda)$ and $g(\lambda)$ intersect twice, say at $\lambda = \lambda_0$ and $\lambda = \lambda_1$, with $0 < \lambda_0 < \lambda_1$. We have $\lambda_0 < 1 < \lambda_1$ if $0 < y < \frac{r}{4}$, while $\lambda_1 = 1$ if $y = \frac{r}{4}$. Further, $f(\lambda) - g(\lambda) < 0$ if either $0 < \lambda < \lambda_0$ or $\lambda > \lambda_1$, whereas $f(\lambda) - g(\lambda) > 0$ if $\lambda_0 < \lambda < \lambda_1$.
- (iii) Let $1 < r \leq 2$ and $\frac{r}{4} < y < 1$. Define

$$\hat{y}(r) \equiv \frac{(r+1)^2}{4r} \left(\frac{r-1}{r+1} \right)^{1-\frac{1}{r}}.$$

We have three cases. First, if $\frac{r}{4} < y < \hat{y}(r)$, then $f(\lambda)$ and $g(\lambda)$ intersect twice, say at $\lambda = \lambda_0$ and $\lambda = \lambda_1$. We have $0 < \lambda_0 < \lambda_1 < 1$. Further, $f(\lambda) - g(\lambda) < 0$ if either $0 < \lambda < \lambda_0$ or $\lambda > \lambda_1$, whereas $f(\lambda) - g(\lambda) > 0$ if $\lambda_0 < \lambda < \lambda_1$. Second, if $y = \hat{y}(r)$, then $f(\lambda)$ and $g(\lambda)$ intersect once, say at $\lambda = \lambda_1 > 0$. We have $\lambda_1 < 1$. Further, $f(\lambda) - g(\lambda) < 0$ if either $0 < \lambda < \lambda_1$ or $\lambda > \lambda_1$. Third, if $\hat{y}(r) < y < 1$, then $f(\lambda)$ and $g(\lambda)$ do not intersect. We have $f(\lambda) - g(\lambda) < 0$ for all $\lambda > 0$.

Proof. Take $0 < r \leq 1$. Note that $f(1) > 4 = g(1)$ if and only if $0 < y < \frac{r}{4}$. Part (i) follows now easily from Lemma A.2. Figure A.1 illustrates the case with $0 < r < 1$ of part (i). The case $r = 1$ can be illustrated similarly.

Next, take $1 < r \leq 2$. Using Lemma A.2, we now have $\lim_{\lambda \downarrow 0} f(\lambda) = 0 < \lim_{\lambda \downarrow 0} g(\lambda) = 1$. Further, $f(\lambda)$ is strictly concave if $1 < r < 2$ and linear if $r = 2$, while $g(\lambda)$ is strictly convex. Again, $f(1) > 4 = g(1)$ if and only if $0 < y < \frac{r}{4}$. Next, $f(\lambda)$ and $g(\lambda)$ have a unique point of intersection if and only if their graphs are tangent, i.e.

$$f'(\lambda) = g'(\lambda) \Leftrightarrow 4\sigma(r-1)\lambda^{r-2} = 2r\lambda^{r-1}(\lambda^r + 1), \quad (\text{A.21})$$

$$f(\lambda) = g(\lambda) \Leftrightarrow 4\sigma\lambda^{r-1} = (\lambda^r + 1)^2, \quad (\text{A.22})$$

where $\sigma \equiv \frac{r}{4y}$. From (A.22), we see that

$$4\sigma = \frac{(\lambda^r + 1)^2}{\lambda^{r-1}}. \quad (\text{A.23})$$

Substituting this into (A.21), we find

$$(r-1)(\lambda^r + 1) = 2r\lambda^r, \quad (\text{A.24})$$

which can be solved for λ , i.e.

$$\lambda = \hat{\lambda}(r) \equiv \left(\frac{r-1}{r+1} \right)^{\frac{1}{r}}. \quad (\text{A.25})$$

Substituting $\hat{\lambda}(r)$ in (A.23) and using $y = \frac{r}{4\sigma}$, we obtain

$$y = \hat{y}(r) = \frac{r(\hat{\lambda}(r))^{r-1}}{((\hat{\lambda}(r))^r + 1)^2} = \frac{(r+1)^2}{4r} \left(\frac{r-1}{r+1} \right)^{1-\frac{1}{r}}. \quad (\text{A.26})$$

It can be verified that $\frac{r}{4} < \hat{y}(r) < 1$. Part (ii) and part (iii) follow now easily. Figure A.2 illustrates the case with $1 < r < 2$. ■

Fig. A.1. Functions $f(\cdot)$ and $g(\cdot)$ defined in (A.15) and (A.16), for the case with $0 < r < 1$ and, respectively, $y < r/4$, $y = r/4$ and $y > r/4$.

Fig. A.2. Functions $f(\cdot)$ and $g(\cdot)$ defined in (A.15) and (A.16), for the case with $1 < r < 2$ and, respectively, $y < r/4$, $y = r/4$ and $y = \hat{y}(r)$.

We also obtain the following auxiliary corollary.

Corollary A.1. Consider $f(\lambda; r, y)$, $f(\lambda; r, y^0)$ and $g(\lambda; r)$ defined according to (A.15) and (A.16), with $0 < y < y^0 < 1$. Given y and y^0 , denote λ_1 defined in parts (i), (ii), and (iii) of Lemma A.3 as, respectively, $\lambda_1(y)$ and $\lambda_1(y^0)$. We then have $\lambda_1(y^0) < \lambda_1(y)$.

Proof. The proof follows easily from Lemma A.2 and (the proof of) Lemma A.3, by noting that $g(\lambda)$ is independent of y , while $f(\lambda; r, y^0) < f(\lambda; r, y)$ for all $\lambda > 0$. ■

Using FOCs (A.12) – (A.14) and these auxiliary results, we establish the next lemma.

Lemma A.4. In any equilibrium of the model with asymmetric information we have $0 < \tilde{x}_L < \frac{r_L}{4} < \tilde{y} < \tilde{x}_H < \frac{r_H}{4}$. Furthermore, there is at most one equilibrium.

Proof. Let $(\tilde{x}_H, \tilde{x}_L, \tilde{y})$ be an equilibrium. Using Lemma A.1, we know that $\tilde{x}_H > 0$, $\tilde{x}_L > 0$ and $\tilde{y} > 0$. Using (A.12), it easily follows that $r_H/(\tilde{\lambda}_H \tilde{y}) > 4$, so $\tilde{x}_H < \frac{r_H}{4}$. Similarly, (A.13) implies that $\tilde{x}_L < \frac{r_L}{4}$. Using (A.11), it also follows that $\tilde{y} < \frac{r_H}{4} < 1$.

Next, taking $f(\cdot)$ from (A.15) and $g(\cdot)$ from (A.16) with $r = r_H$ and $y = \tilde{y}$, consider

$$f(\lambda; r_H, \tilde{y}) - g(\lambda; r_H) = 0 \quad (\text{A.27})$$

and

$$f(\lambda; r_L, \tilde{y}) - g(\lambda; r_L) = 0. \quad (\text{A.28})$$

Note that (A.12) can be rewritten as (A.27) with $\lambda = \tilde{\lambda}_H$, while (A.13) can be rewritten as (A.28) with $\lambda = \tilde{\lambda}_L$.

Using $0 < \tilde{y} < \frac{r_H}{4} < 1$ and parts (i) and (ii) of Lemma A.3, we find the following. First, take $0 < r_H \leq 1$. Then there is a unique λ_1 such that (A.27) holds, with $\lambda_1 > 1$. Note that λ_1 corresponds to a maximum of (8) with $t = H$ and $y = \tilde{y}$, since $f(\lambda; r_H, \tilde{y}) - g(\lambda; r_H, \tilde{y}) > 0$ if $0 < \lambda < \lambda_1$, whereas $f(\lambda; r_H, \tilde{y}) - g(\lambda; r_H, \tilde{y}) < 0$ if $\lambda > \lambda_1$. Hence, we must have $\tilde{\lambda}_H = \lambda_1$, and thus $0 < \tilde{\lambda}_L < 1 < \tilde{\lambda}_H$. Given $\tilde{\lambda}_H$, the value of $\tilde{\lambda}_L$ follows uniquely from (A.14). Second, take $1 < r_H \leq 2$. Then there are two values of λ for which (A.27) holds. They can be denoted by λ_0 and λ_1 , with $0 < \lambda_0 < 1 < \lambda_1$. Only λ_1 corresponds to a maximum of (8) with $t = H$ and $y = \tilde{y}$, while λ_0 corresponds to a minimum. To see this, note that $f(\lambda; r_H, \tilde{y}) - g(\lambda; r_H, \tilde{y}) < 0$ if either $0 < \lambda < \lambda_0$ or $\lambda > \lambda_1$, whereas $f(\lambda; r_H, \tilde{y}) - g(\lambda; r_H, \tilde{y}) > 0$ if $\lambda_0 < \lambda < \lambda_1$. Hence, the unique relevant solution of (A.27) is $\lambda_1 > 1$. It follows again that we must have $\tilde{\lambda}_H = \lambda_1$, and thus $0 < \tilde{\lambda}_L < 1 < \tilde{\lambda}_H$. Given $\tilde{\lambda}_H$, the value of $\tilde{\lambda}_L$ follows again uniquely from (A.14). In sum, $\tilde{x}_H$ and $\tilde{x}_L$ are uniquely defined given $\tilde{y}$, and $0 < \tilde{x}_L < \tilde{y} < \tilde{x}_H$.

Next, observe that $\tilde{y} > \frac{r_L}{4}$. In order to see this, suppose on the contrary, to begin with, that $\tilde{y} < \frac{r_L}{4}$. Using parts (i) and (ii) of Lemma A.3, we then find the following. First, if $0 < r_L \leq 1$, there is a unique λ_1 such that (A.28) holds, with $\lambda_1 > 1$. It follows that we must have $\tilde{\lambda}_L = \lambda_1$. However, this is inconsistent with $0 < \tilde{\lambda}_L < 1 < \tilde{\lambda}_H$. Second, if $1 < r_L < 2$, there are two values of λ such that (A.28) holds. They can be denoted by λ_0 and λ_1 , with $0 < \lambda_0 < 1 < \lambda_1$. We see that only λ_1 corresponds to a maximum of (8) with $t = L$ and $y = \tilde{y}$. It follows that $\tilde{\lambda}_L = \lambda_1$. However, this is inconsistent with $0 < \tilde{\lambda}_L < 1 < \tilde{\lambda}_H$. Hence, $\tilde{y} < \frac{r_L}{4}$ cannot be true. In turn, suppose that $\tilde{y} = \frac{r_L}{4}$. Using parts (i) and (ii) of Lemma A.3 and similar reasoning as before, we now obtain the following. If $0 < r_L \leq 1$, there is a unique λ_1 such that (A.28) holds, with $\lambda_1 = 1$. If $1 < r_L < 2$, there is a unique relevant λ_1 such that (A.28) holds, with $\lambda_1 = 1$. In both instances we obtain again an inconsistency with $0 < \tilde{\lambda}_L < 1 < \tilde{\lambda}_H$. So, $\tilde{y} = \frac{r_L}{4}$ cannot be true. This establishes the first statement of the lemma.

Turning to the second statement, suppose there are two equilibria, $(\tilde{x}_H, \tilde{x}_L, \tilde{y})$ and $(\tilde{\tilde{x}}_H, \tilde{\tilde{x}}_L, \tilde{\tilde{y}})$, or equivalently, $(\tilde{\lambda}_H, \tilde{\lambda}_L, \tilde{y})$ and $(\tilde{\tilde{\lambda}}_H, \tilde{\tilde{\lambda}}_L, \tilde{\tilde{y}})$. We must have $\tilde{\tilde{y}} \neq \tilde{y}$. We may assume without loss of generality that $\tilde{\tilde{y}} > \tilde{y}$. Also note that $\tilde{y} < \frac{r_H}{4} < 1$ and $\tilde{\tilde{y}} < \frac{r_H}{4} < 1$. Using $f(\tilde{\lambda}_H; r_H, \tilde{y}) - g(\tilde{\lambda}_H; r_H, \tilde{y}) = 0$, $f(\tilde{\tilde{\lambda}}_H; r_H, \tilde{\tilde{y}}) - g(\tilde{\tilde{\lambda}}_H; r_H, \tilde{\tilde{y}}) = 0$ and Corollary A.1, it follows that $\tilde{\tilde{\lambda}}_H < \tilde{\lambda}_H$. Next, we also have $\tilde{\tilde{y}} > \frac{r_L}{4}$ and $\tilde{\tilde{y}} > \frac{r_L}{4}$. Using $f(\tilde{\lambda}_L; r_L, \tilde{y}) - g(\tilde{\lambda}_L; r_L, \tilde{y}) = 0$, $f(\tilde{\tilde{\lambda}}_L; r_L, \tilde{\tilde{y}}) - g(\tilde{\tilde{\lambda}}_L; r_L, \tilde{\tilde{y}}) = 0$ and Corollary A.1, we find that $\tilde{\tilde{\lambda}}_L < \tilde{\lambda}_L$. However, we now have a contradiction since (A.14) and

$$\alpha_H \tilde{\lambda}_H + \alpha_L \tilde{\lambda}_L = 1 \quad (\text{A.29})$$

cannot hold simultaneously. We conclude that there is at most one equilibrium. ■

Proceeding, given $0 < r \leq 2$, $0 < \alpha_H < 1$ and $\alpha_L = 1 - \alpha_H$, we define

$$h(\lambda; r) = \frac{r\lambda^{r-1}}{(\lambda^r + 1)^2}, \quad (\text{A.30})$$

for $\lambda \in [1, \frac{1}{\alpha_H}]$, and

$$k(\lambda; r) = \frac{r \left(\frac{1 - \alpha_H \lambda}{\alpha_L} \right)^{r-1}}{\left(\left(\frac{1 - \alpha_H \lambda}{\alpha_L} \right)^r + 1 \right)^2}, \quad (\text{A.31})$$

for $\lambda \in [1, \frac{1}{\alpha_H}]$. We present the following result.

Lemma A.5. *We have the following:*

- (i) $h(\lambda; r)$ and $k(\lambda; r)$ are continuously differentiable in λ .
- (ii) $h(\lambda; r)$ is strictly decreasing in λ .
- (iii) $h(1; r) = \frac{r}{4}$ and $0 < h(\frac{1}{\alpha_H}; r) < \frac{r}{4}$.
- (iv) $k(1; r) = \frac{r}{4}$.
- (v) Take $0 < r_L < r_H \leq 2$. There exists some $\tilde{\lambda} \in (1, \frac{1}{\alpha_H})$ such that

$$h(\tilde{\lambda}; r_H) = k(\tilde{\lambda}; r_L). \quad (\text{A.32})$$

Proof. The proof of parts (i) to (iv) is immediate. In order to show part (v), note that $k(1; r_L) < h(1; r_H)$. We now consider three cases. First, take $0 < r_L < 1$. Then $\lim_{\lambda \uparrow \frac{1}{\alpha_H}} k(\lambda; r_L) = +\infty \left(> h\left(\frac{1}{\alpha_H}; r_H\right) \right)$. It follows that there exists some $\tilde{\lambda} \in (1, \frac{1}{\alpha_H})$ satisfying (A.32). Second, take $r_L = 1$. Then $\lim_{\lambda \uparrow \frac{1}{\alpha_H}} k(\lambda; r_L) = 1 > h\left(\frac{1}{\alpha_H}; r_H\right)$. Again there exists some $\tilde{\lambda} \in (1, \frac{1}{\alpha_H})$ satisfying (A.32). Third, take $1 < r_L < 2$. Then $\lim_{\lambda \uparrow \frac{1}{\alpha_H}} k(\lambda; r_L) = 0 (< k(1; r_L))$. We see that $k'(\lambda; r_L) = 0$ if and only if

$$\left(\frac{1 - \alpha_H \lambda}{\alpha_L} \right)^{r_L} = \frac{r_L - 1}{r_L + 1}. \quad (\text{A.33})$$

Using this, we find that $k(\lambda; r_L)$ has a maximum at $\lambda^{\max} \in (1, \frac{1}{\alpha_H})$, with

$$k(\lambda^{\max}; r_L) = \frac{(r_L - 1)^{1 - \frac{1}{\alpha_L}} (r_L + 1)^{1 + \frac{1}{\alpha_L}}}{4r_L}, \quad (\text{A.34})$$

with $k(\lambda^{\max}; r_L) > \frac{1}{2} > h(1; r_H)$. Again there exists some $\tilde{\lambda} \in (1, \frac{1}{\alpha_H})$ satisfying (A.32). ■

Proof of Proposition 2. Using (A.12) and (A.30), we obtain $\tilde{y} = h(\tilde{\lambda}_H; r_H)$, while (A.13), (A.14) and (A.31) imply $\tilde{y} = k(\tilde{\lambda}_H; r_L)$. Combining, we see that in equilibrium we must have

$$h(\tilde{\lambda}_H; r_H) = k(\tilde{\lambda}_H; r_L). \quad (\text{A.35})$$

Using part (v) of Lemma A.5, we know that there exists some $\tilde{\lambda}_H \in (1, \frac{1}{\alpha_H})$ satisfying (A.35). This demonstrates existence of an equilibrium. Uniqueness of the equilibrium and the second statement of Proposition 2 follow from Lemma A.4.

Finally, consider the last statement of the Proposition. We know that $\mathbb{E}(\tilde{x}) = \tilde{y}$ from (A.11). Next, we show that $\mathbb{E}(\tilde{p}) > \frac{1}{2}$. First, take $r = r_H$ and let player 2 exert effort $\tilde{y}$. If player 1 plays $x_H = \tilde{y}$ in this case, his payoff is equal to $\pi_H = \frac{1}{2} - \tilde{y} > 0$ (since $\tilde{y} < \frac{r_H}{4} \leq \frac{1}{2}$). If player 1 plays his best-response $\tilde{x}_H$ to $\tilde{y}$, his payoff is $\tilde{\pi}_H = \tilde{p}_H - \tilde{x}_H > \pi_H$ (the inequality follows since $\tilde{x}_H \neq \tilde{y}$). Next, take $r = r_L$ and let player 2 exert effort $\tilde{y}$. If player 1 plays $x_L = \tilde{y}$ in this case, his payoff is equal to $\pi_L = \frac{1}{2} - \tilde{y} > 0$. If player 1 plays his best-response $\tilde{x}_L$ to $\tilde{y}$, his payoff is $\tilde{\pi}_L = \tilde{p}_L - \tilde{x}_L > \pi_L$ (the inequality follows since $\tilde{x}_L \neq \tilde{y}$). Combining and using (A.11), we see that

$$\alpha_H \tilde{\pi}_H + \alpha_L \tilde{\pi}_L = \mathbb{E}(\tilde{p}) - (\alpha_H \tilde{x}_H + \alpha_L \tilde{x}_L) = \mathbb{E}(\tilde{p}) - \tilde{y} > \alpha_H \pi_H + \alpha_L \pi_L = \frac{1}{2} - \tilde{y}. \quad (\text{A.36})$$

Hence, we obtain the result. In turn, $\mathbb{E}(\tilde{\pi}_1) > \mathbb{E}(\tilde{\pi}_2)$ follows immediately. ■

References

- Amegashie, J.A., 2006. A contest success function with a tractable noise parameter. *Publ. Choice* 126, 135–144.
- Che, Y.-K., Gale, I., 1997. Rent dissipation when rent seekers are budget constrained. *Publ. Choice* 92, 109–126.
- Clark, D.J., 1997. Learning the structure of a simple rent-seeking game. *Publ. Choice* 93, 119–130.
- Dari-Mattiacci, G., Parisi, F., 2014. Returns to effort in rent-seeking games. *Publ. Choice* 159, 99–104.
- Drugov, M., Ryvkin, D., 2017. Biased contests for symmetric players. *Game. Econ. Behav.* 103, 116–144.
- Einy, E., Haimanko, O., Moreno, D., Sela, A., Shitovitz, B., 2015. Equilibrium existence in Tullock contests with incomplete information. *J. Math. Econ.* 61, 241–245.
- Epstein, G.S., Meale, Y., 2013. Who gains from information asymmetry? *Theor. Decis.* 75, 305–337.
- Ewerhart, C., 2015. Mixed equilibria in Tullock contests. *Econ. Theor.* 60, 59–71.
- Ewerhart, C., Quartieri, F., 2016. Unique Equilibrium in Incomplete Information Contests with Budget Constraints. Working Paper no. 133. University of Zurich, Switzerland.
- Fey, M., 2008. Rent-seeking contests with incomplete information. *Publ. Choice* 135 (3–4), 225–236.
- Grossmann, M., 2014. Uncertain contest success function. *Eur. J. Polit. Econ.* 33, 134–148.
- Hillman, A.L., Van Long, N., 2017. Rent Seeking: the Social Costs of Contestable Benefits. CESifo Working Paper no. 6462, Munich, Germany.
- Hurley, T., Shogren, J.F., 1998a. Effort levels in a Cournot Nash contest with asymmetric information. *J. Publ. Econ.* 69, 195–210.
- Hurley, T., Shogren, J.F., 1998b. Asymmetric information contests. *Eur. J. Polit. Econ.* 14, 645–665.
- Kahana, N., Klunover, D., 2016. Complete rent dissipation when the number of rent seekers is uncertain. *Econ. Lett.* 141, 8–10.
- Konrad, K.A., 2009. *Strategy and Dynamics in Contests*. Oxford University Press, Oxford.
- Malueg, D.A., Yates, A.J., 2004. Rent seeking with private values. *Publ. Choice* 119, 161–178.
- Meale, Y., Nitzan, S., 2016. Discrimination in contests: a survey. *Rev. Econ. Des.* 20, 145–172.
- Münster, J., 2006. Contests with an unknown number of contestants. *Publ. Choice* 129, 353–368.
- Myerson, R.B., Wärneryd, K., 2006. Contests with an unknown number of contestants. *Econ. Theor.* 27, 469–474.
- Nti, K.O., 1999. Rent-seeking with asymmetric valuations. *Publ. Choice* 98, 415–430.
- Pérez-Castrillo, J.D., Verdier, T., 1992. A general analysis of rent-seeking games. *Publ. Choice* 73, 335–350.
- Rai, B.K., Sarin, R., 2009. Generalized contest success functions. *Econ. Theor.* 40, 139–149.
- Ryvkin, D., 2010. Contests with private costs: beyond two players. *Eur. J. Polit. Econ.* 26, 558–567.
- Schoonbeek, L., Winkel, B.M., 2006. Activity and inactivity in a rent-seeking contest with private information. *Publ. Choice* 127 (1), 123–132.
- Skaperdas, S., 1996. Contest success functions. *Econ. Theor.* 7, 283–290.
- Tullock, G., 1980. Efficient rent seeking. In: Buchanan, J.M., Tollison, R.D., Tullock, G. (Eds.), *Toward a Theory of the Rent-seeking Society*. Texas A&M University Press, College Station, pp. 97–112.
- Wang, Z., 2010. The optimal accuracy level in asymmetric contests. *B E J. Theor. Econ.* 10, 1–16.
- Wärneryd, K., 2003. Information in conflicts. *J. Econ. Theor.* 110, 121–136.
- Wärneryd, K., 2013. Common-value contests with asymmetric information. *Econ. Lett.* 120, 525–527.
- Wasser, C., 2013. Incomplete information in rent-seeking contests. *Econ. Theor.* 53, 239–268.
- Yildirim, M., 2015. Accuracy in contests: players' perspective. *Rev. Econ. Des.* 19, 67–90.