University of Groningen

Critical points of the integral map of the charged 3-body problem

Hoveijn, I.; Waalkens, H.; Zaman, M.

Published in:
Indagationes Mathematicae

DOI:
[10.1016/j.indag.2018.09.006](https://doi.org/10.1016/j.indag.2018.09.006)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2019

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Hoveijn, I., Waalkens, H., & Zaman, M. (2019). Critical points of the integral map of the charged 3-body problem. *Indagationes Mathematicae*, 30(1), 165-196. <https://doi.org/10.1016/j.indag.2018.09.006>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Critical points of the integral map of the charged three-body problem

I. Hoveijn*, H. Waalkens, M. Zaman

*Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen,
The Netherlands*

Received 12 July 2018; received in revised form 24 September 2018; accepted 24 September 2018

Communicated by H.W. Broer

Abstract

This is the first in a series of three papers where we study the integral manifolds of the charged three-body problem. The integral manifolds are the fibers of the map of integrals. Their topological type may change at critical values of the map of integrals. Due to the non-compactness of the integral manifolds one has to take into account besides ‘ordinary’ critical points also critical points at infinity. In the present paper we concentrate on ‘ordinary’ critical points and in particular elucidate their connection to central configurations. In a second paper we will study critical points at infinity. The implications for the Hill regions, i.e. the projections of the integral manifolds to configuration space, are the subject of a third paper. © 2018 Royal Dutch Mathematical Society (KWG). Published by Elsevier B.V. All rights reserved.

Keywords: Charged three-body problem; Integral manifold; Critical point; Relative equilibrium; Central configuration

1. Introduction

In this paper we consider a system of N point masses with position vectors $q_i \in \mathbb{R}^3$ and masses $m_i > 0$ whose dynamics is described by Newton’s second law of motion $m_i \ddot{q}_i = F_i(q_1, \dots, q_N)$, $i = 1, \dots, N$, with forces

$$F_i(q_1, \dots, q_N) = \sum_{i \neq j} F_{ij}(q_i, q_j),$$

* Corresponding author.

E-mail address: ihoveijn@zonnet.nl (I. Hoveijn).

where $F_{ij}(q_i, q_j)$ is the force on the point mass i at position q_i due to the point mass j at position q_j which is assumed to be of the form

$$F_{ij}(q_i, q_j) = -\gamma_{ij} \frac{1}{\|q_i - q_j\|^3} (q_i - q_j). \quad (1)$$

Here $\gamma_{ij} = \gamma_{ji} \in \mathbb{R}$ for $i \neq j$. Examples are gravitational forces where $\gamma_{ij} = m_i m_j > 0$, electrostatic forces where $\gamma_{ij} = -Q_i Q_j$ with Q_i being the charge of the point mass i which can be positive or negative, or combinations of gravitational and electrostatic forces. In the gravitational case the forces F_{ij} are always attractive. This case is well studied even though also here there are many open questions. For $N = 2$ and different signs of the charges Q_1 and Q_2 , the electrostatic force is also attractive and repulsive otherwise. For $N \geq 3$, the electrostatic forces will always involve repulsion, namely repulsion only if all charges have the same sign, or coexisting repulsive and attractive forces when the charges do not all have the same sign. So it is to be expected that for $N \geq 3$, systems interacting via electrostatic forces behave qualitatively differently from systems interacting via gravitational forces. We will refer to the point masses as bodies (as is common in celestial mechanics where the point masses represent planets or stars) or particles (as is more appropriate in the case of electrostatic forces where we think of the point masses representing electrons or atomic nuclei). We will mainly be concerned with the case of three bodies or particles, i.e. $N = 3$. The forces are singular when $q_i = q_j$ for some $i \neq j$. We define the *collision set*

$$\Delta_c = \{(q_1, \dots, q_N) \in \mathbb{R}^{3N} \mid q_i = q_j \text{ for some } i < j\}.$$

The configuration space is then given by $\mathcal{X} = \mathbb{R}^{3N} \setminus \Delta_c$ and we will often write q for $(q_1, \dots, q_N)$ in $\mathcal{X}$.

The forces F_i of the form (1) are conservative. In fact, for $V : \mathcal{X} \rightarrow \mathbb{R}$ with

$$V(q) = \sum_{i < j} -\frac{\gamma_{ij}}{\|q_i - q_j\|},$$

we have $F_i = -\partial_{q_i} V$.

We wish to consider the system above as a Hamiltonian dynamical system. Therefore we introduce the *phase space* $M = T^*\mathcal{X} \simeq \mathcal{X} \times \mathbb{R}^{3N}$, the cotangent bundle of $\mathcal{X}$, which is endowed with the symplectic two-form $\omega = \sum_i dp_i \wedge dq_i$ where q and p are the canonical coordinates on $T^*\mathcal{X}$. The equations of motion can now be derived from the function $H : M \rightarrow \mathbb{R}$ given by $H(q, p) = \sum_i \frac{1}{2m_i} \|p_i\|^2 + V(q)$ from $(\dot{q}, \dot{p}) = X_H(q, p)$ where X_H is the vector field defined by $dH(\cdot) = \omega(X_H, \cdot)$. We obtain the familiar Hamilton equations

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = \frac{1}{m_i} p_i \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i} \end{aligned} \quad (2)$$

which are equivalent to Newton's equations above.

The potential V is invariant under the special Euclidean group $\text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$ which is the semi-direct product of the group of rotations $\text{SO}(3)$ and the group of translations $\mathbb{R}^3$, i.e. $V(\text{R}(q_1 + a), \dots, \text{R}(q_N + a)) = V(q_1, \dots, q_N)$ for all $\text{R} \in \text{SO}(3)$ and $a \in \mathbb{R}^3$. Many of the results presented in this paper will only depend on this property. An $\text{SE}(3)$ -invariant potential that only consists of pairwise interactions can be considered as a function of the inter particle distances $r_{ij} = \|q_i - q_j\|$, $1 \leq i < j \leq N$ only (see [20] for $\text{SE}(3)$ -invariants). We use this

property to define an N -body system to which then many of the presented results will apply, and we define the charged N -body problem as a special case for which then more can be said than in the case of a general N -body system.

Definition 1.1. For $M = T^*\mathcal{X}$ where $\mathcal{X} = \mathbb{R}^{3N} - \Delta_c$ with Δ_c being a closed subset of $\mathbb{R}^{3N}$ (the collision set), we call the Hamiltonian system (M, ω, H) an N -body system with masses $m_i > 0$ of the i th body, $i = 1, \dots, N$, if H is of the form kinetic plus potential energy

$$H = T + V,$$

where the kinetic energy is $T(p) = \sum_i \frac{1}{2m_i} p_i^2$ and V is a function of the $\text{SE}(3)$ -invariants $r_{ij} = \|q_i - q_j\|$, $1 \leq i < j \leq N$, only. We will refer to the system that has potential

$$V(q) = \sum_{i < j} -\frac{\gamma_{ij}}{r_{ij}(q)}$$

with $\gamma_{ij} \in \mathbb{R}$ as the *charged N -body system*.

Recall that in case of electrostatic interactions the coefficients γ_{ij} are given by $-Q_i Q_j$ with the Q_i being the charges of the particles. We note that even for $N = 3$, the map $\mathbb{R}^N \rightarrow \mathbb{R}^{N(N-1)/2}$, $(Q_1, \dots, Q_N) \mapsto (Q_1 Q_2, Q_1 Q_3, \dots, Q_1 Q_N, Q_2 Q_3, \dots, Q_2 Q_N, \dots, Q_{N-1} Q_N)$ is not bijective and not even surjective. But we will ignore this fact. Our definition of a charged N -body system also includes the gravitational N -body system. Many of the results will not depend on the special form of the potential that defines a charged N -body system but only on the $\text{SE}(3)$ symmetry that more generally defines an N -body system. These results then also apply, e.g., to an N -atomic molecules where the potential might be a complicated function obtained from the Born–Oppenheimer approximation.

Finding solutions to the equations of motion is in general very difficult. Exploiting the symmetries, the problem of two bodies, $N = 2$, can be reduced to one degree of freedom. The gravitational Kepler problem is the most famous example (see, e.g., [5] for a detailed account of the solution). For $N = 3$, the situation is however already considerably more complicated. Even for the well-studied case of the gravitational three-body problem only few solutions are known. The most prominent ones are the ones found by Euler and Lagrange where the three bodies move along ellipses with the three bodies being at each instant of time on a line or at the vertices of an equilateral triangle, respectively. These solutions belong to the class of so called *homographic* solutions where the bodies move individually along conic sections and together form configurations that remain invariant under similarity transformations. An important building block for such solutions are *central configurations* which are special points in configuration space that have the property that for each body i , the force F_i is proportional to the distance vector of the i th body to the common center of mass where the proportionality constant is the same for all bodies [10].

Definition 1.2. A *central configuration* is a point $q = (q_1, \dots, q_N)$ in configuration space for which there is a *multiplier* $\lambda \in \mathbb{R}$ such that

$$\frac{\partial}{\partial q_i} V = \lambda m_i (q_i - Q) \quad \text{for all } i \in \{1, \dots, N\}.$$

Here $Q = \frac{1}{m} \sum_i m_i q_i$ is the center of mass.

Only fairly recently another class of solutions given by so called *choreographies* like the figure-eight orbit have been found. They were first found numerically by C. Moore [14] and later proved to exist using variational techniques by A. Chenciner and R. Montgomery [7].

Instead of studying individual solutions we restrict ourselves to the study of more global properties of N -body systems. By definition N -body systems have many symmetries. By Noether's theorem (and its generalizations in terms of momentum maps) the symmetries entail the existence of integrals in addition to the Hamiltonian function H , i.e. functions $M \rightarrow \mathbb{R}$ whose values are constant along solution curves of Hamilton's equations (2). The solution curves are hence constrained to the joint level sets of these integrals. What we will be interested in is the geometry of these integral manifolds. For the gravitational three-body problem, the topology of the integral manifolds has been studied in great detail [13]. Using the homogeneity of the gravitational potential it can be shown that they depend on a single scalar parameter for which there are nine critical values at which the topology (apart from one exception) changes. The situation is complicated by the non-compactness of the integral manifolds. This requires one to not only study 'ordinary' critical points of the map of integrals but also critical points at infinity (see the work of Albouy [2] where this notion has been made precise). In a series of three papers we carry out a program that provides the first steps towards a similar study as that in [13] to the charged three-body problem. In the present first paper we will study 'ordinary' critical points of the map of integrals. In the second paper we will study critical points at infinity. In the third paper we study the Hill regions which are the projections of the integral manifolds to the configuration space $\mathcal{X}$.

The main results of the present paper concern the number of critical points related to central configurations in a charged three-body system.

Theorem 1.3. *Consider a charged three-body system as in Definition 1.1. For each value of the masses $m = (m_1, m_2, m_3) \in \mathbb{R}_{>0}^3$, there can be zero, one, two or three critical points associated to collinear central configurations, depending on the values of the parameters γ_{ij} in the potential. There is at most one critical point associated to a non-collinear central configuration.*

This theorem follows from Proposition 4.12 and Theorem 4.15. Central configurations play a subtle role in the study of critical points of the map of integrals. In fact there are two types of critical points of the map of integrals of which one is given by relative equilibria, i.e. equilibria of the system reduced by its symmetries. We will show that for every central configuration (of a charged three-body system) with a positive multiplier, there is a relative equilibrium in the sense that the relative equilibrium (a point in phase space) projects to a central configuration (a point in configuration space), Corollary 3.7. We note that for a charged N -body system, the situation is more complicated than for a gravitational N -body system. In the gravitational case every relative equilibrium projects to a planar central configuration. We will show that this remains to be the case also in the charged case when all γ_{ij} have the same sign (Proposition 3.11). In the case of mixed signs this need not be true as we demonstrate in the third paper [8].

Apart from the vast amount of physics literature on the helium atom with regard to semiclassical quantization (see, e.g., [19,17]), there is relatively little literature on charged N -body system especially compared to the gravitational case. The existing literature in the charged case then most of the time involves some kind of restricted system. Without trying to be complete we list a few starting with the severest restrictions. Solutions in a collinear charged three-body system are studied in [15,11]. Relative equilibria in a helium system with one charged mass fixed and two charged masses moving without restriction are studied in [18]. Relative equilibria

related to central configurations are studied in [16] for three-body systems and in [3] for N -body systems. There are not many results on stability of relative equilibria. We mention [4] for stability of relative equilibria projecting on central configurations in charged three-body systems. In [12] the stability of relative equilibria projecting on a triangular central configuration is related to a geometric property of the triangle. The charged three-body problem restricted to isosceles configurations is studied in [6]. Our results extend those of [16] in the sense that we consider the problem in a larger framework and our results encompass the full parameter space.

The present paper is organized as follows. In Section 2 we use the symmetries of N -body systems to define the map of integrals. Critical points of N -body systems with a general $\text{SE}(3)$ -invariant potential are the subject of Section 3. In Section 4 we consider central configurations in general charged three-body systems as defined above. We single out the central configurations related to critical points in Section 5. Some remarks and an outlook are given in Section 6. Some details are discussed in appendices.

2. Symmetries and the integral map

In the introduction we based our definition of an N -body system as a Hamiltonian system on the invariance of the potential under the action of the special Euclidean group $\text{SE}(3)$. In this section we make the notion of symmetries of Hamiltonian system more formal. We will distinguish between symplectic group actions such as translations and rotations and non-symplectic group actions. The symplectic group actions will lead to the definition of the integral map whose study is the main subject of this paper. The homogeneity of the potential for charged N -body systems gives rise to a non-symplectic group action which facilitates conclusions on the integral manifolds that go further than in the case of an N -body system which does not have this symmetry.

2.1. Symplectic actions

Here we use a general construction for group actions initially defined on the configuration space $\mathbb{R}^{3N}$ only. Here we ignore the fact that we might have to exclude a collision set Δ_c from the configuration space. The reason is that the collision set and its complement are invariant under the group actions of interest. Starting from group actions on $\mathbb{R}^{3N}$ it is then easy to define their restrictions to $\mathcal{X} = \mathbb{R}^{3N} \setminus \Delta_c$. First we lift this action to a (symplectic) action on the phase space $T^*\mathbb{R}^{3N}$. Then we construct a Hamiltonian function for the action of one-parameter subgroups. Throughout we assume that we consider an N -body problem as in Definition 1.1. Our exposition will be very short. For a more complete and detailed account, see for example Abraham and Marsden [1] or Arnol'd [5].

Our starting point is the action of a (smooth) group G on the configuration space $\mathbb{R}^{3N}$. For a fixed element g of the group, the group action is a diffeomorphism Φ_g on $\mathbb{R}^{3N}$. As such the cotangent lift of Φ_g is given by $\Phi_g^\sharp : T^*\mathbb{R}^{3N} \rightarrow T^*\mathbb{R}^{3N} : (q, p) \mapsto (\Phi_g(q), d_{\Phi_g(q)}(\Phi_g^{-1})^*(p))$. This map is symplectic on $T^*\mathbb{R}^{3N}$ by construction, i.e. $\Phi_g^{\sharp*}\omega = \omega$. Now let Φ_t be the action on $\mathbb{R}^{3N}$ of a one-parameter subgroup and let X be the vector field on $\mathbb{R}^{3N}$ generating the flow Φ_t . Then $F : T^*\mathbb{R}^{3N} \rightarrow \mathbb{R} : (q, p) \mapsto \sum_i p_i \xi_i(q)$ is the Hamiltonian of the flow $\Phi_t^\sharp$ on $T^*\mathbb{R}^{3N}$, where the ξ_i are the components of X . This means that F generates the flow $\Phi_t^\sharp$ in the sense that the vector field $X_F = \frac{d}{dt} \Phi_t^\sharp|_{t=0}$ on $T^*\mathbb{R}^{3N}$ satisfies $\omega(X_F, \cdot) = dF$.

The Hamiltonian system (M, ω, H) is said to be invariant under the symplectic action of the group G if for each fixed group element g , the pullback of H by the cotangent lift $\Phi_g^\sharp$ is equal to H , i.e. $\Phi_g^{\sharp*}H = H$. In terms of the function F generating the vector field X_F associated with the

action of a one-parameter subgroup this means $\{F, H\} = 0$ where $\{\cdot, \cdot\}$ is the Poisson bracket of F and H which is defined via the symplectic two-form ω as $\{F, H\} = \omega(X_F, X_H)$ which gives

$$\{F, H\} = \sum_i \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i}.$$

Translation symmetry. The potential V of an N -body system is by definition invariant under the translation group $\mathbb{R}^3$ acting on the configuration space $\mathbb{R}^{3N}$ as $(q_1, \dots, q_N) \mapsto (q_1 + a, \dots, q_N + a)$ for each $a \in \mathbb{R}^3$. The derivative of this action for a fixed group element is the identity. Therefore the cotangent lift of this action to $T^*\mathbb{R}^{3N}$ is given by

$$\Phi_a^\sharp : M \rightarrow M : (q_1, \dots, q_N, p_1, \dots, p_N) \mapsto (q_1 + a, \dots, q_N + a, p_1, \dots, p_N).$$

The translation group is a three-parameter group generated by the translations over $t e_j$ for $t \in \mathbb{R}$ and e_j the j th basis vector of $\mathbb{R}^3$. For each of these, the vector field generating the flow is constant, $X_j = \sum_i \frac{\partial}{\partial q_{i,j}}$. The integrals corresponding to this action are the components of total momentum $P(q, p) = \sum_i p_i$ [1,5]. The Hamiltonian H of the N -body system is invariant under $\Phi_a^\sharp$ as is easily checked: $H(\Phi_a^\sharp(q, p)) = H(q, p)$ for all $a \in \mathbb{R}^3$ and in each point $(q, p) \in M$. The commutativity of $\mathbb{R}^3$ implies $\{P_i, P_j\} = 0$ for $i, j \in \{1, 2, 3\}$.

Since the map P restricted to the levels of H does not have critical values, we may without loss of generality choose any value for P . The total momentum determines the time derivative of the center of mass $Q(q, p) = \frac{1}{m} \sum_i m_i q_i$, where $m = \sum_i m_i$ is the total mass, according to

$$\frac{d}{dt} Q = \{Q, H\} = \frac{1}{m} \sum_i m_i \frac{p_i}{m_i} = \frac{1}{m} P.$$

The implication is that choosing the value zero for P fixes the center of mass. Indeed, the time derivative of Q is zero. Thus the components of the center of mass are also integrals if total momentum equals zero. Since the function Q does not have critical points we may without loss of generality set the value of Q equal to zero as well.

Rotation symmetry. By definition the potential function V of an N -body system is also invariant under the action of the group $\text{SO}(3)$ on configuration space. The action of $\text{SO}(3)$ on $\mathbb{R}^{3N}$ is ‘diagonal’. Let $R \in \text{SO}(3)$ then each q_i is mapped to $R q_i$. The derivative of this map is R and since $(R^{-1})^T = R$ the cotangent lift of the action on $\mathbb{R}^{3N}$ to an action on $T^*\mathbb{R}^{3N}$ is given by

$$\Phi_R^\sharp : M \rightarrow M : (q_1, \dots, q_N, p_1, \dots, p_N) \mapsto (R q_1, \dots, R q_N, R p_1, \dots, R p_N).$$

The group $\text{SO}(3)$ is not commutative, but it is still a three-parameter group generated by rotations around the coordinate axes in $\mathbb{R}^3$. Each of these forms a one-parameter subgroup and the vector field generating the rotations are linear. For example, the rotations R_x around the x -axis are generated by

$$X = \sum_i \left(q_{i,3} \frac{\partial}{\partial q_{i,2}} - q_{i,2} \frac{\partial}{\partial q_{i,3}} \right).$$

So the Hamiltonian function of the flow is $\sum_i (p_{i,2} q_{i,3} - p_{i,3} q_{i,2})$ which is the first component of total angular momentum: the integrals of the $\text{SO}(3)$ -action are the components of angular momentum $L(q, p) = \sum_i q_i \times p_i$. Again the Hamiltonian H of the N -body system is invariant under $\Phi_R^\sharp$ for each $R \in \text{SO}(3)$ and thus each component of L satisfy $\{L_j, H\} = 0$, $j \in \{1, 2, 3\}$. However, since $\text{SO}(3)$ is not commutative, the integrals L_j do not Poisson commute. Instead they generate a Lie subalgebra isomorphic to $\mathfrak{so}(3)$ in the Lie algebra of smooth functions on M with the Poisson bracket.

2.2. Non-symplectic actions

For a charged N -body system, the Hamilton function H is according to Definition 1.1 of the form $H(q, p) = T(p) + V(q)$ where the kinetic energy T is a homogeneous function of degree 2 in p and the potential V is a homogeneous of degree -1 in q . Due to this special property it is useful to also consider a non-symplectic group action called *dilation symmetry*. The dilation group is given by the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ for which one can define an action on the extended phase space $M \times \mathbb{R}$ as

$$\Phi : \mathbb{R}^* \times M \times \mathbb{R} \rightarrow M \times \mathbb{R} : (\mu, q, p, t) \mapsto (\mu^\alpha q, \mu^\beta p, \mu^\gamma t).$$

If the exponents α, β and γ satisfy $\alpha = -2\beta$ and $\gamma = -3\beta$ the equations of motion are invariant. However, neither the Hamiltonian H nor the symplectic form ω are invariant in this case. They transform as $H \mapsto \mu^{2\beta} H$ and $\omega \mapsto \mu^{-\beta} \omega$.

2.3. Integrals and relative equilibria

Let (M, ω, H) be a Hamiltonian system and suppose that $\Phi_s : M \rightarrow M$ is a one-parameter group of symplectic diffeomorphisms such that H is invariant. Furthermore let F be the Hamiltonian function associated with the vector field X_F that generates Φ_s . Then we have

$$0 = \frac{d}{ds} H(\Phi_s) = dH(X_F) = \omega(X_H, X_F) = \{H, F\}.$$

This means that H is preserved by the flow of F . By the skew symmetry of the Poisson bracket, this also means that F is preserved under the flow of H . In fact, $\{H, F\} = 0$ means that the symplectic vector fields X_F and X_H commute, and equivalently the corresponding flows commute. A function F that is preserved by the flow of H has several names: *conserved quantity*, *conserved function*, *constant of motion* or *integral*. The existence of an integral has several important implications for the dynamics. The integral can, e.g., lead to a special type of solutions of the Hamiltonian system associated to the corresponding symmetry. This happens when the vector fields are linearly dependent at some point $(q, p) \in M$, i.e. $X_H(q, p) = \lambda X_F(q, p)$, for some $\lambda \in \mathbb{R}$. Due to the Φ_s -invariance this equation holds for the Φ_s -orbit through (q, p) . The orbit of Φ_s through (q, p) thus coincides with the solution curve of H through (q, p) . Such orbits are called *relative equilibria* because on the reduced system, where Φ_s -orbits are points, the reduced Hamiltonian has an equilibrium point.

From a slightly different point of view we also have the following. If the vector fields X_F and X_H are linearly dependent, the gradients of F and H are also linearly dependent and vice versa. Thus we get the equation $\text{grad}H(q, p) = \lambda \text{grad}F(q, p)$, which can be interpreted as the equation for critical points of H restricted to the levels of F . The conclusion is that relative equilibria are in one to one correspondence with critical points of H restricted to the levels of an integral.

2.4. The integral map of an N -body system

As mentioned above the existence of an integral F restricts the dynamics of H to level sets of the function F . This simplifies the analysis of the (dynamical) problem. In fact the Liouville theorem states that if for a Hamiltonian system there are sufficiently many integrals then the system is *integrable* [5]. More precisely if the system is $2n$ -dimensional, there must be n independent, Poisson commuting integrals. Considered as a Hamiltonian system, the N -body

system is $6N$ -dimensional. In Section 2.1 we have seen that the invariance of a N -body system under translations and rotations leads to 9 integrals, the three components of total momentum P and the center of mass Q which we all assume to have the value zero and the three components of the angular momentum L . Together with the Hamiltonian H this gives 10 integrals. All the 10 integrals commute with H with respect to the Poisson bracket. Even though the integrals do not commute mutually they render the 1-body and 2-body systems integrable. For $N \geq 3$, this however does not give enough integrals to guarantee integrability. Nevertheless the integrals can be used to reduce the dimension of the system. In fact our focus will be on the *integral manifolds*, i.e. the fibers of the *integral map*. We use the 10 integrals of an N -body system to define the *integral map* $\mathcal{F}$ as

$$\mathcal{F} : M \rightarrow \mathbb{R}^{10} : (q, p) \mapsto (H(q, p), L(q, p), P(q, p), Q(q, p)).$$

Rather than in the dynamics of the N -body system we will be interested in the fibers of the integral map. For most values of $\mathcal{F}$, the fibers of $\mathcal{F}$ are manifolds, which we call *integral manifolds*. A value ϕ of $\mathcal{F}$ such that in any open neighborhood of ϕ fibers exist that are not diffeomorphic, is called a *bifurcation value* of $\mathcal{F}$. This means that upon crossing a bifurcation value the topology of the integral manifolds may change. The values of $\mathcal{F}$ at critical points of $\mathcal{F}$ in M are examples of such bifurcation values. Due to the non-compactness of the fibers bifurcations can occur not only at such critical values but also due to critical points at infinity [2]. Our aim is to find the bifurcation values in the spirit of [13] for the charged N -body system with $N = 3$ where in the present paper we will concentrate on bifurcation values due to ‘ordinary’ critical points and in a forthcoming paper we will study critical points at infinity [9].

3. Critical points of the integral map

Consider the N -body system as defined in Definition 1.1. Let us find critical points of the integral map $\mathcal{F}$ by considering the rank of $D\mathcal{F}$ restricted to points in phase space where $P(q, p) = 0$ and $Q(q, p) = 0$. We first formulate and prove a statement on the rank of $D\mathcal{F}$ and after that give an interpretation of the result and draw some conclusions, see also [10].

Proposition 3.1. *The derivative $D\mathcal{F}$ of $\mathcal{F}$ at (q, p) does not have full rank if and only if at least one of the following three conditions holds*

1. *the q_i and p_i , $i = 1, \dots, N$, are multiples of a single vector $e \in \mathbb{R}^3$,*
2. *(q, p) is an equilibrium point, or*
3. *(q, p) is a relative equilibrium with respect to the $SO(3)$ action.*

We prove this proposition using the following two lemmas.

Lemma 3.2. *For $i = 1, \dots, N$, the i th component of the gradient of V is a linear combination of the position vectors $q_1, \dots, q_N$. Specifically we have*

$$\frac{\partial}{\partial q_i} V = \sum_j \alpha_{ij} q_j,$$

where, using $r_{ij} = \|q_i - q_j\|$,

$$\alpha_{ii} = \sum_{i < k} \frac{\partial V}{\partial r_{ik}} \frac{1}{r_{ik}} + \sum_{k < i} \frac{\partial V}{\partial r_{ki}} \frac{1}{r_{ki}},$$

and for $i \neq j$,

$$\alpha_{ij} = \begin{cases} -\frac{\partial V}{\partial r_{ij}} \frac{1}{r_{ij}} & \text{if } i < j \\ -\frac{\partial V}{\partial r_{ji}} \frac{1}{r_{ji}} & \text{if } j < i \end{cases}.$$

Proof. By direct computation we get

$$\begin{aligned} \frac{\partial}{\partial q_i} V &= \sum_{j < k} \frac{\partial V}{\partial r_{jk}} \frac{\partial r_{jk}}{\partial q_i} \\ &= \sum_{j < k} \frac{\partial V}{\partial r_{jk}} \left(\frac{q_j - q_k}{r_{jk}} \delta_{ij} + \frac{q_k - q_j}{r_{jk}} \delta_{ik} \right) \\ &= \sum_{i < k} \frac{\partial V}{\partial r_{ik}} \frac{q_i - q_k}{r_{ik}} + \sum_{j < i} \frac{\partial V}{\partial r_{ji}} \frac{q_i - q_j}{r_{ji}} \\ &= \left(\sum_{i < k} \frac{\partial V}{\partial r_{ik}} \frac{1}{r_{ik}} + \sum_{k < i} \frac{\partial V}{\partial r_{ki}} \frac{1}{r_{ki}} \right) q_i - \sum_{i < j} \frac{\partial V}{\partial r_{ij}} \frac{1}{r_{ij}} q_j - \sum_{i > j} \frac{\partial V}{\partial r_{ji}} \frac{1}{r_{ji}} q_j, \end{aligned}$$

where the sums in the first and second lines are double sums, δ_{ij} (δ_{jk}) in the second line denotes the Kronecker symbol so that the sums in the third and fourth lines become single sums, and in the fourth line we relabeled the summation index. The result follows. $\square$

Lemma 3.3. *The total force on an N -body system is zero, i.e.*

$$\sum_i \frac{\partial}{\partial q_i} V = 0.$$

Proof. We sum the third line of the equation in the proof of the previous lemma over i to get

$$\begin{aligned} \sum_i \frac{\partial}{\partial q_i} V &= \sum_{i < k} \frac{\partial V}{\partial r_{ik}} \frac{q_i - q_k}{r_{ik}} + \sum_{j < i} \frac{\partial V}{\partial r_{ji}} \frac{q_i - q_j}{r_{ji}} \\ &= \sum_{i < k} \frac{\partial V}{\partial r_{ik}} \frac{q_i - q_k}{r_{ik}} + \sum_{i < k} \frac{\partial V}{\partial r_{ik}} \frac{q_k - q_i}{r_{ik}}, \end{aligned}$$

where the sums after the first and second equality symbols are double sums, and we relabeled the summation indices in the second line. The result follows. We note that the result follows even faster from the conservation of the total momentum P , namely

$$0 = \dot{P} = \sum_i \dot{p}_i = \sum_i \{p_i, H\} = \sum_i \frac{\partial}{\partial q_i} V. \quad \square$$

Let us now proceed with the proof of [Proposition 3.1](#).

Proof of Proposition 3.1. The transpose matrix of the derivative of the map $D\mathcal{F}$ at (q, p) is given by

$$(D\mathcal{F}(q, p))^T = \begin{pmatrix} \frac{\partial}{\partial q_1} V & -A_{p_1} & 0 & \frac{m_1}{m} I \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial}{\partial q_N} V & -A_{p_N} & 0 & \frac{m_N}{m} I \\ \frac{p_1}{m_1} & A_{q_1} & I & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{p_N}{m_N} & A_{q_N} & I & 0 \end{pmatrix}, \quad (3)$$

where I is the identity matrix on $\mathbb{R}^3$ and A_x is the skew symmetric matrix associated with the cross product on $\mathbb{R}^3$, namely $A_x(y) = x \times y$ so that

$$A_x = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$

$D\mathcal{F}(q, p)$ does not have full rank if a non-zero vector $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ exists such that $(D\mathcal{F}(q, p))^T \lambda = 0$. The last six columns of $(D\mathcal{F}(q, p))^T$ are clearly independent. Hence, for $D\mathcal{F}(q, p)$ not to have full rank, at least one of λ_0 and λ_1 must be non-zero. We now distinguish two main cases, namely $\lambda_0 = 0$ and $\lambda_0 \neq 0$.

1. Suppose that $\lambda_0 = 0$. Then we get $2N$ equations from $(D\mathcal{F}(q, p))^T \lambda = 0$, namely

$$\begin{cases} 0 = \frac{m_i}{m} \lambda_3 - p_i \times \lambda_1, \\ 0 = \lambda_2 + q_i \times \lambda_1 \end{cases}$$

for $i \in \{1, \dots, N\}$. As $\lambda_1 = 0$ would imply $\lambda_2 = 0$ and $\lambda_3 = 0$ which together with $\lambda_0 = 0$ does not give a critical point it follows that $\lambda_1 \neq 0$. Adding the first N equations and adding the second N equations with weights m_i/m we get

$$\begin{cases} 0 = \lambda_3 \frac{1}{m} \sum_i m_i - \left(\sum_i p_i \right) \times \lambda_1 = \lambda_3, \\ 0 = \lambda_2 \frac{1}{m} \sum_i m_i + \left(\frac{1}{m} \sum_i m_i q_i \right) \times \lambda_1 = \lambda_2, \end{cases}$$

where we used $0 = P = \sum_i p_i$ and $0 = Q = \frac{1}{m} \sum_i m_i q_i$. Therefore $\lambda_2 = 0$ and $\lambda_3 = 0$. We are left with the equations $q_i \times \lambda_1 = 0$ and $p_i \times \lambda_1 = 0$ which imply that the q_i and the $p_i, i = 1, \dots, N$, are multiples of the same vector λ_1 . If, conversely, we assume that the q_i and p_i are multiples of the same vector e , then taking $\lambda = (0, e, 0, 0)$ shows that $\text{rank } D\mathcal{F}$ is less than 10.

2. Now suppose that $\lambda_0 \neq 0$. Without restriction we may assume $\lambda_0 = 1$. From $(D\mathcal{F}(q, p))^T \lambda = 0$ we again get $2N$ equations

$$\begin{cases} 0 = \frac{\partial}{\partial q_i} V + \frac{m_i}{m} \lambda_3 - p_i \times \lambda_1, \\ 0 = \frac{p_i}{m_i} + \lambda_2 + q_i \times \lambda_1, \end{cases}$$

for $i \in \{1, \dots, N\}$. Adding the equations in the same manner as in case 1 we get

$$\begin{cases} 0 = \sum_i \frac{\partial}{\partial q_i} V + \lambda_3 \frac{1}{m} \sum_i m_i - \left(\sum_i p_i \right) \times \lambda_1 = \lambda_3, \\ 0 = \frac{1}{m} \sum_i p_i + \lambda_2 \frac{1}{m} \sum_i m_i + \left(\frac{1}{m} \sum_i m_i q_i \right) \times \lambda_1 = \lambda_2. \end{cases}$$

Here we again use $0 = P = \sum_i p_i$ and $0 = Q = \frac{1}{m} \sum_i m_i q_i$, and $\sum_i \frac{\partial}{\partial q_i} V = 0$ as we showed in Lemma 3.3. So again we find $\lambda_2 = 0$ and $\lambda_3 = 0$ and we are left with the equations

$$\begin{cases} \frac{\partial}{\partial q_i} V = p_i \times \lambda_1, \\ p_i = m_i \lambda_1 \times q_i, \end{cases} \quad (4)$$

For $\lambda_1 = 0$, we see that (q, p) must be an equilibrium point. The case $\lambda_1 \neq 0$, on the other hand, gives critical points of the Hamiltonian restricted to levels of angular momentum. Indeed we have $0 = (D\mathcal{F}(q, p))^T \lambda = \nabla_{(q,p)} H + \lambda_{11} \nabla_{(q,p)} L_1 + \lambda_{12} \nabla_{(q,p)} L_2 + \lambda_{13} \nabla_{(q,p)} L_3$, where the L_j are the components of angular momentum. With the characterization of relative equilibria in Section 2.3 we see that these points are in indeed relative equilibria with respect to the action of $\text{SO}(3)$. $\square$

Let us now comment on Proposition 3.1.

From case 1 in the proof of Proposition 3.1 we see that critical points of this type do not depend on the specific Hamiltonian of the N -body system. These points are in fact critical points of angular momentum. The reason is that in the vector λ only component λ_1 is non-zero and $(D\mathcal{F}(q, p))^T \lambda$ in Eq. (3) thus only selects L . The N bodies are moving on a single line in $\mathbb{R}^3$ through the center of mass. In that case angular momentum is zero. So zero is a critical value. Needless to say that level zero of angular momentum also contains other motions of the N bodies than just motion on a line. The question arises whether an N -body system admits equilibrium points and relative equilibria. In the gravitational case we have the following.

Proposition 3.4. *There are no equilibrium points in the gravitational N -body problem.*

Proof. As V is a homogeneous function of degree -1 of the position vectors we have by Euler's homogeneous function theorem

$$\sum_{i=1}^N q_i \frac{\partial V}{\partial q_i} = -V.$$

As V is a strictly negative function the left hand side of the equation cannot vanish which implies that V has no critical points. Hence this implies that the gravitational N -body problem has no equilibrium points. $\square$

Remark 3.5. In the proof of [Proposition 3.4](#) we used the fact that the potential for the gravitational N -body problem is homogeneous and everywhere negative. For the charged N -body problem the potential is still homogeneous. However the potential does in general not have a constant sign. So the existence of equilibrium points is not excluded by the argument employed in [Proposition 3.4](#). The proof shows that if there are equilibrium solutions for the charged case then they project to the zero-level set of the potential.

Concerning relative equilibria we have the following proposition.

Proposition 3.6. *Planar central configurations with positive multiplier give rise to relative equilibria. More specifically, if there exists a vector $e \in \mathbb{R}^3$ such that the q_i and p_i satisfy*

1. $\langle q_i, e \rangle = 0$,
2. $\frac{\partial}{\partial q_i} V = \langle e, e \rangle m_i q_i$ and
3. $p_i = m_i e \times q_i$.

for $i \in \{1, \dots, N\}$, then the point (q, p) is a relative equilibrium. Conversely, if a relative equilibrium projects onto a central configuration then the central configuration is planar and has a positive multiplier.

Proof. Recall that case 2 of the proof of [Proposition 3.1](#) with $\lambda_1 \neq 0$ gives a relative equilibrium. Now note that Eq. (4) that corresponds to this case is equivalent to

$$\begin{cases} \frac{\partial}{\partial q_i} V = m_i(\lambda_1 \times q_i) \times \lambda_1 = m_i(\langle \lambda_1, \lambda_1 \rangle q_i - \langle \lambda_1, q_i \rangle \lambda_1), \\ p_i = m_i \lambda_1 \times q_i. \end{cases} \quad (5)$$

This means that the vectors $\frac{\partial}{\partial q_i} V$ are in a plane perpendicular to λ_1 . We have $\frac{\partial}{\partial q_i} V = \sum_j \alpha_{ij}(q) q_j$ with the α_{ij} as defined in [Lemma 3.2](#). Since the $\frac{\partial}{\partial q_i} V$ are perpendicular to λ_1 we must have

$$\sum_j \alpha_{ij}(q) \langle q_j, \lambda_1 \rangle = 0. \quad (6)$$

Let (α_{ij}) be the matrix with entries α_{ij} . Then (α_{ij}) is symmetric and by [Lemma 3.3](#) the row sums of (α_{ij}) are zero. The latter implies that $(1, 1, \dots, 1)$ is an element of $\ker(\alpha_{ij})$, which in turn implies that $\langle q_j, \lambda_1 \rangle = c$ independent of j is a solution of Eq. (6). But then

$$0 = \langle \sum_i m_i q_i, \lambda_1 \rangle = \sum_i m_i \langle q_i, \lambda_1 \rangle = c \sum_i m_i$$

implies $c = 0$ because the masses are positive. Thus we are left with the following equations, no longer equivalent to the system in (5),

$$\begin{cases} \frac{\partial}{\partial q_i} V = \langle \lambda_1, \lambda_1 \rangle m_i q_i, \\ p_i = m_i \lambda_1 \times q_i. \end{cases}$$

This means that a critical point (q, p) of this type projects onto a planar central configuration q with a positive multiplier.

Now suppose that a critical point (q, p) , i.e. a solution of Eq. (5), projects onto a central configuration q . Then a μ exists such that $\frac{\partial}{\partial q_i} V = \mu m_i q_i$. Since the $\frac{\partial}{\partial q_i} V$ are perpendicular to λ_1 we must have $\langle \lambda_1, q_i \rangle = 0$. So q must be a planar central configuration. Again using Eq. (5) we see that the multiplier must be positive. $\square$

The proposition does not necessarily provide all relative equilibria. Depending on the potential V there may be more. In our proof this fact is related to the kernel of the matrix A associated to the gradient of the potential, [Lemma 3.2](#). This kernel is at least spanned by the vector $(1, 1, \dots, 1)$. The gravitational potential is an example where this vector is the only spanning vector of $\ker(A)$. Thus in the gravitational case the proposition gives all relative equilibria.

[Propositions 3.1](#) and [3.6](#) relate central configurations, [Definition 1.2](#), and critical points of the integral map given by relative equilibria. The relation is subtle, not only because central configurations reside in configuration space and critical points in phase space. To make the relation more clear we give a list of corollaries. Irrespective of the potential V (apart from being $SE(3)$ invariant) we have the following corollaries.

Corollary 3.7. *For every planar central configuration q with a positive multiplier there exists a critical point (q, p) .*

Corollary 3.8. *If a relative equilibrium (q, p) projects onto a central configuration q , the latter must be a planar central configuration with a positive multiplier.*

If we do know more about the potential V then we have the following.

Corollary 3.9. *If $(1, 1, \dots, 1)$ is the only spanning vector of $\ker(A)$, [Lemma 3.2](#), then every relative equilibrium (q, p) projects onto a planar central configuration with a positive multiplier.*

Corollary 3.10. *If $\dim(\ker(A)) > 1$, then relative equilibria do not necessarily project onto central configurations.*

Determining the kernel of A is in general not easy. We do however have another criterion in the next proposition.

Proposition 3.11. *Suppose the derivatives $\partial_{r_{ij}} V$ of the potential V of an N -body system have equal signs for all $1 \leq i < j \leq N$. Then every relative equilibrium projects onto a planar central configuration.*

The proof of this proposition will be given below. In view of [Corollary 3.8](#) the multiplier of this central configuration must be positive. [Proposition 3.11](#) implicitly states that under the conditions mentioned, the kernel of A is one-dimensional. Needless to say that knowing this kernel is not our ultimate goal. An example to which [Proposition 3.11](#) applies is the gravitational N -body system for which a proof can already be found in [21]. For this system, there is a relative equilibrium for every planar central configuration and every relative equilibrium projects onto a central configuration. In particular all central configurations are planar. This need not be so for a charged N -body system. If in the case of electrostatic interactions all charges have the same sign then there is no relative equilibrium (all forces are repulsive). If the charges have different signs then there are indeed examples of relative equilibria that do not project onto a central configuration, see [8].

Proof of Proposition 3.11. Let (q, p) be a solution of Eq. (5). Suppose $\partial_{r_{ij}} V > 0$ for all $1 \leq i < j \leq N$. Considering the equations in (5) in case 2 of the proof of [Proposition 3.6](#) we need to show that $\partial_{q_i} V$ being orthogonal to a $\lambda \in \mathbb{R}^3$, $\lambda \neq 0$, for $i = 1, \dots, N$, implies that the q_i are orthogonal to λ for all $i = 1, \dots, N$. Then we infer from Eq. (5) that q is a central configuration with a positive multiplier.

Now we show by contradiction that $\partial_{q_i} V$ is orthogonal to a $\lambda \in \mathbb{R}^3$, $\lambda \neq 0$, for $i = 1, \dots, N$. To this end let Π be the plane orthogonal to λ . As we require $Q = 0$, Π contains the origin. Without restriction we may assume that $\lambda = (0, 0, 1)$ (otherwise rotate the coordinate system). Suppose one or more q_i are not contained in the plane Π . From this q_i choose one that is furthest away from Π . Without restriction we can assume that this q_i lies above Π , i.e. has a positive z -component (otherwise rotate the coordinate system). Note that the maximum z can be attained by more than one q_i . However no more than $N - 1$ many can attain the maximal z because otherwise all q_i would have to be in a plane and because of $Q = 0$ this plane would be Π . Without restriction we can take q_1 to be the one with maximal z and q_2 to have a smaller z -component (otherwise relabel). Then the z -component of the force on particle 1 is

$$F_{1z} = -\partial_{z_1} V = -\sum_{i=2}^N \frac{\partial V}{\partial r_{1i}} \frac{z_1 - z_i}{r_{1i}} \leq \frac{\partial V}{\partial r_{12}} \frac{z_1 - z_2}{r_{12}} < 0$$

which contradicts $\partial_{q_i} V$ being orthogonal to $\lambda = (0, 0, 1)$ for all $i = 1, \dots, N$. The same kind of arguments apply to the case $\partial_{r_{ij}} V < 0$ for $1 \leq i < j \leq N$.

4. Central configurations

In the previous section we have seen that for an N -body system, planar central configurations with a positive multiplier give rise to critical points. In this respect three-body systems are exceptional because here all central configurations are planar. More generally central configurations turn up in N -body systems when we look for solutions whose configuration remains similar during the motion, so called *homographic* solutions. This means that during motion only size and orientation of the configuration change. Imposing these conditions on the configuration $(q_1, \dots, q_N)$, $q_i \in \mathbb{R}^3$ leads to the equation for a central configuration in [Definition 1.2](#). Central configurations are invariant under the action of the Euclidean group $SE(3)$, and if the potential is homogeneous like in the case of a charged N -body system then they are also invariant under dilations.

Lemma 4.1. *The equation defining a central configuration in [Definition 1.2](#) is invariant under translations and rotations. For a charged N -body system, it is moreover invariant under dilations.*

Proof. Clearly the defining equation of a central configuration in [Definition 1.2](#) is translation invariant. Since the $\partial_{q_i} V$ transform under rotations like the q_i and a rotation of the q_i also induces a rotation of the center of mass Q the defining equation of a central configuration is invariant under rotations. The potential of the charged N -body system is homogeneous of degree -1 . If we define in this case the action of dilations as $(q, \lambda) \mapsto (tq, t^{-3}\lambda)$ for $t \in \mathbb{R}^*$, the defining equation is also dilation invariant. $\square$

In this section we find the central configurations for the charged three-body system, i.e. for a three-body system with potential

$$V(q) = -\frac{\alpha_3}{\|q_1 - q_2\|} - \frac{\alpha_1}{\|q_2 - q_3\|} - \frac{\alpha_2}{\|q_3 - q_1\|}, \quad (7)$$

where $\alpha_i = \gamma_{jk}$ for $(i, j, k) = (1, 2, 3)$ and cyclic permutations. Recall from the introduction that in the case of electrostatic interactions the map from the charges Q_i to the coefficients $\gamma_{ij} = -Q_i Q_j$ is not onto. Indeed for three bodies, the map $\mathbb{R}^3 \rightarrow \mathbb{R}^3 : x \mapsto y =$

$(-x_2x_3, -x_1x_3, -x_1x_2)$ has image $\{y \in \mathbb{R}^3 \mid y_1y_2y_3 \leq 0\}$, a collection of four octants. As mentioned in the introduction we will ignore this fact.

It turns out that we have to distinguish between collinear and non-collinear central configurations.

4.1. Outline of the method

Our approach differs from that in the literature (see [16]) in particular with respect to how we treat collinear central configurations. We start with identifying the space of collinear configurations. On this space the translation, rotation, dilation and permutation groups act. Then we first find the reduced space of collinear configurations by reducing with respect to translations, rotations and dilations. In case of three bodies the reduced space is one-dimensional and the condition for central configurations reduces to a single equation. However, due to the fact that the masses are undetermined and the potential contains unknown coefficients we have in fact a parameter family of equations. When we use the potential from Eq. (7) the collinear central configurations are determined by the zeros of a fifth degree polynomial. In general it will be impossible to find explicit solutions to this equation. But the solutions depend on the parameters and by using failure of the implicit function theorem we are at least able to find *regions* in parameter space with a constant number of solutions.

We will take a more geometric point of view and consider the equation for central configurations on the product of the parameter space and the reduced space. Then the equation defines a hyper surface. Projecting the latter to the parameter space we obtain the aforementioned regions by locating the projection singularities. (As it turns out, the surface itself has no singularities.) The simplest singularities are folds and where a fold is parallel to the projection direction we get more complicated singularities. We carry out this program in some detail for the potential in Eq. (7).

In the system of three charged bodies, the parameter space is six-dimensional: there are three masses m_i and three coefficients α_i in the potential, $i \in \{1, 2, 3\}$. However, the reduced equation for the collinear central configurations is homogeneous in both the m_i and the α_i . Thus each set of three can be reduced to the ratios $[m_1 : m_2 : m_3]$ and $[\alpha_1 : \alpha_2 : \alpha_3]$. At this point we prefer to consider the parameter family of equations as a parameter family (parameterized by the m_i) of parameter families (parameterized by the α_i). This splitting is very useful in identifying the aforementioned regions in the parameter plane with coordinates $(\beta_1, \beta_2) = (\frac{\alpha_1}{\alpha_3}, \frac{\alpha_2}{\alpha_3})$ (assuming $\alpha_3 \neq 0$). But when we consider permutations of the bodies, the picture becomes slightly more complicated, see [Appendix B](#).

To determine the regions we consider the fifth degree polynomial mentioned before, as a parameter family for each m (the aforementioned parameter family of parameter families). As it turns out the topology of the regions in the $[\alpha_1 : \alpha_2 : \alpha_3]$ -space does not depend on m . Thus we are left with the problem of classifying the number of zeros of a polynomial depending on two parameters β_1 and β_2 . Regions in the parameter plane with a constant number of zeros are bounded by curves where the polynomial has double zeros. From a geometrical point of view, these are fold curves. Therefore we also expect isolated cusp points which we find indeed.

To find the non-collinear central configurations we also first apply reduction with respect to translational and rotational symmetry on configuration space. We will see that the condition to have a central configuration is equivalent to having a critical point of the potential restricted to levels of the moment of inertia. The latter condition can easily be stated in the reduced configuration space. Using the SE(3) invariant mutual body distances r_{ij} as coordinates the computation of non-collinear central configurations is straightforward.

Fig. 1. Regions in the (β_1, β_2) -parameter plane with different numbers of real zeros of the polynomial f in Eq. (8) in the intervals I_i . The regions are numbered 1 to 13 and in Table 1 the numbers of real zeros are listed. Segments of different thickness of the bounding curve correspond to images of the intervals I_1 , I_2 and I_3 . When compactifying the parameter plane by adding one point at infinity, the branches in the left picture meet at infinity as shown in the right picture where the ‘origin’ represents the point added at infinity. The points $u = -1$, $u = 0$ and $u = \infty$ correspond to two body collisions.

4.2. Collinear central configurations

In this section we consider collinear central configurations for charged three-body systems and we carry out the program sketched in the previous section. The main result for three bodies is the following.

Theorem 4.2. *Consider a three-body system with potential as in Eq. (7). For all values of the masses $m = (m_1, m_2, m_3) \in \mathbb{R}_{>0}^3$ there are thirteen different open regions in the (β_1, β_2) parameter plane with a constant number of collinear central configurations. Depending on the region there can be 1, 3 or 5 of these.*

The reduced space $\mathbb{R} \cup \{\infty\}$ of collinear configurations has a natural splitting into three intervals $I_1 := (\infty, -1)$, $I_2 := (-1, 0)$ and $I_3 := (0, \infty)$, see Section 4.2.2. In Fig. 1 together with Table 1 the number of real zeros of the reduced equation on each of the intervals I_i on the reduced space is indicated. Each of these intervals corresponds to a certain order of the bodies on a line, again see Section 4.2.2.

4.2.1. The space of collinear configurations

Let us first identify the space of collinear configurations for an N -body system. The point $q = (q_1, \dots, q_N)$ is a collinear configuration if the differences to the center of mass $q_i - Q$ are multiples of the same vector, i.e. there is a unit vector $v \in \mathbb{R}^3$ such that $q_i - Q = r_i v$, with $r_i \in \mathbb{R}$ and $|r_i| = \|q_i - Q\|$ for $i \in \{1, \dots, N\}$. The r_i are not independent as $\sum_i m_i r_i = 0$. A collinear

Table 1*The number of zeros of f in each region in the intervals I_1 , I_2 and I_3 .*

Region	Zeros	Region	Zeros
1	(0, 0, 1)	7	(1, 3, 1)
2	(0, 2, 1)	8	(1, 2, 0)
3	(0, 1, 0)	9	(1, 0, 0)
4	(2, 1, 0)	10	(2, 0, 1)
5	(3, 1, 1)	11	(1, 0, 2)
6	(1, 1, 1)	12	(0, 1, 2)
		13	(1, 1, 3)

configuration is thus determined by $N - 1$ many $r_i \in \mathbb{R}$, the center of mass $Q \in \mathbb{R}^3$, and a vector v on the unit sphere S^2 . Accordingly, the space of collinear configurations is topologically equivalent to $\mathbb{R}^{N-1} \times \mathbb{R}^3 \times S^2$. On this space we have four group actions, namely actions of translations, rotations, dilations and permutations. Whereas the translations and rotations act on the factors $\mathbb{R}^3$ and S^2 , respectively, the dilations and permutations act on the first factor $\mathbb{R}^{N-1}$.

4.2.2. The reduced space of collinear configurations

In the following we will reduce the space of collinear configurations by the symmetries. In the case of a charged three-body system, the reduced space will be one-dimensional which will simplify the method of finding collinear central configurations.

Lemma 4.3. *For a charged N -body system, the space of collinear configurations reduced with respect to translations, rotations and dilations is topologically equivalent to $\mathbb{RP}^{N-2}$.*

Proof. The space of collinear configurations is $\mathbb{R}^{N-1} \times \mathbb{R}^3 \times S^2$. First we reduce the translations which act on the second factor. The space of orbits of the action of the translation group $\mathbb{R}^3$ is again $\mathbb{R}^3$ and as the reduced space we can take the point where the center of mass Q is at the origin. The rotations act on the third factor. Every two points on S^2 are $\text{SO}(3)$ -equivalent, i.e. there is only one $\text{SO}(3)$ -orbit on S^2 . Therefore the reduced space is a single point which we can choose to be, e.g., the unit vector e_1 . On the remaining factor $\mathbb{R}^{N-1} \cong \{(r_1, \dots, r_N) \in \mathbb{R}^N \mid \sum_i m_i r_i = 0\}$ the dilations act as $r_i \mapsto tr_i$ for $t \in \mathbb{R}^*$. The orbits are lines through the origin. Then the reduced space is topologically equivalent to $\mathbb{RP}^{N-2}$. $\square$

On the reduced (with respect to translations and rotations) space of collinear configurations of charged three-body systems we can introduce on the space $\{(r_1, r_2, r_3) \in \mathbb{R}^3 \mid m_1 r_1 + m_2 r_2 + m_3 r_3 = 0\}$ we may also use the differences $x := r_2 - r_3$, $y := r_3 - r_1$ and $z := r_1 - r_2$ as coordinates, to obtain $V_3 := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$. The latter, as a vector space is isomorphic to $\mathbb{R}^2$ and thus the reduced space with respect to dilations is topologically equivalent to $\mathbb{RP}^1$ which in turn is homeomorphic to the circle S^1 . On the latter, regarded as $\mathbb{R} \cup \{\infty\}$, we take u , defined by $y = ux$, as a coordinate.

Definition 4.4. Let I_1 , I_2 and I_3 be three intervals in $\mathbb{R} \cup \{\infty\}$, defined as: $I_1 := (\infty, -1)$, $I_2 := (-1, 0)$ and $I_3 := (0, \infty)$.

The intervals I_i correspond to orderings of the bodies with body i in the middle, see Table 2. The boundaries of I_1 , I_2 and I_3 , i.e. the points -1 , 0 and ∞ , correspond to collisions of bodies 1 and 2, 1 and 3 and bodies 2 and 3, respectively.

Fig. 2. Left: The reduced space $\mathbb{R} \cup \{\infty\}$ of collinear configurations with the intervals I_1 , I_2 and I_3 . Inside the circle collisions of pairs are indicated at the special values of u . Right: A collinear configuration with two relative distances x and y .

Table 2

Intervals in reduced space of collinear configurations and order of bodies.

Interval	Order
I_1	(2, 1, 3) or (3, 1, 2)
I_2	(1, 2, 3) or (3, 2, 1)
I_3	(1, 3, 2) or (2, 3, 1)

4.2.3. Polynomial equation for collinear central configurations on the reduced space

Now that we know the reduced space of collinear configurations we derive a reduced equation for the collinear central configurations. Using the same coordinates as in the previous section we find a single equation for u . In deriving this equation we also encounter the intermediate coordinates x , y and z . Their signs have a relation with the order of the bodies as is implicit in the discussion of the previous section. Not every choice is allowed since $x + y + z \equiv 0$. Here we take $x > 0$, $y > 0$ and $z < 0$, corresponding to the order (1, 3, 2), see Fig. 2. Permuting the bodies we get the other possible sign choices, see Appendix B. On the reduced space we have a single coordinate u defined by $y = ux$ and the identity for x , y and z yields $z = -(1 + u)x$. Finally we have the following result on collinear central configurations.

Proposition 4.5. *On the reduced space of collinear configurations the equation for central configurations reads $f(u) = 0$, where*

$$f(u) = \alpha_1 f_1(u) + \alpha_2 f_2(u) + \alpha_3 f_3(u)$$

with

$$\begin{aligned} f_1(u) &:= m_1 u^2 (1 + u)^2 (m_2 + m_2 u + m_3 u), \\ f_2(u) &:= -m_2 (1 + u)^2 (m_1 + m_3 + m_1 u), \\ f_3(u) &:= -m_3 u^2 (m_2 - m_1 u). \end{aligned} \quad (8)$$

The polynomial f depends on six parameters: α_1 , α_2 , α_3 , m_1 , m_2 and m_3 .

Proof. We start the proof with the components of $\frac{\partial V(r)}{\partial r_i} = -m_i \lambda r_i$ for $i \in \{1, 2, 3\}$. We will eventually eliminate λ , then the resulting equations will be linear in α_i . Taking linear combinations of the above equations with coefficients $(m_2, -m_1, 0)$, $(0, m_3, -m_2)$ and

$(-m_3, 0, m_1)$, we get equations depending on differences $r_i - r_j$ only. Following the reduction steps in the proof of [Lemma 4.3](#) we switch to variables $x = r_2 - r_3$, $y = r_3 - r_1$ and $z = r_1 - r_2$. Using the choice of signs $x > 0$, $y > 0$ and $z < 0$ we get

$$\begin{cases} \alpha_1 \frac{m_2 + m_3}{x^2} - \alpha_2 \frac{m_2}{y^2} + \alpha_3 \frac{m_3}{z^2} - \lambda m_2 m_3 x = 0, \\ \alpha_2 \frac{m_1 + m_3}{y^2} - \alpha_1 \frac{m_1}{x^2} + \alpha_3 \frac{m_3}{z^2} - \lambda m_1 m_3 y = 0, \\ \alpha_3 \frac{m_1 + m_2}{z^2} + \alpha_1 \frac{m_1}{x^2} + \alpha_2 \frac{m_2}{y^2} + \lambda m_1 m_2 z = 0. \end{cases} \quad (9)$$

The sum of these equations with coefficients $(m_1, m_2, -m_3)$ yields $\lambda m_1 m_2 m_3 (x + y + z) = 0$ which is true because $x + y + z$ is identical to zero. However this implies that the equations are linearly dependent (which is not surprising). Eliminating λ , we will be left with a single equation. Now if (x, y, z) is a solution, then (tx, ty, tz) for $t \in \mathbb{R}^*$ is also a solution (dilation symmetry). Like in [Lemma 4.3](#) we set $y = ux$ and $z = -x - ux$. Then we get a single equation for u only which can be written as $f(u) = 0$ as given in the proposition. This polynomial f depends on the parameters α_i and m_i . $\square$

Note that the polynomial f is homogeneous (degree one) in the α_i and homogeneous (degree two) in the m_i . This in turn implies that scaling the parameters $(\alpha_1, \alpha_2, \alpha_3) \mapsto (t\alpha_1, t\alpha_2, t\alpha_3)$ and $(m_1, m_2, m_3) \mapsto (sm_1, sm_2, sm_3)$ with $t, s \in \mathbb{R}^*$ does not change the solutions u of $f(u) = 0$.

Our ultimate goal is to obtain the number of collinear central configurations for each value of the parameters α_i and m_i . In order to do that we have to know how the permutation group of the bodies acts on the reduced space. This will be postponed to [Appendix B](#). For the moment our aim is to divide the parameter space into regions with a constant number of real solutions of $f(u) = 0$. Since the parameter space is six-dimensional we make some simplifications. First we fix the values of the masses m_i at arbitrary values. Then we scale the parameters α_i according to $(\alpha_1, \alpha_2, \alpha_3) \mapsto (\beta_1, \beta_2, 1) = \frac{1}{\alpha_3}(\alpha_1, \alpha_2, \alpha_3)$ assuming $\alpha_3 \neq 0$. Thus we are left with a two-parameter family of polynomials. The discriminant set of f is now a collection of curves in the (β_1, β_2) -parameter plane. On these curves f has double zeros, so by crossing one of these curves the number of real zeros jumps by two. Using a more detailed analysis we will be able to tell how the number of zeros in each interval I_1 , I_2 and I_3 changes.

Proposition 4.6. *The intersection of the discriminant set of a general fifth degree polynomial and the two parameter family f consists of three curves: both coordinate axes $\beta_1 = 0$ and $\beta_2 = 0$ and a curve Γ . These curves separate open regions in the (β_1, β_2) -parameter plane. In each region the number of solutions of $f(u) = 0$ in the intervals I_1 , I_2 and I_3 is constant. The curve Γ is parameterized by $u \in \mathbb{R} \cup \{\infty\}$ and depends on the m_i . A parameterization $c : \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R}^2$ of Γ is given by*

$$c(u) = \left(\frac{f_3(u)f_2'(u) - f_2(u)f_3'(u)}{f_2(u)f_1'(u) - f_1(u)f_2'(u)}, \frac{f_1(u)f_3'(u) - f_3(u)f_1'(u)}{f_2(u)f_1'(u) - f_1(u)f_2'(u)} \right), \quad (10)$$

where f_1 , f_2 and f_3 are defined in Eq. (8). The curves and regions are shown in [Fig. 5](#) for the special case $m_1 = m_2$ and $m_3 = 1$.

Proof. We first consider two special cases. Expanding the polynomial f using the definition in Eq. (8) we have:

$$\begin{aligned} f(u) &= \alpha_1 m_1 (m_2 + m_3) u^5 \\ &\quad + \alpha_1 m_1 (3m_2 + 2m_3) u^4 \\ &\quad + (\alpha_1 m_1 (3m_2 + m_3) - \alpha_2 m_1 m_3 + \alpha_3 m_1 m_3) u^3 \\ &\quad + (\alpha_1 m_1 m_2 - \alpha_2 m_2 (3m_1 + m_3) - \alpha_3 m_2 m_3) u^2 \\ &\quad - \alpha_2 m_2 (3m_1 + 2m_3) u \\ &\quad - \alpha_2 m_2 (m_1 + m_3). \end{aligned}$$

From this expression we infer that if α_1 tends to zero, two zeros of f tend to infinity and if α_2 tends to zero, two zeros of f tend to zero. Therefore on the lines $\alpha_1 = 0$ and $\alpha_2 = 0$, f has double zeros. In general f has double zeros on the discriminant set defined by the equations

$$\begin{cases} f(u) = 0, \\ f'(u) = 0. \end{cases}$$

The equations are linear in the α_i and we solve them for α_1 and α_2 by using Cramer's rule. Furthermore we use the decomposition of f defined in Eq. (8). In the last step we replace the α_i by β_i according to $(\alpha_1, \alpha_2, \alpha_3) \mapsto (\beta_1, \beta_2, 1) = \frac{1}{\alpha_3}(\alpha_1, \alpha_2, \alpha_3)$ assuming $\alpha_3 \neq 0$. Thus we find the parameterization c of Γ in Eq. (10). $\square$

Remark 4.7.

1. Details about special points on the curve Γ , like singular points and points at infinity, can be found in [Appendix A](#).
2. Since we parameterize the curve Γ with the zeros $u \in \mathbb{R}$ of f we do not have to worry about pairs of complex conjugate double zeros which in principle are also produced by our method.

4.2.4. The number of zeros of the polynomial f

In the previous paragraph we found regions in the (β_1, β_2) -parameter plane with a constant number of real solutions of $f(u) = 0$. Since these regions are bounded by the coordinate axes and the curve Γ of double zeros of f , the number of zeros changes by two upon crossing one of them. As soon as we know the number of real solutions in one region we can find the numbers in the other regions by the next lemma.

Lemma 4.8. *On crossing the β_1 -axis the number of zeros in I_2 and I_3 both increase or decrease by one, and on crossing the β_2 -axis the number of zeros in both I_1 and I_3 increase or decrease by one. The curve Γ is the union of images of the intervals I_i by the parameterization c . On crossing the image of I_i , the number of real zeros in I_i changes by two.*

Proof. On the β_1 -axis the polynomial f has a double zero at $u = 0$, and on the β_2 -axis the f has a double zero at $u = \infty$. On a part of Γ which is the image of I_i , f has a double zero in I_i . $\square$

To use [Lemma 4.8](#) we determine the number of real solutions in one region.

Lemma 4.9. *Choose the masses as $m_1 := \mu$, $m_2 := \mu$ and $m_3 := 1$. In region 1, for example, for $\beta_1 = \beta_2 = 1$, the polynomial f has one real zero in the interval I_3 .*

Proof. With the masses and parameters as stated the polynomial factorizes as $f(u) = \mu(1 + \mu)(u - 1)(1 + 3u + 5u^2 + 3u^3 + u^4 + \mu(1 + u)^4)$. Then f has a real zero at $u = 1 \in I_3$ and the other zeros of f are complex (in fact two reciprocal complex conjugate pairs). $\square$

Using these two lemmas we can easily construct the numbers of zeros as indicated in Fig. 1. Starting in region 1, f has only one real zero which lies in I_3 . Crossing the β_2 -axis into region 11, the number of zeros in both I_1 and I_3 increases or decreases by one. Since in region 1 there are no zeros of f in I_1 , the number of zeros increases and the number of zeros of f for (I_1, I_2, I_3) changes from $(0, 0, 1)$ to $(1, 0, 2)$. Starting again in region 1 but now crossing the β_1 -axis into region 12, the number of zeros of f changes from $(0, 0, 1)$ to $(0, 1, 2)$. Let us now start in region 12, crossing the curve Γ occurs on the image of I_3 and the number of zeros on I_3 changes by two. However we cannot decide whether it is an increase or a decrease. Therefore we start again in region 12 but now cross the β_2 -axis into region 13. Then the number of zeros in both I_1 and I_3 changes by one. Since in region 12 there are no zeros of f in interval I_1 , the change must be an increase. Reasoning in this manner we obtain all triples in Table 1.

4.2.5. The number of collinear central configurations

From the previous sections we know the number of solutions of $f(u) = 0$ in each interval I_1 , I_2 and I_3 . In Appendix B we consider the action of the permutation group on the reduced space of collinear configurations and the parameter space. This leads to the following. Fixing the order of the bodies such that the differences $x = r_2 - r_3$ and $y = r_3 - r_1$ have equal sign is equivalent to requiring $u \in I_3$ on the reduced space. This means that only solutions of $f(u) = 0$ in I_3 correspond to collinear central configurations of three bodies ordered as $(1, 3, 2)$ or $(2, 3, 1)$. This is the choice we made in Section 4.2.3.

Lemma 4.10. *If the bodies are ordered as $(1, 3, 2)$ or $(2, 3, 1)$, collinear central configurations correspond to solutions of $f(u) = 0$ in the interval I_3 of the reduced space of collinear configurations. Depending on the values of the parameters there are 0, 1, 2 or 3 collinear central configurations in this ordering.*

4.2.6. Miscellaneous remarks

In the following we give several remarks on the previous analysis.

Remark 4.11.

1. The permutation group consists of six elements. But we see only three different orders in the reduced space of collinear configurations. The reason is that, for example, the permutation $(1, 2, 3) \mapsto (3, 2, 1)$ can be regarded as a rotation over π in configuration space.
2. Using the same method for the gravitational three-body system we find a single equation for the collinear central configurations on the reduced space of collinear configurations. In fact we find a three-parameter family of equations where the masses of the bodies are the parameters, namely

$$f(u) = (m_2 + m_3)u^5 + (3m_2 + 2m_3)u^4 + (3m_2 + m_3)u^3 - (3m_1 + m_3)u^2 - (3m_1 + 2m_3)u - (m_1 + m_2).$$

Using the fact that f is homogeneous in m we may divide by, for example, m_3 or equivalently set $m_3 = 1$. Thus we consider a two-parameter family. The discriminant set of

Fig. 3. Left: The surface defined by $f(u, m) = 0$ for a three-particle system with gravitational interaction. The fold lines (red) are projected on the (m_1, m_2) -parameter plane. To obtain $f(u, m)$ take $\alpha_i = m_j m_k$ with $(i, j, k) = (1, 2, 3)$ and cyclic permutations in Eq. (8). Right: The surface defined by $f(u) = 0$ for a charged three-body system. The fold lines (red) are projected on the (β_1, β_2) -parameter plane. To obtain $f(u)$ take $\alpha_1 = \beta_1, \alpha_2 = \beta_2$ and $\alpha_3 = 1$ in Eq. (8). Note that in the projection on the right only the segment in and near the first quadrant of the schematic picture in Fig. 1 are visible because in reality Fig. 1 is very distorted.

f is a set of curves in the (m_1, m_2) -parameter plane. These can be viewed as fold lines under the projection of the surface defined by $f(u) = 0$ in the u -direction, see Fig. 3. In this case however it is not hard to show that the curves constituting the discriminant set stay outside the first quadrant in the (m_1, m_2) -parameter plane. Since the masses are positive there is a fixed number of collinear central configurations. To find this number we only need to check one example. Taking $m_1 = m_2 = 1$ we see that there is only one collinear central configuration for $u = 1$.

Lemma 4.10 follows immediately from Table 1. To find the total number of collinear central configurations we also have to consider different orderings of the bodies. We find these by permutations and the properties of the polynomial f in Proposition 4.5 under permutations. The details are elaborated in Appendix B and the result is given in Corollary B.4, implying that we just have to add the numbers of solutions for each region in Table 1. Thus the final result follows from Lemma C.1.

Proposition 4.12. Consider a three body system with potential V as in Eq. (7). For each value of the masses $m = (m_1, m_2, m_3) \in \mathbb{R}_{>0}^3$, there can be 1, 3 or 5 collinear central configurations depending on the values of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

We note that this is a stronger result than the one in [16] which states that the number is between 1 and 5.

4.3. Non-collinear central configurations

To find the non-collinear central configurations we also apply reduction with respect to translation and rotation symmetry. To find the equation for central configurations in reduced coordinates we use the characterization of central configurations as critical points of the potential restricted to levels of the moment of inertia.

4.3.1. The reduced configuration space

Ignoring the collision set Δ_c the configuration space of a three-body system is $\mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. Reduction with respect to the group of translations $\mathbb{R}^3$ gives the translation reduced space with topology $\mathbb{R}^6$ which is commonly chosen as the space $\{q \in \mathbb{R}^9 \mid Q = \frac{1}{m} \sum_i m_i q_i = 0\}$, i.e. the subspace of $\mathbb{R}^9$ on which the center of mass Q is vanishing. Instead the reduction with respect to translations can also be used to achieve $q_1 = 0$. We will take this point of view. For a non-collinear configuration, one can then achieve by rotation about the origin that q_2 is located on the positive x -axis (the first axis in a Euclidean space $\mathbb{R}^3$ which has the origin at q_1). By another rotation about the x -axis one can achieve that q_3 is contained in the upper half of the (x, y) -plane (with y corresponding to the second axis of the Euclidean space). The configuration space reduced by translations and rotations hence has the topology $\mathbb{R}_+^3$. As coordinates on $\mathbb{R}_+^3$ we can take the mutual body distances $r_{ij} = \|q_i - q_j\|$ which are non-negative and need to satisfy the triangle inequality. As the reduced configuration space we hence get $\{(r_{12}, r_{23}, r_{13}) \in \mathbb{R}_+^3 \mid r_{ij} + r_{jk} \geq r_{ki} \text{ for mutually different } i, j, k \in \{1, 2, 3\}\}$. The latter is a cone over a triangle, containing three singular planes which intersect in three singular lines which meet in a singular point at the vertex.

4.3.2. Non-collinear central configurations on the reduced space

In order to obtain an equation for central configurations in the reduced configuration space it is useful to note that central configurations can be obtained as critical points of the potential V restricted to the levels of constant moment of inertia. The (polar) moment of inertia is defined as

$$I(q) = \sum_i m_i \|q_i - Q\|^2.$$

Then we have the following.

Lemma 4.13. *A central configuration is a critical point of V restricted to the levels of I .*

Proof. The proof follows immediately from the Lagrange multiplier theorem. $\square$

This characterization is useful on the reduced space on which both V and I can be expressed as functions of the SE(3) invariants r_{ij} . The potential V from Eq. (7) is already expressed in invariants. For I , we have the following.

Lemma 4.14. *On the reduced space*

$$I(r_{12}, r_{13}, r_{23}) = \frac{1}{2(m_1 + m_2 + m_3)} (m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2).$$

Using Lemma 4.13 we readily obtain the non-collinear central configurations from $(\partial_{r_{12}} V, \partial_{r_{13}} V, \partial_{r_{23}} V)$ and $(\partial_{r_{12}} I, \partial_{r_{13}} I, \partial_{r_{23}} I)$ being parallel as $r_{ij} = \left(\frac{\alpha_k (m_1 + m_2 + m_3)}{m_i m_j \lambda} \right)^{\frac{1}{3}}$ where λ is the Lagrange

multiplier and $k \neq i, k \neq j$. Note however, that the r_{ij} still have to satisfy the triangle inequalities. In the gravitational case we have $\alpha_k = m_i m_j$ which give equal $r_{ij} = \left(\frac{m}{\lambda}\right)^{\frac{1}{3}}$. This is the equilateral triangle solution named after Lagrange. If the potential is purely electrostatic we have $r_{ij} = \left(-\frac{m}{\lambda} \frac{Q_i Q_j}{m_i m_j}\right)^{\frac{1}{3}}$ where the Q_i are the charges. Only when the charges have equal sign, a non-linear central configuration may exist. In our case, only when the α have equal sign, a non-collinear central configuration may exist. Even then the triangle inequalities are not automatically satisfied for all α and m . In summary we have the following theorem.

Theorem 4.15. *A charged three-body system with a potential as in Eq. (7) has at most one non-collinear central configuration. A necessary (but not sufficient) condition for the existence of a non-collinear central configuration is that the α_i in Eq. (7) have the same sign.*

5. Critical points in the charged 3-body system

Let us now consider critical points of the charged three-body system related to central configurations. Since every central configuration in a three-body system is planar we know from [Corollary 3.7](#) that to each central configuration with a positive multiplier we can associate a critical point. We already determined the central configurations in the previous section, but we still have to determine the signs of the multipliers. To this end we use the following lemma.

Lemma 5.1. *For a homogeneous potential of degree -1 and a central configuration q with multiplier λ it holds that*

$$V(q) = -\lambda I(q),$$

where $I(q)$ is the (polar) moment of inertia around the center of mass Q .

Proof. The equation is obtained from scalar multiplication of the equation defining a central configuration ([Definition 1.2](#)) with $q_i - Q$, the summation over i and using Euler's homogeneous function theorem. $\square$

Thus the sign of the multiplier can be read off from the sign of the potential.

Proposition 5.2. *In a charged three-body system with potential (7) there are, depending on the parameters $(\alpha_1, \alpha_2, \alpha_3)$, zero, one, two or three critical points associated to collinear central configurations. There are no critical points associated to non-collinear central configurations.*

Proof. For critical points associated to collinear central configurations, we refer to [Appendix D](#) on the sign of the potential. For critical points associated to non-collinear central configurations we note that the relative distances of the bodies are given by $r_{ij} = \left(-\frac{m}{\lambda} \frac{Q_i Q_j}{m_i m_j}\right)^{\frac{1}{3}}$ where the Q_i are the charges. This implies that the charges must have equal signs and that the multiplier λ must be negative. Now it follows from [Corollary 3.8](#) that there can be no critical point associated to a non-collinear central configuration. $\square$

6. Remarks and outlook

In this paper we studied the charged three-body problem. Finding solutions of the equations of motion is a formidable task. As solutions are confined to the joint level sets of the integrals

Fig. 4. Schematic plot illustrating the relation of critical points of the integral map to central configurations (see main text) for a) a (general) charged N -body system, b) a gravitational N -body system, and c) a (general) three-body system with electrostatic interactions.

(constants of motion) some information on the solutions can be obtained from the geometry of the level sets, or in other words, the integral manifolds which are the fibers of the integral map. In the gravitational case the topology of the fibers has been studied by McCord, Meyer and Wang [13]. The present paper comprises the first steps towards a generalization of their results to more general potentials. The topology of the integral manifolds may change at a critical value of the integral map. In the present article we studied the critical values for charged N -body systems and in particular for the case $N = 3$. The gravitational N -body system is contained in our definition of a charged N -body system as a special case (Definition 1.1). In the gravitational case it is known that critical points given by relative equilibria always project to planar central configurations [13,10]. In this paper we have seen that the situation is more subtle in the more general, charged case.

In order to summarize the results of this paper we consider the schematic plot in Fig. 4. The disks represent the set of critical points of the integral map. In Propositions 3.1 and 3.6 we have seen that the critical points are of two different types.

The left halves (region 1) of the disks contain critical points which on the space of vanishing center of mass and total momentum are given by *collinear phase space points* where the position vectors and momentum vectors of all bodies point in the same direction (assuming the center of mass to be at the origin). These are critical points of the angular momentum and independent of the particular Hamiltonian of the N -body system (where we define an N -body system according to Definition 1.1). The right halves of the disks represent critical points which are relative equilibria, i.e. critical points of the Hamiltonian restricted to the level set of constant angular momentum. The relative equilibria result from choosing suitable momenta for collinear (region 2) or non-collinear planar (region 3) central configurations, both with positive multiplier, or equivalently negative potential energy (Lemma 5.1). Here the momenta are chosen in such a way that they give rise to a rigid rotation of the corresponding central configuration. Region 4 represents relative equilibria that do not project to central configurations.

For the gravitational N -body system, region 4 has disappeared, see Fig. 4(b). In the gravitational N -body system all relative equilibria project to planar central configurations, and conversely every planar central configuration can be turned into a relative equilibrium by a proper choice of momenta. For the gravitational three-body problem, region 2 consists of the rigid rotations of collinear central configurations already found by Euler and region 3 consists of the rigid rotation of an equilateral triangle found by Lagrange.

A three-body system interacting via electrostatic forces has no relative equilibria resulting from non-collinear planar central configurations, i.e. compared to the general case in Fig. 4(a) region 3 has disappeared, see Fig. 4(c). This is the contents of Proposition 5.2 which also states that there are (up to similarity transformations) at most three relative equilibria associated with

Fig. 5. Left: Schematic picture of the curve Γ . The arrows indicate how the three branches are connected at infinity. For $u \in \{\xi_-, \xi_0, \xi_+\}$, Γ has points at infinity, for $u \in \{\eta_-, \eta_0, \eta_+\}$, Γ has cusp points. Right: Special values of u on the reduced space.

collinear central configurations. Moreover, the proof of [Proposition 3.6](#) shows that there might be relative equilibria that do not project to central configurations (we will give an example in [\[8\]](#)).

For a discussion of the stability of the relative equilibria of the charged three-body problem we refer to [\[4\]](#). Instead our focus in a forthcoming paper will be bifurcations of the non-compact integral manifolds due to critical points of the integral map at infinity [\[9\]](#). For the gravitational case, such critical points have been studied by Albouy [\[2\]](#). Specific examples and the manifestation of the bifurcations of the integral manifolds in terms of their configuration space projections, the so-called Hill regions, will be given in [\[8\]](#).

As the number of critical points of the integral manifolds of a charged three-body system differs in general from the gravitational case the topology of the integral manifolds will in general also be different. This is furthermore complicated by the fact that the charged case has more parameters and that opposed to the gravitational case not all relative equilibria project to central configurations. It remains to be seen whether the different nature of these critical points in the charged case also imply different kinds of bifurcations of the topology of the integral manifolds. As the bifurcations of the integrals manifolds in the gravitational case could be understood in [\[13\]](#) to a large extent from their projections to configuration space which is also the space in which the central configurations exist it is to be expected that this will be different in the charged case.

Acknowledgments

We thank Lei Zhao for useful discussions at an early stage of this research. Furthermore we wish to thank an anonymous referee for his suggestions which both directly and indirectly helped to improve the quality of this article. M. Zaman was supported by the Higher Education Commission of Pakistan, grant number No.SO(Trg)HE/1-10/Ph.D Scholarships/2012.

Appendix A. Special points of the curve Γ

On the curve Γ the polynomial f has double zeros. Therefore Γ is a fold curve and we may expect to find cusps. Moreover the parameterization of Γ is by rational functions. So we may also

expect points at infinity on Γ . For the next exposition, we write the parameterization explicitly (cf Eq. (10))

$$c(u) = \left(\frac{m_3}{m_1} \frac{g_1(u)}{g_3(u)}, \frac{m_3}{m_2} \frac{g_2(u)}{g_3(u)} \right) \text{ for } u \in \mathbb{R}, \quad (11)$$

where

$$\begin{aligned} g_1(u) &:= m_1(3m_1 + m_2 + m_3)u^2 - m_1(-3m_1 + m_2 - 3m_3)u - 2m_2(m_1 + m_3) \\ g_2(u) &:= u^3[-2m_1(m_2 + m_3)u^2 + m_2(-m_1 + 3(m_2 + m_3))u \\ &\quad + m_2(m_1 + 3m_2 + m_3)] \\ g_3(u) &:= (1 + u)^3[2m_1(m_2 + m_3)u^2 + (3m_3(m_2 + m_3) + m_1(4m_2 + 3m_3))u \\ &\quad + 2m_2(m_1 + m_3)]. \end{aligned}$$

To keep the computations tractable we for now set $m_1 := \mu$, $m_2 := \mu$ and $m_3 := 1$. The consequences of this choice are:

1. When $m_1 = m_2 = \mu$ the curve Γ has a symmetry property. Let $T(\beta_1, \beta_2) = (\beta_2, \beta_1)$ then $Tc(u) = c(\frac{1}{u})$. This will be explained in more detail in [Appendix B](#).
2. A consequence of the previous is that if η is a value such that $c(\eta)$ is a singular point, then $c(\eta^{-1})$ is also a singular point. If Γ has an odd number of singular points, one must occur for $u = 1$ or $u = -1$. The singular points of Γ are found by solving $c'(u) = 0$.
3. Similarly, if ξ is a value such that $c(\xi)$ is a point at infinity (taking an appropriate limit), then $c(\xi^{-1})$ is also a point at infinity. In case Γ has an odd number of such points, one must occur for $u = 1$ or $u = -1$.

With these choices for the masses m_i we have the following result.

Lemma A.1. *The curve Γ defined in Eq. (11) has six special points.*

- (i) *There are three points at infinity defined by the zeros of g_3 (see (11)) namely at u equal to*

$$\xi_{\pm} = \frac{-3 - 6\mu - 4\mu^2 \pm \sqrt{9 + 36\mu + 44\mu^2 + 16\mu^3}}{4\mu(1 + \mu)} \text{ or } \quad (12)$$

$$\xi_0 = -1.$$

For $\xi_{\pm}$, the corresponding points on Γ are regular points at infinity. Note that $\xi_{-}\xi_{+} = 1$. For ξ_0 the corresponding point is a degenerate cusp at infinity, with equivalent local parameterization $t \mapsto (t^3 + O(t^4), t^4 + O(t^5))$. The curve Γ has three branches, namely $c((\xi_{-}, \xi_0))$, $c((\xi_0, \xi_{+}))$ and $c((\xi_{+}, \infty) \cup (\infty, \xi_{-}))$, see [Fig. 5](#).

- (ii) *There are three singular points at u taking one of the values*

$$\eta_{\pm} = \frac{-5 - 12\mu - 8\mu^2 \pm \sqrt{21 + 80\mu + 92\mu^2 + 32\mu^3}}{2(1 + 5\mu + 4\mu^2)} \text{ or } \quad (13)$$

$$\eta_0 = 1.$$

Each singular point is a regular cusp with an equivalent local parameterization $t \mapsto (t^2 + O(t^3), t^3 + O(t^4))$. Note that $\eta_{-}\eta_{+} = 1$.

The inequalities $\xi_{-} < \eta_{-} < \xi_0 < \eta_{+} < \xi_{+} < \eta_0$, valid for all $\mu > 0$, imply that the six special points on Γ are alternately cusps and points at infinity.

Proof. Points at infinity are found from solving $g_3(u) = 0$. Since g_3 has an easy factorization, see Eq. (11), we immediately find the solutions presented in the lemma. For each solution, we have $g_1(\xi, m) \neq 0$ and $g_2(\xi, m) \neq 0$ so that we find indeed points at infinity. The singular points are found from solving $c'(\eta) = (0, 0)$. After some computations we find that η must be a zero of the polynomial

$$f_3 f_2' f_1'' - f_2 f_3' f_1'' - f_3 f_1' f_2'' + f_1 f_3' f_2'' - f_2 f_1' f_3'' - f_1 f_2' f_3'',$$

where the f_i are defined in Eq. (8). For our choice $m_1 = m_2 = \mu$ and $m_3 = 1$, this polynomial factorizes as

$$(u - 1)u^2(u + 1)^2(1 + 5\mu + 4\mu^2 + 5u + 12\mu u + 8\mu^2 u + u^2 + 5\mu u^2 + 4\mu^2 u^2)$$

from which we readily find the values of η . For a local study of the singular points, we look at the Taylor series of $A[c(\eta + t) - c(\eta)]$, where A is a linear transformation that locally puts Γ in a standard form, namely $A = \text{mat}(Jv, v)$ where v is the tangent vector of Γ at the singularity, $v := \frac{d}{dt}(c(\eta + t) - c(\eta))|_{t=0}$, and J is the matrix of a rotation over $\frac{\pi}{2}$. For each value of η , we find

$$A[c(\eta + t) - c(\eta)] = (\gamma_1(\mu)t^3 + O(t^4), \gamma_2(\mu)t^2 + O(t^3))$$

where the γ_i are functions of μ . In general they are non-zero for positive values of μ . Their explicit expressions are quite involved except for $\eta = 1$. Then we have $\gamma_1(\mu) = \frac{7+8\mu}{24+80\mu+64\mu^2}$ and $\gamma_2(\mu) = \frac{3(7+8\mu)}{8(3+4\mu)^2}$. For a local study of points at infinity, we look at the Taylor series of $\frac{c}{\|c\|^2}$. Let ξ be a value such that $c(\xi)$ “is” a point at infinity. Then we find

$$\frac{c(\xi + t)}{\|c(\xi + t)\|^2} = (\gamma_1(\mu)t + O(t^2), \gamma_2(\mu)t + O(t^2))$$

for $\xi = \xi_-$ and $\xi = \xi_+$. So these are regular points since the γ_i are positive for m positive. For $\xi = -1$ however, we find after a linear transformation

$$A \frac{c(\xi + t)}{\|c(\xi + t)\|^2} = \left(\frac{3+2\mu}{4}t^3 + O(t^4), \frac{3}{16}(3+2\mu)^2 t^4 + O(t^5) \right).$$

Even for general values of m_1, m_2 and m_3 we find

$$A(m_1, m_2, m_3) \frac{c(\xi + t)}{\|c(\xi + t)\|^2} = (a_1(m_1, m_2, m_3)t^3 + O(t^4), a_2(m_1, m_2, m_3)t^4 + O(t^5)),$$

where the a_i are positive. Therefore at $\xi = -1$ we have a singular point at infinity. $\square$

We present a schematic picture of the curve Γ in Fig. 5 which in reality is very distorted. The picture however serves to show the global structure and the alternation of singular points and points at infinity. Moreover from this picture we get regions in the parameter plane with a constant number of real solutions of f .

Appendix B. The action of the permutation group

The main purpose of this section is to derive Corollary B.4 which states that all central configurations can be found from equation $f(u) = 0$ in Proposition 4.5. To reach this conclusion we have to consider the action of the permutation group on the reduced space of collinear configurations. The permutation group S_3 of three bodies is generated by $\pi_1 = (1 \ 2)$ and $\pi_2 = (2 \ 3)$. Then the other elements are $\pi_0 = \text{id}$, $\pi_3 = \pi_1 \circ \pi_2$, $\pi_4 = \pi_2 \circ \pi_1$ and $\pi_5 = \pi_1 \circ \pi_2 \circ \pi_1$. Consider the product space of collinear configurations and the space of

Table 3Permutation of the intervals I_1 , I_2 and I_3 by h_1 and h_2 .

	I_1	I_2	I_3
h_1	I_2	I_1	I_3
h_2	I_3	I_2	I_1

parameters $(r, \alpha, m) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. Let S_3 act on each factor in the same standard way, then we immediately have the following.

Lemma B.1. *The equation for (collinear) central configurations in Definition 1.2 with potential V from Eq. (7) is invariant with respect to the action of S_3 defined above.*

Now we consider the reduced space of collinear configurations. Then there is an induced S_3 -action on this space. Using the intermediate variables $x = r_2 - r_3$, $y = r_3 - r_1$ and $z = r_1 - r_2$ and the coordinate u on the reduced space, defined by $y = ux$ we readily obtain the following result.

Lemma B.2. *The induced S_3 -action on the reduced space of collinear configurations is determined by the homomorphisms h_i with: $h_0 = id$, $h_1(u) = \frac{1}{u}$, $h_2(u) = -(1+u)$, $h_3 = h_1 \circ h_2$, $h_4 = h_2 \circ h_1$ and $h_5 = h_1 \circ h_2 \circ h_1$.*

Note that the three disjoint intervals I_1 , I_2 and I_3 in $\mathbb{R} \cup \{\infty\}$, 4.4, are permuted by the h_i , see Table 3.

On the reduced space of collinear configurations equation $f(u) = 0$, see (8), determines the collinear central configurations. The polynomial f depends on parameters m and α and we explicitly write $f(u; \alpha, m)$ to stress that fact. In deriving f we made a choice for the signs of the intermediate variables x , y and z . Permuting the bodies will also change the signs of the transformed x , y and z . From the very beginning we could have absorbed these signs into the parameters α_i . This would give Eqs. (9) a more symmetric appearance ($\alpha_3 \mapsto -\alpha_3$), but obfuscate the sign dependence. Therefore we chose fixed signs, but this means that the S_3 -action on α is non-standard on the reduced space. In fact we have $\phi_0 = id$, $\phi_1(\alpha) = (\alpha_2, \alpha_1, \alpha_3)$, $\phi_2(\alpha) = (\alpha_1, -\alpha_3, -\alpha_2)$, $\phi_3 = \phi_1 \circ \phi_2$, $\phi_4 = \phi_2 \circ \phi_1$ and $\phi_5 = \phi_1 \circ \phi_2 \circ \phi_1$. A short computation immediately reveals the following.

Lemma B.3. *The polynomial f transforms under the generators of the S_3 action on (r, α, m) according to*

$$\begin{aligned} -u^5 f(h_1(u); \phi_1(\alpha), \pi_1(m)) &= f(u; \alpha, m), \\ -f(h_2(u); \phi_2(\alpha), \pi_2(m)) &= f(u; \alpha, m). \end{aligned}$$

Thus we have the following.

Corollary B.4. *The collinear central configurations for permuted bodies follow from equation $f(u) = 0$ in Proposition 4.5 with the particular sign choice, using the transformations from Lemma B.3.*

Before finding the discriminant set of f we simplified the problem by using the fact that f is homogeneous of degree 1 in α and assuming $\alpha_3 \neq 0$ we scaled $(\alpha_1, \alpha_2, \alpha_3) \mapsto \frac{1}{\alpha_3}(\alpha_1, \alpha_2, \alpha_3)$.

Then we get an S_3 -action ψ on the two-parameter space with coordinates $(\beta_1, \beta_2) = \beta(\alpha) = (\frac{\alpha_1}{\alpha_3}, \frac{\alpha_2}{\alpha_3})$ defined by $\psi_i(\beta(\alpha)) = \beta(\phi_i(\alpha))$. In particular we have the generators

$$\begin{aligned}\psi_1(\beta_1, \beta_2) &= (\beta_2, \beta_1), \\ \psi_2(\beta_1, \beta_2) &= (-\frac{\beta_1}{\beta_2}, \frac{1}{\beta_2}).\end{aligned}$$

One of the components of the discriminant set of f is the curve Γ . Using the lemmas and definitions above we arrive at the final result.

Corollary B.5. *Under permutations of the bodies, the curve Γ transforms via its parameterization c as*

$$\psi_i\left(c(h_i(u); \pi_i(m))\right) = c(u; m)$$

for $i \in \{0, 1, 2, 3, 4, 5\}$.

Looking at the action of the maps ψ_i we also have the following corollary.

Corollary B.6. *The topology of the regions does not depend on the masses $m = (m_1, m_2, m_3) \in \mathbb{R}_{>0}^3$.*

Appendix C. Counting the collinear central configurations

In Section 4.2 we found the number of collinear central configurations for the order $(1, 3, 2)$ (or $(2, 3, 1)$), i.e. body three in the middle. Now we also wish to find the number of collinear central configurations for the same parameter values, but with body two or body one in the middle. These configurations can be obtained from the order $(1, 3, 2)$ by the permutations $\pi_2 = (23)$ and $\pi_5 = (13)$ respectively. From Lemma B.3 we know that on the open intervals I_1 , I_2 and I_3 equation $f(u, \phi_i(\alpha), \pi_i(m)) = 0$ is equivalent with $f(h_i^{-1}(u), \alpha, m) = 0$. Therefore the number of collinear central configurations with a permuted order is given by the number of solutions of $f(u, \alpha, m)$ in the intervals I_2 and I_1 respectively. Thus we have the following lemma.

Lemma C.1. *For each value of the masses $m = (m_1, m_2, m_3) \in \mathbb{R}_{>0}^3$, there can be 1, 3 or 5 collinear central configurations depending on the values of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.*

Appendix D. The sign of the potential

In this section we only consider *collinear* central configurations. For brevity we drop *collinear*. We wish to determine the sign of the potential V at a central configuration. The latter are determined by a fifth degree polynomial f , see Eq. (8), and thus cannot be found explicitly. We can however find central configurations at which the potential is zero because the polynomial f depends on parameters. In fact this way we find regions in the (β_1, β_2) -parameter plane, see Appendix A, where the numbers of central configurations with positive and negative potential values do not change.

To this end we eliminate u from the equations

$$\begin{cases} f(u) = 0, \\ U(u) = 0 \end{cases}$$

Table 4

The number of zeros of f in each region in the intervals I_1 , I_2 and I_3 . Together with the signs of the potential at each zero. Region 6 is split into four subregions with different signs of the potential, namely $(+, +, -)$, $(-, +, +)$, $(+, -, +)$ and $(+, +, +)$. The last subregion is bounded only by the curve defined by Eq. (14).

Region	Zeros	Signs	Region	Zeros	Signs
1	(0, 0, 1)	(, , -)	7	(1, 3, 1)	(+, - - -, +)
2	(0, 2, 1)	(, - -, -)	8	(1, 2, 0)	(+, ++,)
3	(0, 1, 0)	(, +,)	9	(1, 0, 0)	(+, ,)
4	(2, 1, 0)	(++, +,)	10	(2, 0, 1)	(-- -, , -)
5	(3, 1, 1)	(- - -, +, +)	11	(1, 0, 2)	(+, , --)
6	(1, 1, 1)		12	(0, 1, 2)	(, +, --)
			13	(1, 1, 3)	(+, +, - - -)

where U is the potential on the reduced space of collinear configurations. For sake of completeness note that

$$U(u) = -\frac{\beta_1 u^2 + (1 + \beta_1 + \beta_2)u + \beta_2}{u(1 + u)}.$$

After eliminating u we find a relation for β defining a curve bounding the aforementioned regions

$$(\beta_1 - \beta_2)^2 + 2(\beta_1 + \beta_2) + 1 = 0. \quad (14)$$

Note that this equation does not involve the masses. The curve defined by this equation is a parabola located in the third quadrant in Fig. 1. The parabola is symmetric with respect to reflections at the diagonal and tangent to the β_1 and β_2 axes at $(\beta_1, (\beta_2) = (-1, 0))$ and $(\beta_1, (\beta_2) = (0, -1))$, respectively. It is not hard to see that it has no intersections with the curve Γ . Thus the regions of constant numbers of zeros of f in the intervals I_1 , I_2 and I_3 are not subdivided any further apart from region 6. The latter splits into four subregions bounded by the curve in Eq. (14) and the coordinate axes.

Now we count the number of central configurations with negative value of the potential in the interval $I_3 = (0, \infty)$ only since the others follow by the action of the permutations, see Appendix B. Taking any point from the various regions we find Table 4.

References

- [1] R.H. Abraham, J.E. Marsden, *Foundations of Mechanics*, second ed., Addison-Wesley, 1987.
- [2] A. Albouy, Integral manifolds of the n -body problem, *Invent. Math.* 114 (1993) 463–488.
- [3] F. Alfaro, E. Pérez-Chavela, Relative equilibria in the charged n -body problem, *Canad. Appl. Math. Q.* 10 (2003) 1–13.
- [4] F. Alfaro, E. Pérez-Chavela, Linear stability of relative equilibria in the charged three body problem, *J. Differential Equations* 245 (2008) 1923–1944.
- [5] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1980.
- [6] P. Atela, R.I. McLachlan, Global behavior of the charged isosceles three-body problem, *Int. J. Bifur. Chaos* 4 (4) (1994) 865–884.
- [7] A. Chenciner, R. Montgomery, A remarkable periodic solution of the three-body problem in the case of equal masses, *Ann. of Math.* 152 (3) (2000) 881–901.
- [8] I. Hoveijn, H. Waalkens, M. Zaman, Bifurcations of Hill regions in charged three-body problems, 2018 (in preparation).
- [9] I. Hoveijn, H. Waalkens, M. Zaman, Critical points at infinity in the charged three-body problem, 2018 (in preparation).
- [10] J. Llibre, R. Moeckel, C. Simó, *Central Configurations, Periodic Orbits and Hamiltonian Systems*, in: *Advanced Courses in Mathematics — CRM Barcelona*, Birkhäuser, 2015.

- [11] J. Llibre, D.J. Tonon, Symmetric periodic orbits for the collinear charged 3-body problem, *J. Math. Phys.* 58 (2017).
- [12] J.E. Mansilla, C. Vidal, Geometric interpretation for the spectral stability in the charged three-body problem, *Celest. Mech. Dyn. Astr.* 113 (2012) 205–213.
- [13] C.K. McCord, K.R. Meyer, Q. Wang, The Integral Manifolds of the Three Body Problem, in: *Memoirs of the American Mathematical Society*, no. 628, AMS, 1998.
- [14] C. Moore, Braids in classical dynamics, *Phys. Rev. Lett.* 70 (24) (1993) 3675–3679.
- [15] A.C. Ortega, M. Falconi, Schubart solutions in the charged collinear three-body problem, *J. Dynam. Differential Equations* 28 (2016) 519–532.
- [16] E. Pérez-Chavela, D. Saari, A. Susin, Z. Yan, Central configurations in the charged three body problem, *Contemp. Math.* 198 (1996) 137–155.
- [17] K. Richter, G. Tanner, D. Wintgen, Classical mechanics of two-electron atoms, *Phys. Rev. A* 48 (1993) 4182–4196.
- [18] F. Rupp, J. Scheuerle, Classification of a class of relative equilibria in three body Coulomb systems, *Discrete Contin. Dyn. Syst.* (2011).
- [19] G. Tanner, K. Richter, J.-M. Rost, The theory of two-electron atoms: between ground state and complete fragmentation, *Rev. Modern Phys.* 72 (2000) 497–544.
- [20] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1997.
- [21] A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, 1941.