ON THE ASYMPTOTIC BEHAVIOR OF THE COEFFICIENTS OF ASYMPTOTIC POWER SERIES AND ITS RELEVANCE TO STOKES PHENOMENA*

G. K. IMMINK†

Abstract. This paper discusses the relevance of the asymptotic behavior of the coefficients of asymptotic power series for the study of Stokes phenomena. By way of illustration a connection problem is considered in the theory of linear difference equations.

Key words. asymptotic expansion, isomorphism of Malgrange, Cauchy-Heine transform, saddle-point method, Stokes phenomenon, linear analytic functional equation, difference equation

AMS(MOS) subject classifications. 30El 5, 39

Introduction. In this paper we extend and apply ideas of Malgrange [10] and Ramis [12] concerning the connection between Stokes phenomena, in a wider sense, and formal power series. We start with an illustrative example.

Let \( y \) be an analytic function on the Riemann surface of \( \log z \), with the following properties.

(i) \( y \) admits an asymptotic expansion of the form \( \sum_{n=0}^{\infty} y_n z^{-n} \) as \( z \to \infty \) in the sector \( S: -\pi/2 < \arg z < 5\pi/2 \).

(ii) \( y(z) - y(z e^{2\pi i}) = c e^{-z}, \quad c \in \mathbb{C}^* \).

The second property implies that the asymptotic behavior of \( y \) changes abruptly as \( \arg z \) becomes larger than \( 5\pi/2 \) or less than \( -\pi/2 \). Such a change in asymptotic behavior will be called a Stokes phenomenon.

Now consider the function \( h \) defined by

\[
\begin{align*}
h(z) &= \frac{1}{2\pi i} \int_0^\infty \frac{e^{-t}}{t-z} \, dt, \quad 0 < \arg z < 2\pi. \\
\end{align*}
\]

\( h \) is a Cauchy-Heine transform of \( e^{-z} \) (cf. [12]). By deformation of the path of integration it may be continued analytically to the Riemann surface of \( \log z \). With the aid of residue calculus we readily verify that

\[
(0.1) \quad h(z) - h(z e^{2\pi i}) = e^{-z}.
\]

Moreover, \( h \) admits the asymptotic expansion \( \sum_{n=1}^{\infty} h_n z^{-n} \) as \( z \to \infty, \quad z \in S \), where

\[
(0.2) \quad h_n = -\frac{1}{2\pi i} \int_0^\infty e^{-it} t^{-n-1} \, dt, \quad n \in \mathbb{N}.
\]

From (ii) and (0.1) we conclude that

\[
y(z e^{2\pi i}) - ch(z e^{2\pi i}) = y(z) - ch(z).
\]

Thus it turns out that \( y - ch \) is a single-valued analytic function, admitting an asymptotic expansion of the form \( \sum_{n=0}^{\infty} a_n z^{-n} \) as \( z \to \infty, \quad z \in S \), where

\[
(0.3) \quad a_n = y_n - ch_n.
\]

* Received by the editors January 19, 1989; accepted for publication (in revised form) February 14, 1990.

† University of Groningen, Institute of Econometrics, P.O. Box 800, 9700AV, Groningen, the Netherlands.
This implies that \( y - ch \) is holomorphic at \( \infty \) and, consequently, \( \sum_{n=0}^{\infty} a_n z^{-n} \) is a convergent power series. From (0.2) and (0.3) it now follows that

\[
c = -2\pi i \lim_{n \to \infty} \frac{y_n}{(n-1)!}.
\]

Apparently, the constant \( c \), which plays a central role in the Stokes phenomenon occurring in this example, is intimately related to the asymptotic behavior of the coefficients \( y_n \). It is this relationship that forms the subject of this paper.

We shall consider the following situation. Suppose we are given a number of sectors \( S_\nu, \nu \in \{1, \ldots, N\} \), which cover a neighborhood of \( \infty \) and a corresponding number of functions \( y_\nu \) with the following properties: \( y_\nu \) is analytic in \( S_\nu \) and represented asymptotically by a series of the form \( \sum_{n=0}^{\infty} \hat{y}_n z^{-n} \) (independent of \( \nu \)) as \( z \to \infty, z \in S_\nu \), \( \nu \in \{1, \ldots, N\} \). Moreover, assume that

\[
(0.4) \quad y_{\nu+1}(z) - y_\nu(z) = \sum_{j=1}^{m} c_j \varphi_j^\nu(z), \quad z \in S_\nu \cap S_{\nu+1}, \quad \nu \in \{1, \ldots, N\},
\]

where \( S_{N+1} = e^{2\pi i} S_1 \), \( y_{N+1}(z) = y_1(ze^{-2\pi i}) \), \( c_j \in \mathbb{C} \), and the \( \varphi_j^\nu \) belong to a certain class of analytic functions. We shall establish a relation between the complex numbers \( c_j \) and the asymptotic behavior of \( \hat{y}_n \) for \( n \to \infty \). In some applications this relation may be exploited to "compute" at least part of the numbers \( c_j \) from the coefficients \( \hat{y}_n \) (cf. [9] and Remark 2 herein).

If the \( y_\nu \) represent (sectorial models of) a resurgent function, our results could be derived from the work of Ecalle (cf. [4]). For the present purpose, however, this assumption is not needed and we shall establish the relation mentioned above in a more direct manner.

The argument is essentially the same as the one we used in [9]. It is based on the Propositions 1.1–1.3 herein. Proposition 1.1 concerns the properties of Cauchy–Heine transforms of functions like the \( \varphi_j^\nu \) in (0.4). Proposition 1.2 enables us to construct, from the Cauchy–Heine transforms of the \( \varphi_j^\nu \), analytic functions \( H_\nu \) with the same properties as the \( y_\nu \) and only differing from the \( y_\nu \) by a convergent power series in \( 1/z \). The coefficients of the asymptotic expansion \( \hat{H} \) of the \( H_\nu \) are given by the expression

\[
\hat{H}_n = -\frac{1}{2\pi i} \sum_{\nu=1}^{N} \sum_{j=1}^{m} c_j \int_{y_\nu} \varphi_j^\nu(t) t^{-n-1} dt, \quad y_\nu \subset S_\nu.
\]

Under certain conditions, like those mentioned in Proposition 1.3, the saddle-point method may be applied to the integral

\[
\int_{y_\nu} \varphi_j^\nu(t) t^{-n-1} dt
\]

to obtain its asymptotic behavior for \( n \to \infty \). The main result is stated in Theorem 1.4. In § 2 this result is applied to a connection problem in the theory of homogeneous linear difference equations.

1. **The general argument.** Let \( \mathbb{C}_\infty \) denote the Riemann surface of \( \log z \). Let \( z_0 \in \mathbb{C}_\infty \), \( \alpha, \beta \in \mathbb{R}, \alpha < \beta \). By \( S(\alpha, \beta) \) we denote the sector

\[
S(\alpha, \beta) = \{ z \in \mathbb{C}_\infty : \alpha < \arg z < \beta \}
\]

and by \( S(z_0, \alpha, \beta) \) the set

\[
S(z_0, \alpha, \beta) = \{ z \in \mathbb{C}_\infty : \alpha < \arg (z - z_0) < \beta, |z| > |z_0| \}.
\]

This will also be called a sector.
If $S$ is a sector of the form $S = S(z_0, \alpha, \beta)$, then $S$ will denote the sector $S(z_0, \alpha, \beta + 2\pi)$.

Let $S = S(z_0, \alpha, \beta)$, $S' = S(z_1, \alpha', \beta')$ with $\alpha < \alpha' < \beta' < \beta$. We shall write $S' \Subset S$ whenever $z_1 \in S$ and $S' \Subset S(z_0, \alpha', \beta')$.

Let $h = \sum_{n=0}^{\infty} h_n z^{-n}$ be a formal power series in $z^{-1}$, $S$ a sector of the type (1.1), and $h$ a function on $S$. We say that $h$ is represented asymptotically by $\hat{h}$ as $z \to \infty$ in $S$, and write

$$h(z) \sim \sum_{n=0}^{\infty} h_n z^{-n}, \quad z \to \infty \text{ in } S$$

if, for every $S' \Subset S$ and every $N \in \mathbb{N}$,

$$R_N(h; z) = h(z) - \sum_{n=0}^{N-1} h_n z^{-n} = O(z^{-N}), \quad z \to \infty, \quad z \in S'.$$

Any function $\varphi$ which is analytic in a sector $S$ and represented asymptotically by zero (i.e., the series with coefficients equal to zero) as $z \to \infty$ in $S$, may be written as the difference of two determinations of its Cauchy-Heine transform. The following proposition, due to Ramis, is concerned with the asymptotic properties of this Cauchy-Heine transform.

**Proposition 1.1** (cf. [12, Prop. 4.2]). Let $\alpha$ and $\beta$ be real numbers such that $\alpha < \beta$, $z_0 \in S(\alpha, \beta)$, and let $\varphi$ be an analytic function on $S = S(z_0, \alpha, \beta)$. Suppose there exist positive numbers $M_n$, $n \in \mathbb{N}$, such that

$$\sup_{z \in S} |z^n \varphi(z)| < M_n, \quad n \in \mathbb{N}.$$

Then the function $h$ defined by

$$h(z) = \frac{1}{2\pi i} \int_{\gamma} \varphi(\zeta) \frac{d\zeta}{\zeta(z-z)} \in S(z_0, \Theta, \Theta + 2\pi),$$

where $\gamma$ is a half line in $S$ from $z_0$ to $\infty$ with direction $\Theta$, has the following properties:

(i) $h$ can be continued analytically to $S$,

(ii) $h(z) - h(z e^{2\pi i}) = \varphi(z)$ for all $z \in S$,

(iii) $h$ is represented asymptotically by

$$\sum_{n=0}^{\infty} -\frac{1}{2\pi i} \left( \int_{\gamma} \varphi(\zeta) \zeta^{n-1} d\zeta \right) z^{-n},$$

as $z \to \infty$ in $S$. Moreover, for every $S' \Subset S$ there exists a positive constant $C_S$ such that

$$\sup_{z \in S'} |z^n R_n(h; z)| \leq C_S M_{n+1}, \quad n \in \mathbb{N}.$$

**Proof.** Let us suppose that $S$ is a convex set, i.e., $\beta - \pi / 2 < \arg z_0 < \alpha + \pi / 2$. In that case every half line from $z_0$ to $\infty$ with direction $\Theta \in (\alpha, \beta)$ lies in $S$. If $\gamma$ has direction $\Theta$, $h$ is obviously analytic in $S(z_0, \Theta, \Theta + 2\pi)$. The analytic continuation to $S$ is obtained by varying $\Theta$. Part (ii) follows immediately from Cauchy’s theorem.

Now let $S' = S(z_1, \alpha', \beta') \Subset S$. Then there is a number $\varepsilon \in (0, \pi / 2)$ such that $\alpha + \varepsilon < \arg (z - z_0) < \beta + 2\pi - \varepsilon$ for all $z \in S'$. Let $z \in S'$ and choose $\Theta \in (\alpha, \beta)$ in such
a way that $\Theta + \varepsilon < \arg (z - z_0) < \Theta + 2\pi - \varepsilon$. Let $\gamma_\Theta$ be the half line from $z_0$ to $\infty$ with direction $\Theta$. For all $\zeta \in \gamma_\Theta$ the following inequality holds:

\begin{equation}
|\zeta - z| > |z - z_0| \sin \varepsilon > |z| \left(1 - \left|\frac{z_0}{z_1}\right|\right) \sin \varepsilon.
\end{equation}

It is easily seen that

\begin{equation}
z^nR_n(h; z) = \frac{z}{2\pi i} \int_{\gamma_\Theta} \frac{\varphi(\zeta)}{\zeta - z} \frac{\zeta^{n-1}}{\varphi(\zeta)} d\zeta,
\end{equation}

$n \in \mathbb{N}$.

With (1.2) and (1.3) it follows that

\begin{equation}
|z^nR_n(h; z)| < \frac{1}{2\pi \sin \varepsilon} \left(1 - \left|\frac{z_0}{z_1}\right|\right)^{-1} \sup_{\Theta \in (\alpha, \beta)} \left|\int_{\gamma_\Theta} \frac{d\zeta}{\zeta^2}\right| M_{n+1}.
\end{equation}

Hence

\begin{equation}
\sup_{z \in S'} |z^nR_n(h; z)| < \frac{1}{2\pi \sin \varepsilon} \left(1 - \left|\frac{z_0}{z_1}\right|\right)^{-1} \sup_{\Theta \in (\alpha, \beta)} \left|\int_{\gamma_\Theta} \frac{d\zeta}{\zeta^2}\right| M_{n+1}
\end{equation}

and this proves (iii).

If $S$ is not convex the above argument must be adapted in an obvious manner.

**Proposition 1.2** (cf. [10], [12]). Let $N \in \mathbb{N}$. Let $\alpha, \beta, \nu \in \{1, \ldots, N\}$, be real numbers such that $\alpha_1 < \beta_1 < \beta_{n+1} < \beta_{n+2} < \beta_1 + 2\pi$. Let $z_1 \in S(\alpha_1, \beta_1)$ and $S' = S(z_1, \alpha_1, \beta_1)$. Suppose that, for every $\nu \in \{1, \ldots, N\}$, we are given an analytic function $\varphi(\zeta)$ on $S'$ with the property that $\varphi(\zeta) \sim 0$ as $\zeta \to \infty$ in $S'$.

Let

\begin{equation}
h_\nu(z) = \frac{z}{2\pi i} \int_{\gamma_\nu} \frac{\varphi(\zeta)}{\zeta - z} d\zeta,
\end{equation}

$S(z_\nu, \Theta_\nu, \Theta_\nu + 2\pi)$, $\nu \in \{1, \ldots, N\}$, where $\gamma_\nu$ is a half line in $S'$ from $z_\nu$ to $\infty$ with direction $\Theta_\nu$ and let

\begin{equation}
H_\nu(z) = \sum_{\mu = 1}^{\nu-1} h_\mu(z) + \sum_{\mu = \nu}^{N} h_\mu(z e^{2\pi i}) \quad \text{if } \nu \in \{2, \ldots, N\},
\end{equation}

\begin{equation}
H_1(z) = \sum_{\mu = 1}^{N} h_\mu(z e^{2\pi i}) \quad \text{and} \quad H_{N+1}(z) = \sum_{\mu = 1}^{N} h_\mu(z).
\end{equation}

The functions $H_\nu$ have the following properties:

(i) For every $\nu \in \{1, \ldots, N + 1\}$ there exists a $\tilde{z}_\nu \in S(\alpha_\nu, \beta_\nu)$ such that $\tilde{z}_N = \tilde{z}_1 e^{2\pi i}$ and $H_\nu$ is analytic on $S_\nu = S(z_\nu, \alpha_\nu, \beta_\nu)$.

(ii) $H_{\nu+1}(z) - H_\mu(z) = \varphi(\zeta)$ for all $z \in S_\nu \cap S_{\nu+1}$, $\nu \in \{1, \ldots, N\}$, and $H_{N+1}(z) = H_1(z e^{-2\pi i})$ for all $z \in e^{2\pi i} S_1$.

(iii) $H_\nu$ admits an asymptotic power series expansion $\tilde{H}_\nu$ independent of $\nu$, as $z \to \infty$ in $S_\nu$.

Moreover, if $\tilde{H}_\nu$, $\nu = 1, \ldots, N + 1$, are functions with the same properties, then there exists a function $h$, holomorphic at $\infty$, such that

\begin{equation}
\tilde{H}_\nu - H_\nu = h \quad \text{for all } \nu \in \{1, \ldots, N\}.
\end{equation}

**Proof.** From Proposition 1.1(i) we deduce that $H_\nu$ is analytic in

\[ \bigcap_{\mu = 1}^{\nu-1} S(z_\mu, \alpha_{\mu+1}, \beta_\mu + 2\pi) \bigcap_{\mu = \nu}^{N} S(z_\mu, \alpha_{\mu+1} - 2\pi, \beta_\mu) \quad \text{if } \nu \in \{2, \ldots, N\}, \]
in

\[ \bigcap_{\mu = 1}^{N} S(z_\mu, \alpha_{\mu+1} - 2\pi, \beta_\mu) \text{ if } \nu = 1, \]

and in

\[ \bigcap_{\mu = 1}^{N} S(z_\mu, \alpha_{\mu+1} + 2\pi) \text{ if } \nu = N + 1, \]

and this set contains a sector of the form \( S(\xi, \alpha, \beta) \) for a suitable choice of \( \xi \). Part (ii) follows immediately from Proposition 1.1(ii) by observing that

\[ H_{\nu+1}(z) - H_\nu(z) = h_\nu(z) - h_\nu(z e^{2\pi i}) \text{ for all } \nu \in \{1, \ldots, N\}. \]

Furthermore, Proposition 1.1(iii) implies that \( H_\nu(z) \sim \sum_{n=0}^{\infty} H_n z^{-n} \), as \( z \to \infty \) in \( S_\nu \), where

\[(1.4) \ H_n = -\frac{1}{2\pi i} \sum_{\nu=1}^{N} \int_{\gamma_\nu} \varphi_\nu(\zeta) \zeta^{n-1} \, d\zeta, \quad n \in \mathbb{N}. \]

Now suppose that \( \tilde{H}_\nu, \nu = 1, \ldots, N + 1 \), are functions with the properties (i)-(iii) mentioned in Proposition 1.2. Then there exist \( z'_\nu \in S(\alpha_\nu, \beta_\nu) \) such that both \( H_\nu \) and \( \tilde{H}_\nu \) are analytic on \( S \cup S(z'_\nu, \alpha_\nu, \beta_\nu) \) and we have

\[ \tilde{H}_{\nu+1}(z) - \tilde{H}_\nu(z) = H_{\nu+1}(z) - H_\nu(z), \quad z \in \tilde{S}_\nu \cap \tilde{S}_{\nu+1}, \quad \nu \in \{1, \ldots, N\} \]

and

\[ \tilde{H}_{N+1}(z) - \tilde{H}_1(z e^{-2\pi i}) = H_{N+1}(z) - H_1(z e^{-2\pi i}), \quad z \in \tilde{S}_{N+1}. \]

It follows that

\[ \tilde{H}_{\nu+1} - H_{\nu+1} = \tilde{H}_\nu - H_\nu \quad \text{for all } \nu \in \{1, \ldots, N\} \]

and, moreover,

\[ \tilde{H}_{N+1}(z) - H_{N+1}(z) = \tilde{H}_1(z e^{-2\pi i}) - H_1(z e^{-2\pi i}). \]

Hence the function \( h = \tilde{H}_1 - H_1 \) can be continued analytically to a reduced neighborhood of \( \infty \). Furthermore, property (iii) implies that \( h \) admits an asymptotic power series expansion in \( z^{-1} \) as \( z \to \infty \) in a neighborhood of \( \infty \) and, consequently, \( h \) is analytic in a full neighborhood of \( \infty \).

The next proposition concerns the asymptotic behavior of integrals of the type

\[ \int_{\gamma} \varphi(z) z^n \, dz, \]

where \( \gamma \) is a half line and \( \varphi \) is an analytic function with the property that \( \varphi(z) \to 0 \) as \( z \to \infty \) in some sector \( S \) containing \( \gamma \). The conditions (iii)-(v) below are purely technical and have been chosen in such a way that the result follows by a straightforward application of the saddle-point method. They might be relaxed or replaced by other conditions. We have merely tried to define a class of functions for which this method works.

**Proposition 1.3 (cf. [2, Thm. 7, Remark 6]).** Let \( \alpha \) and \( \beta \) be real numbers such that \( \alpha < \beta \), \( z_0 \in S(\alpha, \beta) \), and let \( \psi \) be an analytic function on \( S = S(z_0, \alpha, \beta) \) with the property that

(i) \( \exp \psi(z) \sim 0 \) as \( z \to \infty \) in \( S \).

Let \( g : S \times \mathbb{N} \to \mathbb{C} \) be defined by

\[ g(z, n) = \psi(z) + n \log z. \]
Suppose there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) the following conditions hold:

(ii) The equation \( \partial g / \partial z = 0 \) has a solution \( s_n \in S \) such that the half line \( \gamma_n \) from \( z_0 \) to \( \infty \) through \( s_n \) is contained in \( S \). Moreover, \( s_n \to \infty \) as \( n \to \infty \).

(iii) There exists a number \( \Theta \in (0, \pi/2) \) such that

\[
\left| \arg - s_n \frac{\partial^2 g}{\partial z^2}(s_n, n) \right| \equiv \Theta
\]

and, furthermore, \( s_n^2(\partial^2 g / \partial z^2)(s_n, n) \to \infty \) as \( n \to \infty \).

(iv) There exist positive numbers \( \varepsilon_0 \) and \( K \) such that

\[
\left| z \frac{\partial^3}{\partial z^3} g(z, n) \left\{ \frac{\partial^2}{\partial z^2} g(z, n) \right\}^{-1} \right| \equiv K
\]

if \( |z - s_n| < \varepsilon_0 |s_n| \).

(v) Let \( \alpha_n = \arg (s_n - z_0) - \arg s_n \). There exists a positive number \( \varepsilon_1 \), a function \( n_1 : (0, \varepsilon_1) \to \mathbb{N} \), a bounded function \( g_1 : (0, \varepsilon_1) \times (-1, 0) \to \mathbb{R} \), and a function \( g_2 : (0, \varepsilon_1) \times (0, \infty) \to \mathbb{R} \) such that, for all \( \varepsilon \in (0, \varepsilon_1) \), \( \exp g_2(\varepsilon, \cdot) \in L(0, \infty) \), and, for all \( n \geq n_1(\varepsilon) \),

\[
\Re \left\{ g(s_n(1 + \varepsilon e^{i\alpha_n}), n) - g(s_n(1 - \varepsilon e^{i\alpha_n}), n) \right\} \equiv g_1(\varepsilon, \tau)
\]

if \( \tau \in (-1, z_0/s_n, -\varepsilon) \), whereas

\[
\Re \left\{ g(s_n(1 + \varepsilon e^{i\alpha_n}), n) - g(s_n(1 + \varepsilon e^{i\alpha_n}), n) \right\} \equiv g_2(\varepsilon, \tau)
\]

if \( \tau \in (\varepsilon, \infty) \).

Furthermore, let \( f \) be a bounded analytic function on \( S \) and suppose there exists a positive number \( \varepsilon \) such that

(vi) \( \sup_{z \in I_n(\varepsilon)} |f(z)| - 1| \to 0 \) if \( n \to \infty \), where \( I_n(\varepsilon) \) denotes the segment between \( s_n(1 - \varepsilon e^{i\alpha_n}) \) and \( s_n(1 + \varepsilon e^{i\alpha_n}) \).

Let

\[
\varphi(z) = f(z) \exp \psi(z),
\]

and

\[
J_n = \frac{1}{2\pi i} \int_{\gamma} \varphi(z) z^n \, dz
\]

where \( \gamma \) is a half line in \( S \) from \( z_0 \) to \( \infty \). Then we have

\[
J_n = \left\{ 2\pi s_n^2 \frac{\partial^2 g}{\partial z^2}(s_n, n) \right\}^{-1/2} s_n \exp g(s_n, n)(1 + o(1)), \quad n \to \infty,
\]

where \( \arg \{s_n^2(\partial^2 g / \partial z^2)(s_n, n)\}^{-1/2} \in (-\pi, 0) \).

Proof. We shall closely follow the proof of Theorem 7 in [2]. Let \( n \geq n_0 \). Due to (i), (ii) and the properties of \( f \), we may replace \( \gamma \) by \( \gamma_n \). Let \( \varepsilon > 0 \). We begin by considering the integrand on the segment \( I_n(\varepsilon) \). We put

\[
\left| \frac{\partial^2 g}{\partial z^2}(z, n) - \frac{\partial^2 g}{\partial z^2}(s_n, n) \right| = h(z).
\]

From (iv) we deduce that

\[
h(z) \equiv K \left| \frac{\partial^2 g}{\partial z^2}(s_n, n) \right| \int_{s_n}^{z} \left| \frac{d\zeta}{\zeta} \right| + K \int_{s_n}^{z} h(\zeta) \left| \frac{d\zeta}{\zeta} \right|.
\]
provided \(|z - s_n| < \varepsilon_0|s_n|\). With the aid of Gronwall's generalized inequality (cf. [5, p. 36]) we find
\[
h(z) \leq \tilde{K} \left| \frac{z - s_n}{s_n} \right| \left| \frac{\partial^2 g}{\partial z^2} (s_n, n) \right|,
\]
where \(\tilde{K}\) is a positive constant, provided \(|z - s_n| < \varepsilon_0|s_n|\). Hence it follows that, for all \(\varepsilon \in (0, \varepsilon_0)\),
\[
(1.5) \quad \frac{\partial g}{\partial z} (z, n) = (z - s_n) \frac{\partial^2 g}{\partial z^2} (s_n, n)(1 + \varepsilon O(1)), \quad n \to \infty
\]
\[
(1.6) \quad g(z, n) - g(s_n, n) = \frac{1}{2} (z - s_n)^2 \frac{\partial^2 g}{\partial z^2} (s_n, n)(1 + \varepsilon O(1)), \quad n \to \infty,
\]
uniformly on \(I_n(\varepsilon)\). Here \(O(1)\) is uniformly bounded in \(\varepsilon\).

We introduce a new variable \(w\) by means of
\[
(1.7) \quad \frac{1}{2} w^2 = g(s_n, n) - g(s_n(1 + \tau e^{i\alpha_n}), n), \quad |\tau| \leq \varepsilon.
\]
Due to (1.6) we have
\[
w^2 = -\tau^2 e^{2i\alpha_n} s_n \frac{\partial^2 g}{\partial z^2} (s_n, n)(1 + \varepsilon O(1))
\]
and we remove the ambiguity in the definition of \(w\) by demanding that
\[
(1.8) \quad w = \tau e^{i\alpha_n} \left( -s_n \frac{\partial^2 g}{\partial z^2} (s_n, n) \right)^{1/2} (1 + \varepsilon O(1)),
\]
where the square root has its principal value. Equations (1.7) and (1.5) imply that
\[
\frac{dw}{d\tau} = -s_n e^{i\alpha_n} \frac{\partial g}{\partial z} (s_n(1 + \tau e^{i\alpha_n}), n) = -s_n^2 \tau e^{2i\alpha_n} \frac{\partial^2 g}{\partial z^2} (s_n, n)(1 + \varepsilon O(1)).
\]
Consequently,
\[
(1.9) \quad \frac{dw}{d\tau} = e^{i\alpha_n} \left( -s_n \frac{\partial^2 g}{\partial z^2} (s_n, n) \right)^{1/2} (1 + \varepsilon O(1)).
\]
Let \(w_{\pm}\) correspond to \(\tau = \pm \varepsilon\). From (1.8) it follows that
\[
(1.10) \quad w_{\pm} = \pm \varepsilon e^{i\alpha_n} \left( -s_n \frac{\partial^2 g}{\partial z^2} (s_n, n) \right)^{1/2} (1 + \varepsilon O(1)).
\]
From (1.7), (1.9), (1.10), and condition (vi) of Proposition 1.3 we deduce that
\[
\int_{s_n(1-\varepsilon e^{i\alpha_n})}^{s_n(1+\varepsilon e^{i\alpha_n})} \varphi(z) z^n dz
\]
\[
= s_n e^{i\alpha_n} \exp g(s_n, n) \int_{w_{-}}^{w_{+}} e^{-w^2/2} \left( \frac{dw}{d\tau} \right)^{-1} dw(1 + o(1))
\]
\[
= \left( -s_n \frac{\partial^2 g}{\partial z^2} (s_n, n) \right)^{-1/2} s_n \exp g(s_n, n) \int_{w_{-}}^{w_{+}} e^{-w^2/2} dw(1 + o(1))(1 + \varepsilon O(1)).
\]
Using (1.10) and condition (iii) and noting that \(\lim_{n \to \infty} \alpha_n = 0\), we conclude that, for every \(\varepsilon > 0\),
\[
\lim_{n \to \infty} \int_{w_{-}}^{w_{+}} e^{-w^2/2} dw = \sqrt{2\pi}.
\]
Hence
\[
\frac{1}{2\pi i} \int_{s_n(1+\epsilon e^{i\omega_n})} \varphi(z) z^n \, dz = \left(2\pi s_n^2 \frac{\partial^2 g}{\partial z^2}(s_n, n)\right)^{-1/2} s_n \\
\cdot \left(\exp g(s_n, n) + o(1)\right) \left(1 + \epsilon O(1)\right),
\]
where \(\arg \{s_n^2(\partial^2 g/\partial z^2)(s_n, n)\} - 1/2 \in (-\pi, 0)\).

Next we consider the integral
\[
J^+_n(\epsilon) = \int_{s_n(1+\epsilon e^{i\omega_n})} \varphi(z) z^n \, dz.
\]
From (1.7), condition (v), and the properties of \(f\) we deduce that, for \(n \geq n_1(\epsilon)\),
\[
|J^+_n(\epsilon)| \leq C \left| s_n \exp \left\{ g(s_n, n) - \frac{1}{2} w^2 \right\} \right| \int_{\epsilon}^{\infty} \exp g_\tau(\epsilon, \tau) \, d\tau
\leq C_1 \left| s_n \exp \left\{ g(s_n, n) - \frac{1}{2} w^2 \right\} \right|,
\]
where \(C\) and \(C_1\) are positive constants. In view of (1.10) and condition (iii) this implies that
\[
J^+_n(\epsilon) \approx \left(2\pi s_n^2 \frac{\partial^2 g}{\partial z^2}(s_n, n)\right)^{-1/2} s_n \exp g(s_n, n) o(1), \quad n \to \infty
\]
and the same property holds for the integral over the segment between \(z_0\) and \(s_n(1-\epsilon e^{i\omega_n})\). Combining the above estimates we find
\[
J_n = \left(2\pi s_n^2 \frac{\partial^2 g}{\partial z^2}(s_n, n)\right)^{-1/2} s_n \exp g(s_n, n)(1 + \epsilon O(1))(1 + o(1)), \quad n \to \infty.
\]
Since this is true for every sufficiently small \(\epsilon\) the result follows.

**Theorem 1.4.** Let \(N \in \mathbb{N}\). Let \(\alpha_\nu, \beta_\nu, \nu \in \{1, \cdots, N\}\), be real numbers such that
\[\alpha_\nu \leq \alpha_{\nu+1} < \beta_\nu \leq \beta_{\nu+1}\]
if \(\nu < N\) and \(\alpha_N \equiv \alpha_{N+1} \equiv \alpha_1 + 2\pi < \beta_N \equiv \beta_{N+1} \equiv \beta_1 + 2\pi\). Let \(\bar{z}_\nu \in S(\alpha_\nu, \beta_\nu)\) and \(S_\nu = S(\bar{z}_\nu, \alpha_\nu, \beta_\nu), \nu = 1, \cdots, N, S_{N+1} = e^{2\pi i} S_1\). Suppose that, for each \(\nu \in \{1, \cdots, N\}\), we are given an analytic function \(y_\nu\) on \(S_\nu\), admitting an asymptotic expansion \(\sum_{n=0}^{\infty} \hat{y}_n z^{-n}\) as \(z \to \infty\) in \(S_\nu\), independent of \(\nu\). Let
\[
y_{\nu+1}(z) = y_\nu(z e^{-2\pi i}), \quad z \in S_{\nu+1},
\]
and
\[
\varphi_\nu(z) = y_{\nu+1}(z) - y_\nu(z), \quad z \in S_\nu \cap S_{\nu+1}, \quad \nu \in \{1, \cdots, N\}.
\]
Suppose that for every \(\nu \in \{1, \cdots, N\}\) there exists a sector \(\tilde{S}_\nu \subset S_\nu \cap S_{\nu+1}\), a positive integer \(m(\nu)\), and, for every \(j \in \{1, \cdots, m(\nu)\}\), analytic functions \(\psi^\nu_j\) and \(f^\nu_j\) on \(\tilde{S}_\nu\), satisfying the conditions of Proposition 1.3, and a complex number \(c^\nu_j\) such that
\[
\varphi_\nu(z) = \sum_{j=1}^{m(\nu)} c^\nu_j f^\nu_j(z) \exp \psi^\nu_j(z), \quad z \in \tilde{S}_\nu.
\]
Let \(g^\nu_j(z, n) = \psi^\nu_j(z) + n \log z\), let \(s^\nu_{n,j}\) denote its saddle point, and let
\[
M^\nu_j(n) = \left\{2\pi (s^\nu_{n,j})^2 \frac{\partial^2 g^\nu_j}{\partial z^2}(s^\nu_{n,j}, n)\right\}^{-1/2} s^\nu_{n,j} \exp g^\nu_j(s^\nu_{n,j}, n), \quad j \in \{1, \cdots, m(\nu)\}, \quad \nu \in \{1, \cdots, N\},
\]
where \( \arg \left\{ (s^{2\alpha}_{n^2})^2 (\hat{\alpha}^2 \hat{\alpha}^z) (s^{2\alpha}_{n^2}) \right\}^{-1/2} \in (-\pi, 0) \). Then there exists a convergent power series \( \sum_{n=0}^{\infty} h_nz^{-n} \) such that

\[
(1.11) \quad \hat{y}_n = h_n - \sum_{\nu=1}^{N} \sum_{j=1}^{m(\nu)} c^\nu_j \{ M^\nu_j (n-1) (1 + o(1)) \}, \quad n \to \infty.
\]

**Proof.** There exists \( z_\nu \in S_\nu \cap S_{\nu+1} \) such that \( S_\nu \cap S_{\nu+1} \) contains the sector \( S' = S(z_\nu, \alpha_{\nu+1}, \beta_\nu) \). As \( y_\nu \) and \( y_{\nu+1} \) admit the same asymptotic expansion, it follows that

\[
(\varphi_\nu (z) = y_{\nu+1} (z) - y_\nu (z) \sim o(1) \quad as \quad z \to \infty \quad in \quad S', \quad \nu \in \{ 1, \cdots, N \}).
\]

Obviously, the functions \( y_\nu \) possess the properties (i)-(iii) mentioned in Proposition 1.2.

According to Proposition 1.2 there exists a function \( h_\nu \) holomorphic at \( \infty \), such that \( y_\nu = h_\nu + H_\nu \) for all \( \nu \in \{ 1, \cdots, N \} \). Let \( \sum_{n=0}^{\infty} h_nz^{-n} \) be the power series expansion of \( h_\nu \). With (1.4) we find

\[
\hat{y}_n = h_n - \sum_{\nu=1}^{N} \sum_{j=1}^{m(\nu)} \frac{c^\nu_j}{2\pi i} \int_{\gamma_\nu} f^{\nu}_j (z) \exp \psi^{\nu}_j (z) z^{n-1} dz, \quad n \in \mathbb{N},
\]

where \( \gamma_\nu \) is a half line \( S_\nu, \nu \in \{ 1, \cdots, N \} \).

The proof is completed by application of Proposition 1.3 to each term of the sum in the right-hand side of the above identity.

**Remark 1.** If the \( y_\nu \) as well as the functions \( f^{\nu}_j \exp \psi^{\nu}_j \) are solutions of some homogeneous linear functional equation, the numbers \( c^\nu_j \) play a role similar to the Stokes multipliers in the theory of linear differential equations.

**Remark 2.** If one of the functions \( M^\nu_j \) in (1.11) dominates the rest for \( n \to \infty \), the corresponding coefficient \( c^\nu_j \) may be determined from the asymptotic behavior of \( \hat{y}_n \).

**Remark 3.** Propositions 1.1 and 1.2 may also be used to obtain estimates of the growth of the remainder terms \( R_\nu (y_\nu ; z) \) as \( n \to \infty \). This will be illustrated by the application to linear difference equations in the next section.

**Example.** The nonlinear differential equation

\[
(1.12) \quad \frac{dy}{dz} = a + y + \frac{b}{z^2} y^3, \quad a, b \notin \mathbb{C}^\star
\]

possesses three formal solutions of the form \( \sum_{n=-1}^{\infty} \hat{y}_n z^{-n} \). The coefficients \( \hat{y}_n \) can be determined from the recursive relations

\[
-2\hat{y}_{n+2} + (n+4)\hat{y}_{n+1} + b \sum_{m_1 + m_2 + m_3 = n} \hat{y}_{m_1} \hat{y}_{m_2} \hat{y}_{m_3} = 0, \quad n \geq -1,
\]

\[
(\hat{y}_{-1})^2 = -\frac{1}{b}, \quad \hat{y}_0 = -\frac{1}{2} \hat{y}_{-1}
\]

and

\[
\hat{y}_{n+2} + (n+1)\hat{y}_{n+1} + b \sum_{m_1 + m_2 + m_3 = n} \hat{y}_{m_1} \hat{y}_{m_2} \hat{y}_{m_3} = 0, \quad n \geq 1,
\]

\[
\hat{y}_{-1} = \hat{y}_0 = \hat{y}_1 = 0, \quad \hat{y}_2 = -a.
\]

Let \( \hat{y} \) denote one of the formal solutions and let \( S \) be a sector of aperture less than \( \pi \). It is a well-known fact that there exists a solution of (1.12), analytic in \( S \) and
represented asymptotically by $\hat{y}$ as $z \to \infty$ in $S$, uniformly on $S$ (cf. [13]). Suppose that $y_1$ and $y_2$ are two solutions with these properties. Obviously,

$$
\frac{d}{dz}(y_1 - y_2) = y_1 - y_2 + \frac{b}{z^2}(y_1^2 + y_1 y_2 + y_2^2)(y_1 - y_2).
$$

Let $\hat{y} = \sum_{n=-1}^{\infty} \hat{y}_n z^{-n}$ and suppose the coefficients $\hat{y}_n$ satisfy (1.13). Then we have

$$
y_1^2 + y_1 y_2 + y_2^2 = -\frac{3}{b}(z^2 - z) + h(z),
$$

where $h$ is a bounded analytic function on $S$, admitting an asymptotic expansion as $z \to \infty$ in $S$. Inserting (1.16) into (1.15) we obtain

$$
\frac{d}{dz}(y_1 - y_2) = \left(-2 + \frac{3}{z} + \frac{b}{z^2} h(z)\right)(y_1 - y_2)
$$

and this implies that

$$
y_1 - y_2 = c e^{-2z} z^3 \left(1 + O\left(\frac{1}{z}\right)\right)
$$

as $z \to \infty$ in $S$, where $c$ is a complex number. Hence it follows that (1.12) has a unique solution $y^-$, analytic in a left half plane and represented asymptotically by the series $\sum_{n=-1}^{\infty} \hat{y}_n z^{-n}$ as $z \to \infty$ in this half plane. Moreover, it is easily seen that $y^-$ may be continued analytically to a sector of the form $S(z_1, -\pi/2, 3\pi/2)$, with $z_1 \in \mathbb{C}_\infty$, without a change in asymptotic behavior.

Furthermore, we have

$$
y^-(z) - y^-(ze^{2\pi i}) = c^- e^{-2z} z^3 \left(1 + O\left(\frac{1}{z}\right)\right),
$$

as $z \to \infty$ in $S(-3\pi/2 + \epsilon, -\pi/2 - \epsilon)$ for any $\epsilon \in (0, \pi/2)$. Applying Theorem 1.4 we find

$$
c^- = -2\pi i \lim_{n \to \infty} \frac{2^{n+3} \hat{y}_n}{(n+2)!}.
$$

In a similar manner it is shown that (1.12) possesses a unique solution $y^+$ analytic in $S(z_2, -\pi/2, 5\pi/2)$ for some $z_2 \in \mathbb{C}_\infty$ and represented asymptotically by the series $\sum_{n=-1}^{\infty} \hat{y}_n z^{-n}$ determined by (1.14), as $z \to \infty$ in this sector. Moreover, it turns out that

$$
y^+(z) - y^+(ze^{2\pi i}) = c^+ e^{z} \left(1 + O\left(\frac{1}{z}\right)\right),
$$

as $z \to \infty$ in $S(-\pi/2 + \epsilon, \pi/2 - \epsilon)$ for any $\epsilon \in (0, \pi/2)$. Application of Theorem 1.4 now yields the relation

$$
c^+ = 2\pi i \lim_{n \to \infty} \frac{(-1)^{n+1} \hat{y}_n}{(n-1)!}.
$$

2. An application to linear difference equations. We consider the $m$th-order homogeneous linear difference equation

$$
\sum_{j=0}^{m} a_j(z) y(z+j) = 0,
$$

where $a_j \in \mathbb{C}\{z^{-1}\}$, $j = 1, \ldots, m$ (or, equivalently, a system of $m$ first-order difference equations). The "generic" case is when the characteristic equation of (2.1) has $m$ distinct roots. This case has been treated in [8]. Here we shall deal with a more singular class of equations.
Under certain conditions, (2.1) possesses \( m \) linearly independent formal solutions of the form
\[
\hat{y}_j(z) = \hat{h}_j(z) z^{\rho_j} \exp (d_j \log z + \mu_j z), \quad j = 1, \ldots, m, \tag{2.2}
\]
where \( \hat{h}_j(z) = \sum_{n=0}^{\infty} \hat{h}_{jn} z^{-n} \) with \( \hat{h}_{j0} = 1 \), \( \rho_j \in \mathbb{C} \), \( d_j \in \mathbb{Q} \) and \( \mu_j \in \mathbb{C} \) for all \( j \in \{1, \ldots, m\} \) (cf. [3], [11]).
We put
\[
\rho_i - \rho_j = \rho_{ij}, \quad d_i - d_j = d_{ij}, \quad \text{and} \quad \mu_i - \mu_j = \mu_{ij}, \quad i, j \in \{1, \ldots, m\}
\]
and we assume that, for all \( i, j \in \{1, \ldots, m\} \) such that \( i \neq j \) and \( d_{ij} = 0 \),
\[
\text{Re } \mu_{ij} \neq 0. \tag{2.3}
\]
For merely technical reasons we further assume that
\[
\text{Im } \mu_{ij} \in \{0, -d_{ij} \pi \} \mod 2\pi \quad \text{if } i \neq j, \quad i, j \in \{1, \ldots, m\}, \tag{2.4}
\]
but this condition can easily be removed. For all \( i, j \in \{1, \ldots, m\} \) such that \( i \neq j \) we shall denote by \( n_{ij} \) the integer determined by
\[
0 < \text{Im } \mu_{ij} + 2n_{ij} \pi < 2\pi \quad \text{if } d_{ij} \leq 0, \tag{2.5}
\]
\[
0 < \text{Im } \mu_{ij} + (2n_{ij} + d_{ij}) \pi < 2\pi \quad \text{if } d_{ij} > 0.
\]
Let \( S_1, \ldots, S_7 \) be sectors of the following form:
\[
S_1 = S(R e^{-i(\pi/2)}, -\pi, 0), \quad S_2 = e^{i(\pi/2)} S_1, \quad S_3 = S_4 = e^{i\pi} S_1,
\]
\[
S_5 = e^{i(3\pi/2)} S_1 \quad \text{and} \quad S_6 = S_7 = e^{2i\pi} S_1,
\]
where \( R > 0 \). If \( R \) is chosen sufficiently large, equation (2.1) possesses, for each \( j \in \{1, \ldots, m\} \) and \( \nu \in \{1, 3, 4, 6, 7\} \), a unique solution \( y^\nu_j \), represented asymptotically by \( \hat{y}_j \) as \( z \to \infty \), uniformly on \( 0 < \text{Im } \mu_{ij} + 2n_{ij} \pi < 2\pi \) if \( d_{ij} \leq 0 \),
\[
0 < \text{Im } \mu_{ij} + (2n_{ij} + d_{ij}) \pi < 2\pi \quad \text{if } d_{ij} > 0. \tag{2.6}
\]
for every \( \delta \in (0, \pi/2) \) (cf. [6, Thm. 2.4.5]; note that this is a stronger statement than \( y^\nu_j \sim \hat{y}_j \) as \( z \to \infty \) in \( S_\nu \)). Moreover, we have
\[
y^4_j - y^3_j = p^3_{ij} y^3_j, \quad y^7_j - y^6_j = p^6_{ij} y^6_j, \tag{2.7}
\]
where \( p^3_{ij} \) and \( p^6_{ij} \) are periodic functions of period 1 with the property that
\[
\lim_{z \to \infty} p^3_{ij}(z) = \lim_{z \to -\infty} p^3_{ij}(z) = 0, \quad j \in \{1, \ldots, m\}. \tag{2.8}
\]
Furthermore, for each \( j \in \{1, \ldots, m\} \), equation (2.1) possesses a unique solution \( y^2_j \), analytic in \( S_2 \) and represented asymptotically by \( \hat{y}_j \) as \( z \to \infty \) in \( S_2 \), such that
\[
\lim_{z \to \infty} p^1_{ij}(z) \exp \{-2(n_{ij} - 1) \pi iz\} \quad \text{and} \quad \lim_{z \to -\infty} p^2_{ij}(z) \exp \{-2n_{ij} \pi iz\}
\]
exist for all \( i, j \in \{1, \ldots, m\} \) such that \( i \neq j \).
Similarly, for each \( j \in \{1, \cdots, m\} \), there exists a unique solution \( y^*_j \), analytic in \( S \) and represented asymptotically by \( y^*_j \) as \( z \to \infty \) in \( S \), such that

\[
y^*_j - y^*_j = \sum_{i=1}^{m} p^*_j y^*_i,
\]

where \( p^*_j \) and \( y^*_j \) are periodic functions of period 1 with the following properties:

\[
p^*_j = p^*_j \equiv 0 \quad \text{if} \quad d^*_j < 0 \quad \text{or} \quad d^*_j = 0 \quad \text{and} \quad \Re \mu^*_j \leq 0,
\]

(2.10) \( \lim_{\im z \to \infty} p^*_j(z) \exp \{-2n^*_j \pi i z\} \quad \text{and} \quad \lim_{\im z \to -\infty} p^*_j(z) \exp \{-2(n^*_j - 1) \pi i z\} \)

exist for all \( i, j \in \{1, \cdots, m\} \) such that \( i \neq j \).

Now let

\[
h^*_j(z) = y^*_j(z)z^{-p^*_j} \exp \left(-d^*_j \log z - \mu^*_j z\right), \quad j \in \{1, \cdots, m\}, \quad \nu \in \{1, \cdots, 7\}.
\]

Obviously, \( h^*_j \) is represented asymptotically by \( h^*_j \) as \( z \to \infty \) in \( S \) for all \( j \in \{1, \cdots, m\} \) and all \( \nu \in \{1, \cdots, 7\} \). Moreover, if \( \nu \in \{1, 3, 4, 6, 7\} \), the asymptotic expansion is uniformly valid on (2.6) for every \( \delta \in (0, \pi/2) \). The uniqueness of \( h^*_j \) implies that

(2.11) \( h^*_j(z) = h^*_j(z e^{2\pi i}) \) for all \( j \in \{1, \cdots, m\} \).

Furthermore, we have, for all \( j \in \{1, \cdots, m\} \) and \( \nu \in \{1, \cdots, 6\} \),

(2.12) \( h^*_{j+1}(z) - h^*_j(z) = \sum_{i=1}^{m} p^*_j(z) h^*_i(z) z^{\mu^*_i} \exp \left(d^*_i \log z + \mu^*_i z\right) \).

For all \( i, j \in \{1, \cdots, m\} \) and all \( \nu \in \{1, \cdots, 6\} \) we define an integer \( n^*_j \) and complex numbers \( c^*_j \) and \( \mu^*_j \) as follows:

(2.13) \[
\begin{aligned}
n^*_j &= \max \left\{ n \in \mathbb{Z} : \lim_{\im z \to \infty} p^*_j(z) \exp (-2n \pi i z) \text{ exists} \right\} \quad \text{if} \quad \nu \in \{2, 3, 4\} \quad \text{and} \quad p^*_j \neq 0, \\
&= \min \left\{ n \in \mathbb{Z} : \lim_{\im z \to -\infty} p^*_j(z) \exp (-2n \pi i z) \text{ exists} \right\} \quad \text{if} \quad \nu \in \{1, 5, 6\} \quad \text{and} \quad p^*_j \neq 0, \\
&= 0 \quad \text{otherwise},
\end{aligned}
\]

(2.14) \[
c^*_j = \begin{cases} 0 & \text{if} \quad p^*_j = 0, \\ \lim_{\im z \to \infty} p^*_j(z) \exp (-2n^*_j \pi i z) & \text{otherwise}, \end{cases}
\]

(2.15) \[
\mu^*_j = \mu^*_j + 2n^*_j \pi i z.
\]

Furthermore, we define analytic functions \( f^*_j \) and \( \varphi^*_j \) by

(2.16) \[
f^*_j(z) = \begin{cases} 0 & \text{if} \quad c^*_j = 0, \\
(c^*_j)^{-1} p^*_j(z) \exp (-2n^*_j \pi i z) h^*_i(z) & \text{otherwise}, \end{cases}
\]

(2.17) \[
\varphi^*_j(z) = p^*_j(z) h^*_i(z) z^{\mu^*_i} \exp \left(d^*_i \log z + \mu^*_i z\right).
\]

Obviously,

(2.18) \[
\varphi^*_j(z) = c^*_j f^*_j(z) z^{\mu^*_i} \exp \left(d^*_i \log z + \mu^*_i z\right), \quad i, j \in \{1, \cdots, m\}, \quad \nu \in \{1, \cdots, 6\}.
\]

In order to check whether the conditions of Theorem 1.4 are satisfied, we will first study the properties of the function \( g : S \times \mathbb{N} \to \mathbb{C} \) defined by

\[
g(z, n) = dz \log z + \mu z = (n + \rho) \log z,
\]
where \( d \in \mathbb{Q}, \mu \in \mathbb{C}, \rho \in \mathbb{C} \), and \( S \) is one of the sectors \( S_\nu \cap S_{\nu+1}, \nu \in \{1, \ldots, 6\} \). From (2.18), (2.3), (2.7)-(2.10), and the definitions (2.13)-(2.15) we conclude that the following cases need to be considered:

1. \( d = 0, \rho = 0, \mu = 2m\pi i, m \in \mathbb{N}, S = S_3, \)
2. \( d = 0, \rho = 0, \mu = -2m\pi i, m \in \mathbb{N}, S = S_6, \)
3. \( d = 0, \Re \mu < 0, \Im \mu < 0, S = S_1 \cap S_2, \)
4. \( d = 0, \Re \mu < 0, \Im \mu > 0, S = S_2 \cap S_3, \)
5. \( d = 0, \Re \mu > 0, \Im \mu > 0, S = S_4 \cap S_5, \)
6. \( d = 0, \Re \mu > 0, \Im \mu < 0, S = S_5 \cap S_6, \)
7. \( d < 0, \Im \mu < 0, S = S_1 \cap S_2, \)
8. \( d < 0, \Im \mu > 0, S = S_2 \cap S_3, \)
9. \( d > 0, \Im \mu + d\pi > 0, S = S_4 \cap S_5, \)
10. \( d > 0, \Im \mu + d\pi < 0, S = S_5 \cap S_6. \)

In the first six cases, \( \partial g/\partial z = 0 \) has a unique solution \( s_n \) given by

\[
(2.19) \quad s_n = -\frac{n + \rho}{\mu}.
\]

Hence

\[
(2.20) \quad \arg s_n = \arg \left( -\frac{1}{\mu} \right) (1 + o(1)), \quad n \to \infty.
\]

Furthermore, we have

\[
(2.21) \quad \frac{\partial^2 g}{\partial z^2}(s_n, n) = -\frac{\mu^2}{n + \rho}, \quad s_n \frac{\partial^2 g}{\partial z^2}(s_n, n) = -n - \rho,
\]

\[
(2.22) \quad z \frac{\partial^3 g}{\partial z^3}(z, n) \left\{ \frac{\partial^2 g}{\partial z^2}(z, n) \right\}^{-1} = -2.
\]

Let \( S' \subseteq S \). In each of the cases 1-6 there exists a positive number \( \delta \) such that

\[
\cos (\arg z + \arg s_n) < -\frac{\delta}{|\mu|} \quad \text{for all } z \in S'.
\]

This implies that, for all \( z \in S' \),

\[
\Re g(z, n) \equiv -\delta|z| + (n + \Re \rho) \log |z| - \Im \rho \arg z.
\]

Hence we easily deduce the existence of positive constants \( A_s \) and \( C_s \) such that

\[
(2.23) \quad \sup_{z \in S'} \exp g(z, n) < C_s A_s^n n^n.
\]

Now consider the cases 7-10. There \( d \neq 0 \) and the saddle point \( s_n \) is a solution of the equation

\[
(2.24) \quad s_n \left( \log s_n + \frac{\mu}{d} + 1 \right) = -\frac{n + \rho}{d}.
\]

Let \( h \) be the inverse of the function \( z \to z \log z \) (cf. [9, Ex. III], [4, § 3.6]). It has the following asymptotic behavior:

\[
(2.25) \quad h(z) = \frac{z}{\log z} (1 + o(1)), \quad z \to \infty.
\]
From (2.24) we deduce

\[ s_n = \exp\left(-\frac{\mu}{d} - 1\right) h\left(-\frac{n + \rho}{d} \exp\left(\frac{\mu}{d} + 1\right)\right) \]

\[(2.26) \]

\[ = -\frac{n + \rho}{d} \left\{ \log h\left(-\frac{n + \rho}{d} \exp\left(\frac{\mu}{d} + 1\right)\right) \right\}^{-1}. \]

With (2.25) it follows that

\[(2.27) \]

\[ s_n = -\frac{n}{d \log n} (1 + o(1)), \quad n \to \infty. \]

Equating the imaginary parts on both sides of (2.24), we get

\[ \text{Im } s_n \left( \log |s_n| + \frac{\text{Re } \mu}{d} + 1 \right) + \text{Re } s_n \left( \text{arg } s_n + \frac{\text{Im } \mu}{d} \right) = \frac{\text{Im } \rho}{d}. \]

With (2.27) we find

\[ \text{Im } s_n = \frac{n}{d (\log n)^2} \left( \text{arg } s_n + \frac{\text{Im } \mu}{d} \right) (1 + o(1)), \quad n \to \infty. \]

Hence

\[ \text{Im } s_n = \left\{ \begin{array}{ll}
\frac{n}{d^2 (\log n)^2} \text{Im } \mu (1 + o(1)), & n \to \infty \quad \text{if } d < 0, \\
\frac{n}{d^2 (\log n)^2} (\text{Im } \mu + d \pi) (1 + o(1)), & n \to \infty \quad \text{if } d > 0.
\end{array} \right. \]

(2.28)

Furthermore, we have

\[ \frac{\partial^2 g}{\partial z^2} (s_n, n) = \frac{d^2}{s_n^2} \frac{n + \rho}{n + \rho} = -\frac{d^2}{s_n^2} \log h\left(-\frac{n + \rho}{d} \exp\left(\frac{\mu}{d} + 1\right)\right) (1 + o(1)), \quad n \to \infty \]

(2.29)

and hence

\[ s_n^2 \frac{\partial^2 g}{\partial z^2} (s_n, n) = -n (1 + o(1)), \quad n \to \infty. \]

(2.30)

We easily verify that

\[ z \frac{\partial^3 g}{\partial z^3} (z, n) \left\{ \frac{\partial^2 g}{\partial z^2} (z, n) \right\}^{-1} = -\frac{2(n + \rho) - dz}{n + \rho - dz} \]

(2.31)

and the expression on the right-hand side is obviously uniformly bounded on the half plane \(-d \text{ Re } z > 0\) and thus on \(S\), provided \(n \geq n_0\), where \(n_0\) is some sufficiently large number.

Let \(S' \subset S\). In each of the cases considered this implies the existence of a positive number \(\delta\) such that

\[ d \cos \text{ arg } z < -\delta \quad \text{for all } z \in S'. \]

Let \(0 < \varepsilon < \delta\). Then there exists a positive constant \(C\) such that

\[ \exp g(z, n) < C \exp (-\varepsilon |z| \log |z|)|z|^n, \quad z \in S'. \]
The expression to the right of the inequality sign attains its maximum as $|z| = h(ne/e)/e$ and the maximum value is equal to

$$
\exp\left(-2n + \frac{e}{h} h\left(\frac{ne}{e}\right)\right) h\left(\frac{ne}{e}\right)^{n}
$$

In view of (2.25) it follows that there exist positive constants $A_{S'}$ and $C_{S'}$ such that

$$
(2.32) \quad \sup_{z \in S'} |\exp g(z, n)| < C_{S'} A_{S'} \left(\frac{n}{\log n}\right)^{n}
$$

With the aid of (2.19), (2.21), (2.26), and (2.29) we can derive an explicit expression for the function $M_n : \mathbb{N} \to \mathbb{C}$ given by

$$
(2.33) \quad M(n) = \left(2\pi s_n \frac{\partial^2 g}{\partial z^2}(s_n, n)\right)^{-1/2} s_n \exp g(s_n, n),
$$

where $\arg\left(s_n^2(\partial^2 g/\partial z^2)(s_n, n)^{-1/2}\right) \in (-\pi, 0)$, in each of the cases considered above. With (2.30) we find

$$
M(n) = \left\{ \begin{array}{ll}
-2\pi(n + \rho))^\frac{1}{2} \exp(-n - \rho) \left(\frac{n + \rho}{\mu}\right)^{n + \rho + 1} & \text{if } d = 0, \\
(-2\pi n)^{-1/2} \exp \{(n + \rho)\chi(n)^{-1} - 1\} \left(\frac{n + \rho}{d\chi(n)}\right)^{n + \rho + 1} (1 + o(1)) & \text{if } d \neq 0,
\end{array} \right.
$$

where $\chi(n) = \log h((n + \rho)/d \exp(\mu/d + 1))$. Let us define a function $M_{d, \mu} : \mathbb{C} \to \mathbb{C}$ by

$$
(2.34) \quad -2\pi M_{d, \mu}(s) = \left\{ \begin{array}{ll}
\Gamma(s)(-\mu)^{-s} & \text{if } d = 0, \\
\Gamma(s) \exp \left\{ \frac{s}{\log h(-s/d \exp(\mu/d + 1))} \right\} \left(-d \log \left(\frac{s}{d \exp(\mu/d + 1)}\right)\right)^{-s} & \text{if } d \neq 0,
\end{array} \right.
$$

Using Stirling's formula and the properties of the function $h$, we readily verify that

$$
(2.35) \quad -M(n - 1) = M_{d, \mu}(n + \rho)(1 + o(1)), \quad n \to \infty.
$$

Now let $\nu \in \{1, \cdots, 6\}$, $\tilde{z}_\nu \in S_\nu \cap S_{\nu+1}$, and let $\tilde{S}_\nu$ be a sector of the following form:

$$
\tilde{S}_\nu = S\left(\tilde{z}_\nu, \left(\frac{\nu}{3} - \frac{5}{6}\right) \pi + \delta, \left(\frac{\nu}{3} - \frac{1}{6}\right) \pi\right) \quad \text{if } \nu \in \{1, 4\},
$$

$$
\tilde{S}_\nu = S\left(\tilde{z}_\nu, \frac{\nu - 2}{3} \pi, \left(\frac{\nu}{3} - \frac{1}{6}\right) \pi\right) \quad \text{if } \nu \in \{2, 5\},
$$

$$
\tilde{S}_\nu = S\left(\tilde{z}_\nu, \frac{\nu - 1}{3} \pi + \delta, \frac{\nu}{3} \pi - \delta\right) \quad \text{if } \nu \in \{3, 6\},
$$

where $\delta \in (0, \pi/2)$. Let $i, j \in \{1, \cdots, m\}$ such that $c_{ij} \neq 0$, and let

$$
g_{ij}(z, n) = d_{ij} \log z + \mu_{ij} \log(n + \rho_{ij}) \log z, \quad z \in \tilde{S}_\nu, \quad n \in \mathbb{N}.
$$

From (2.19)-(2.22) and (2.27)-(2.31) we deduce that conditions (ii)-(iv) of Proposition 1.3 are satisfied, provided $\delta$ is chosen sufficiently small. We readily verify that condition (v) holds as well (with $z_0 = \tilde{z}_\nu$).
Next, we consider the function $f^\nu_\gamma$ defined by (2.16). The asymptotic properties of $h^\nu_\gamma$ imply that
\begin{equation}
\lim_{z \to \infty} h^\nu_\gamma(z) = 1 \quad \text{uniformly on } \tilde{S}^\nu.
\end{equation}
Furthermore, from (2.14) and the fact that $p^\nu_\gamma$ is analytic on either a lower or an upper half plane it follows that
\begin{equation}
|p^\nu_\gamma(z) \exp (-2n^\nu_\gamma \pi i z) - c^\nu_\gamma| \leq K \exp (-2\pi |\text{Im } z|), \quad z \in \tilde{S}^\nu,
\end{equation}
where $K$ is a positive constant. From (2.36) and (2.37) it is obvious that $f^\nu_\gamma$ is bounded on $\tilde{S}^\nu$. Moreover, with the aid of (2.20) it is easily seen that, in the case that $d_{ij} = 0$, $f^\nu_\gamma$ satisfies condition (vi) of Proposition 1.3. Now suppose that $\nu \in \{1, 2, 4, 5\}$ and $d_{ij} \neq 0$. Formulas (2.4) and (2.28) imply that $|\text{Im } s_n| \to \infty$ as $n \to \infty$, where $s_n$ denotes the saddle point of $g^\nu_\gamma(z, n)$. With (2.36) and (2.37) it follows that, also in this case, condition (vi) of Proposition 1.3 is fulfilled.

Apparently, all conditions of Theorem 1.4 are satisfied. Applying this theorem and using (2.33) and (2.35), we obtain the following result.

**Theorem 2.1.** For each $j \in \{1, \cdots, m\}$ there exists a convergent power series \(\sum_{n=0}^\infty h_{jm} z^{-n}\) such that
\begin{equation}
\hat{h}_j = h_j + \sum_{n=1}^m \sum_{\nu=1}^6 c^\nu_{ij} (M_{d_{ij}, \nu}(n + p^\nu_\gamma)(1 + o(1))), \quad n \to \infty,
\end{equation}
where $c^\nu_{ij}$ and $M_{d_{ij}, \nu}$ are defined by (2.14) and (2.34), respectively.

With the aid of Propositions 1.1 and 1.2 we are able to estimate the growth of the remainder terms $R_n(h^\nu_\gamma; z)$ for $n \to \infty, j \in \{1, \cdots, m\}$. Let $\nu \in \{1, \cdots, 6\}, S_{\nu} \cap S_{\nu+1}$ is a sector of the form $S(z_\nu, \alpha_{\nu}, \beta_{\nu})$. We begin by considering the functions $h^\nu_\gamma$ defined by
\begin{equation}
h^\nu_\gamma(z) = \frac{z}{2\pi i} \int_{\gamma_{\nu}} \frac{\varphi^\nu_\gamma(\xi)}{\xi (\xi - z)} d\xi, \quad i, j \in \{1, \cdots, m\}, \quad \nu \in \{1, \cdots, 6\},
\end{equation}
where $\gamma$ is a half line in $S_{\nu} \cap S_{\nu+1}$ from $z_\nu$ to $\infty$ and $\varphi^\nu_\gamma$ is defined by (2.17).

**Proposition 2.2.** Let $i, j \in \{1, \cdots, m\}, \nu \in \{1, \cdots, 6\}$. The function $h^\nu_\gamma$ defined by (2.38) is analytic in $S_{\nu} \cap S_{\nu+1}$ and represented asymptotically by
\begin{equation}
\sum_{n=0}^\infty -\frac{1}{2\pi i} \left( \int_{\gamma_{\nu}} \varphi^\nu_\gamma(\xi) \xi^{-n-1} d\xi \right) z^{-n}
\end{equation}
as $z \to \infty$ in $S_{\nu} \cap S_{\nu+1}$. Moreover, for every $S' \subseteq S_{\nu} \cap S_{\nu+1}$ there exist positive constants $A_{S'}$ and $C_{S'}$ such that, for all $n \in \mathbb{N}$,
\begin{equation}
\sup_{z \in S'} |z^n R_n(h^\nu_\gamma; z)| \leq \begin{cases} C_{S'} A_{S'}^\nu n! & \text{if } d_{ij} = 0, \\ C_{S'} A_{S'}^\nu (n / \log n)^n & \text{if } d_{ij} \neq 0. \end{cases}
\end{equation}

**Proof.** The first two statements follow immediately from Proposition 1.1 and the properties of $\varphi^\nu_1$. Now let $S' \subseteq S_{\nu} \cap S_{\nu+1}$ of the form $S' = S(\tilde{z}_\nu, \tilde{\alpha}_{\nu}, \tilde{\beta}_{\nu})$ such that $S' \subseteq \tilde{S}^\nu$. Let $\tilde{\gamma}_\nu$ be a half line in $S'$ from $\tilde{z}_\nu$ to $\infty$ and
\begin{equation}
\tilde{h}^\nu_\gamma(z) = \frac{z}{2\pi i} \int_{\tilde{\gamma}_{\nu}} \frac{\varphi^\nu_\gamma(\xi)}{\xi (\xi - z)} d\xi.
\end{equation}
As $h^\nu_\gamma - \tilde{h}^\nu_\gamma$ is holomorphic at $\infty$, it is obviously sufficient to prove (2.39) for $\tilde{h}^\nu_\gamma$ instead of $h^\nu_\gamma$. Using (2.18), (2.23), (2.29) and (2.32) and noting that, due to (2.16), (2.36), and (2.37),
$f''_j$ is bounded on $S'$, we conclude that there exist positive numbers $A_{S'}$ and $C_{S'}$ such that, for all $n \in \mathbb{N}$,

$$\sup_{z \in S'} |z^n \varphi''_j(z)| \leq \begin{cases} 
C_{S'} A_{S'}^n - \frac{n!}{n} & \text{if } d_j = 0, \\
C_{S'} A_{S'}^n (n/(\log n))^n & \text{if } d_j \neq 0.
\end{cases}$$

The result now follows by application of Proposition 1.1.

**Theorem 2.3** (cf. also [7]). Let $j \in \{1, \ldots, m\}$, $\nu \in \{1, \ldots, 6\}$. For every $S' \subseteq S$, there exist positive constants $A_{S'}$ and $C_{S'}$ such that

$$\sup_{z \in S'} |z^n h'_j(z)| < C_{S'} A_{S'}^n n!,$$ 

Moreover, if the numbers $c''_{\nu j}$ defined by (2.14) vanish for all $i \in \{1, \ldots, m\}$ such that $d_i = 0$ and all $\mu \in \{1, \ldots, 6\}$, then there exist positive constants $C_{S'}$ and $A_{S'}$ such that

$$\sup_{z \in S'} |z^n R_n(h'_j; z)| < C_{S'} A_{S'}^n \left(\frac{n}{\log n}\right)^n.$$ 

**Proof.** Using (2.11), (2.12), and the definitions (2.17) and (2.38), and applying Proposition 1.2, we conclude that there exists a function $h_j$, holomorphic at $\infty$, such that

$$h'_j(z) = h_j(z) + \sum_{i=1}^{\nu-1} \left\{ \sum_{\mu=1}^{\nu} h''_{\nu j}(z) + \sum_{\mu=1}^{\nu} h''_{\mu j}(z e^{2\pi i}) \right\}.$$ 

Thus the statements of the theorem are seen to be an immediate corollary of Proposition 2.2.

To conclude this section we shall apply the above results to the second-order difference equation

$$(z + 2)^2 + \alpha (z + 2) + \beta \gamma(z + 2) - ((z + 1)^2 + \gamma(z + 1)^2 + \gamma(z + 1) + \delta\gamma(z + 1) + \sigma\gamma(z) = 0,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}, \sigma \in \mathbb{C}^\ast$ (this is a particular case of the class of equations considered in [1]). This equation possesses two formal solutions $\hat{y}_1$ and $\hat{y}_2$ of the form

$$\hat{y}_1(z) = \hat{h}_1(z) z^{-\alpha - 2},$$

$$\hat{y}_2(z) = \hat{h}_2(z) z^{-\gamma - 2} \exp \{-2z \log z + (2 + \log \sigma)z\},$$

where $\hat{h}_j(z) = \sum_{n=0}^{\infty} \hat{h}_j n z^{-n}$ with $\hat{h}_j = 1, j = 1, 2$. Thus we have

$$\rho_{12} = 2\gamma - \alpha = -\rho_{21}, \quad d_{12} = 2 = -d_{21}, \quad \mu_{12} = -(2 + \log \sigma) = -\mu_{21}.$$ 

Assumption (2.4) is equivalent to

$$\arg \sigma \neq 0 \mod 2\pi.$$ 

We shall choose $\arg \sigma \in (0, 2\pi)$. With (2.5) it follows that $n_{12} = n_{21} = 0$. Hence, by (2.9) the following limits exist:

$$\lim_{\Im z \to -\infty} p^{1}_{21}(z) \exp 2\pi iz \quad \text{and} \quad \lim_{\Im z \to -\infty} p^{3}_{21}(z).$$

From these and other considerations, based on the particular form of the equation, it can be deduced that the periodic functions $p^{1}_{21}, p^{3}_{21}, p^{1}_{11}$ and $p^{3}_{11}$ must be of the following form:

$$p^{1}_{11}(z) = \frac{c^{3}_{11} \exp 2\pi iz + (\exp 2\pi i y - \exp 2\pi i \alpha)}{(1 - \exp 2\pi i (z - a))(1 - \exp 2\pi i (z - b))},$$

$$p^{3}_{11}(z) = (1 + p^{3}_{11}(z))^{-1} \exp 2\pi i (\gamma - \alpha) - 1,$$
ASYMPTOTIC POWER SERIES AND STOKES PHENOMENA

(2.42)

\[ p_{21}^1(z) = -p_{21}^2(z) = \frac{-c_{21}^2 + c_{21}^1 \exp 2\pi i(z - \gamma)}{1 + \{c_{11}^3 - \exp(-2\pi a) - \exp(-2\pi b)\} \exp 2\pi i z + \exp 2\pi i(\gamma + 2z)}, \]

where \(a\) and \(b\) denote the roots of the polynomial \(z^2 + az + \beta\), and \(c_{21}^1\), \(c_{21}^2\), and \(c_{11}^3\) are defined by (2.14). From (2.7), (2.9), and (2.10) it is seen that \(c_{11}^3 = 0\) for \(\nu \in \{1, 2, 4, 5\}\) and \(c_{21}^2 = 0\) for \(\nu \in \{3, 4, 5, 6\}\). According to Theorem 2.1 there exists a convergent power series \(\sum_{n=0}^{\infty} h_{1n} z^{-n}\) such that

\[ \hat{h}_{1n} = h_{1n} + c_{11}^3 M_{0, \mu_1}^1(n) (1 + o(1)) + c_{11}^6 M_{0, \mu_1}^1(n) (1 + o(1)) \]

\[ + c_{21}^2 M_{-2, \mu_2}^1(n + \alpha - 2\gamma)(1 + o(1)) \]

\[ + c_{21}^3 M_{-2, \mu_2}^2(n + \alpha - 2\gamma)(1 + o(1)), \quad n \to \infty. \]

From (2.40)-(2.42) we deduce, with (2.13), that \(n_{11}^3 = -n_{11}^6 = 1\), \(n_{21}^2 = -1\), \(n_{21}^2 = 0\) and hence, with (2.15), that

\[ \mu_{11}^3 = -\mu_{11}^6 = 2\pi i, \quad \mu_{21}^3 = 2 + \log \sigma - 2\pi i, \quad \mu_{21}^6 = 2 + \log \sigma. \]

Using (2.34), we find

\[ M_{0, \mu_1}^1(n) = (-1)^n M_{0, \mu_1}^6(n) = \Gamma(n)(-2\pi i)^{-n-1}. \]

As the dominating terms in (2.43) are the ones with coefficients \(c_{11}^3\) and \(c_{11}^6\) we conclude that

\[ c_{11}^3 + c_{11}^6 = -\lim_{n \to \infty} \frac{\hat{h}_{12n}(2\pi i)^{2n+1}}{(2n-1)!}, \]

\[ c_{11}^3 - c_{11}^6 = \lim_{n \to \infty} \frac{\hat{h}_{12n+1}(2\pi i)^{2n+2}}{(2n)!}. \]

If \(c_{11}^3 = 0\), then, by (2.41), \(p_{11}^6 = \exp 2\pi i(\gamma - \alpha) - 1\) and, in view of (2.8), this implies \(c_{11}^6 = 0\) and \(\gamma - \alpha \in \mathbb{Z}\). In that case (2.42) becomes

\[ p_{21}^1(z) = -p_{21}^2(z) = \frac{-c_{21}^2 + c_{21}^1 \exp 2\pi i(z - \alpha)}{(1 - \exp 2\pi i(z - a))(1 - \exp 2\pi i(z - b))} \]

(where we have used the identity \(a + b = -\alpha\)), and the coefficient \(c_{21}^2\), \(\nu \in \{1, 2\}\), of the dominating term in (2.43) may be determined from the asymptotic behavior of \(\hat{h}_{1n}\) for \(n \to \infty\).

On the other hand, if \(c_{11}^3 \neq 0\), then the coefficients \(c_{21}^1\) and \(c_{21}^2\) cannot be determined by this method.

REFERENCES


