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PERTURBED NONLINEAR OSCILLATIONS*

B. KAPER†

Abstract. For a class of nonlinear oscillation problems described by a second order ordinary differential equation containing a small nonnegative perturbation parameter ε we determine asymptotic solutions which are uniformly valid as $\varepsilon \downarrow 0$ on intervals of the order of magnitude ε^{-1} . These asymptotic solutions are obtained from formal asymptotic solutions subject to a condition in order that they approximate the exact solution asymptotically. These formal asymptotic solutions are constructed in the form of a finite generalized asymptotic power series involving functions of the periodic two variable type.

1. Introduction. We consider a class of perturbed nonlinear oscillations described by the second order ordinary differential equation containing a small nonnegative perturbation parameter ε

$$(1.1a) \quad x'' + F(x, \varepsilon t) + \varepsilon f(x, x', \varepsilon t, \varepsilon) = 0, \quad t \geq 0,$$

with initial conditions

$$(1.1b) \quad x(0, \varepsilon) = \alpha_1(\varepsilon), \quad x'(0, \varepsilon) = \alpha_2(\varepsilon).$$

The force term F and its perturbation εf include the independent time variable t in a slowly varying way. In general we may expect the solution of problem (1.1) to be an oscillating function with slowly varying amplitude and "frequency". Asymptotically we may distinguish two different time scales: a *local (or rapid) time scale* in which the solution of problem (1.1) is periodic with a period of the asymptotic order of magnitude one and a *slow (or stretched) time scale*, characterized by the slow variable $\tau = \varepsilon t$, which accounts for the slow modulations of the oscillation. In order to include these large scale variations in a study of the solutions we apply the change of variable $\tau = \varepsilon t$ to problem (1.1). Setting $x(\tau/\varepsilon, \varepsilon) = \tilde{x}(\tau, \varepsilon)$ and omitting the tilde ($\sim$) at the same time we get

$$(1.2a) \quad \varepsilon^2 x'' + F(x, \tau) + \varepsilon f(x, \varepsilon x', \tau, \varepsilon) = 0, \quad \tau \geq 0,$$

$$(1.2b) \quad x(0, \varepsilon) = \alpha_1(\varepsilon), \quad \varepsilon x'(0, \varepsilon) = \alpha_2(\varepsilon),$$

where the primes indicate differentiation with respect to τ .

With respect to the perturbation parameter ε we determine asymptotic solutions of arbitrary order of the class of oscillation problems (1.2) uniformly valid on the finite τ -interval $I = [0, L]$. A function $\tilde{\phi}_N$ is called an N th order asymptotic solution of (1.2) on I if

$$|x(\tau, \varepsilon) - \tilde{\phi}_N(\tau, \varepsilon)| \leq K\varepsilon^{N+1}, \quad |\varepsilon\{x'(\tau, \varepsilon) - \tilde{\phi}'_N(\tau, \varepsilon)\}| \leq K\varepsilon^{N+1}, \quad \tau \in I,$$

where x represents the exact solution of (1.2) and K is some positive number independent of τ and ε .

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Asymptotic solutions will be obtained from so-called formal asymptotic solutions. A function ϕ_N is an N th order formal asymptotic solution of (1.2) if

$$\varepsilon^2 \phi_N'' + F(\phi_N, \tau) + \varepsilon f(\phi_N, \varepsilon \phi_N', \tau, \varepsilon) \stackrel{\text{def}}{=} \varepsilon^{N+1} g_N(\tau, \varepsilon) = O(\varepsilon^{N+1}), \quad \tau \in I,$$

$$\alpha_1(\varepsilon) - \phi_N(0, \varepsilon) = O(\varepsilon^{N+1}), \quad \alpha_2(\varepsilon) - \varepsilon \phi_N'(0, \varepsilon) = O(\varepsilon^{N+1})$$

(g_N is called the residual function of ϕ_N). Moreover, we let the residual function g_N satisfy some condition in order that ϕ_N approximate the exact solution uniformly on I . For the construction of such an N th order formal asymptotic solution ϕ_N we anticipate the form of ϕ_N as a finite generalized asymptotic power series for the construction of which we apply a two variable technique; cf. Cole [3], Nayfeh [9]. Apart from the variable τ , which describes the slow modulations, a second variable p is introduced which accounts for the local periodic behavior of the solution. This technique has been applied to a class of weakly nonlinear oscillation problems (for which the force term F is linear in x) by Cole and Kevorkian [4], Mitropol'skii [8] and recently by Hoogstraten and Kaper [5]. In [7], Kuzmak extended the Cole–Kevorkian method to the class of nonlinear oscillation problems (1.2) with a quasi-linear perturbation function

$$\varepsilon^2 x'' + F(x, \tau) + \varepsilon^2 \tilde{f}(x, \tau) x' = 0, \quad \tau \in I.$$

He constructed a first order formal asymptotic solution ϕ_1 of the form

$$\phi_1(\tau, \varepsilon) = y_0(p, \tau) + \varepsilon y_1(p, \tau), \quad p = \varepsilon^{-1} q(\tau),$$

where $q'(\tau) = O(1)$. In his article he did not include a proof of asymptotic correctness of ϕ_1 with respect to the exact solution. For the class of oscillation problems (1.2) this form does not contain sufficient degrees of freedom to satisfy the boundedness condition to be imposed on the coefficients of a generalized asymptotic power series.

Mitropol'skii indicated in his book [8] two asymptotic methods for the class of problems (1.2), an indirect method of successive transformations and a direct method by substitution of the series

$$\phi(\tau, \varepsilon) \sim u_0(\tau, \psi, a) + \varepsilon u_1(\tau, \psi, a) + \cdots,$$

where a and ψ should be determined from the system of equations

$$\begin{aligned} \frac{da}{d\tau} &\sim a_0(\tau, a) + \varepsilon a_1(\tau, a) + \cdots, \\ \frac{d\psi}{d\tau} &\sim \varepsilon^{-1} \omega(\tau, a) + \psi_0(\tau, a) + \cdots. \end{aligned}$$

The first term u_0 is a periodic solution of the τ parameter equation

$$(1.3) \quad \omega^2(\tau) u_{0,\psi\psi} + F(u_0, \tau) = 0,$$

where the subscripts ψ denote partial differentiation with respect to ψ .

This method is an extension of a method developed for a class of weakly nonlinear oscillation problems. In the Introduction of [5] Hoogstraten and Kaper already mentioned the disadvantage of this method with respect to the different

nonlinear first order equations for the amplitude a to be solved in order to obtain asymptotic solutions of different orders. As the coefficients in the equations defining $u_1, u_2, \dots$ are periodic functions Mitropol'skii does not concern himself with this method and restricts himself to the somewhat laborious indirect method of transformations to new variables. This method is by its contents equivalent to the method of averaging.

In this paper we arrive at an N th order asymptotic solution ϕ_N of (1.2) of the form

$$(1.4a) \quad \phi_N(\tau, \varepsilon) = \eta(\tau) + A_0(\tau)\Phi_0(p, \tau) + \sum_{\nu=1}^N \varepsilon^\nu [A_\nu(\tau)z_2^*(p, \tau) + \Phi_\nu(p, \tau)],$$

$$p = \varepsilon^{-1}S(\tau, \varepsilon),$$

$$(1.4b) \quad S(\tau, \varepsilon) = \int_0^\tau \omega(\sigma) d\sigma + \varepsilon \sum_{j=0}^N \varepsilon^j S_j(\tau).$$

The first term $U_0 = \eta + A_0\Phi_0$ represents the solution of a τ -parameter equation of the type (1.3) whereas the functions $U_\nu = A_\nu z_2^* + \Phi_\nu$ satisfy the inhomogeneous linear equations

$$(1.5) \quad \omega^2(\tau)U_{\nu,pp} + F_x(U_0, \tau)U_\nu = -\gamma_\nu, \quad \nu = 1, \dots, N$$

(F_x denotes the partial derivative of F respect to x). U_0 is 2π -periodic and oscillates between $\eta \pm A_0$, z_2^* is one of the two linearly independent homogeneous solutions $z_{1,2}^*$ of (1.5) and Φ_ν represents that particular solution of (1.5) which satisfies

$$\int_0^{2\pi} \Phi_\nu(p, \tau) z_i^*(p, \tau) dp = 0, \quad i = 1, 2.$$

The expansion (1.4) of ϕ_N includes as a particular case the expansion developed by Hoogstraten and Kaper [5] for weakly nonlinear oscillation problems if we identify $\Phi_0(p, \tau) = z_2^*(p, \tau) = \cos p$. A detailed description of the construction method can be found in § 4 of this paper. In § 3 we deduce conditions on the residual function g_N in order that ϕ_N approximate the exact solution uniformly on the interval I . In the final part of § 4 we show that the construction of ϕ_N can be performed as an algorithm yielding successively a sequence of asymptotic solutions of increasing order. In § 5 we shall apply the algorithm to a perturbed Duffing equation with slowly varying coefficients in order to obtain a zeroth order asymptotic solution. To a certain extent the results could be compared to the first order formal asymptotic solution ϕ_1 obtained by Kuzmak of the Duffing equation with slowly varying coefficients and without a perturbation function (cf. [7]). We shall show some limitations of Kuzmak's form in dealing with a more general class of problems as well as in dealing with higher order (formal) asymptotic solutions.

2. Notations and definitions. Throughout this paper all quantities are assumed to be real, ε is a small nonnegative perturbation parameter. For vectors and matrices the subscript ε is used to indicate dependence on ε . All considerations apply for sufficiently small positive ε , unless stated otherwise. The positive

numbers δ_0 , ε_0 and M_0 are generic constants, that is, they are not necessarily the same number each time they appear. By I we denote the closed interval $[0, L]$, where L is an arbitrary fixed positive number. The norm of a vector or a matrix is defined as the sum of the absolute values of the elements. The linear space of real $n \times n$ matrices with unit element E is denoted by $\mathbb{R}^{n \times n}$.

The asymptotic order symbol 0 has its usual meaning and is always understood to be related to the limit process $\varepsilon \downarrow 0$. All asymptotic order relations involving a function of τ hold uniformly for $\tau \in I$, unless stated otherwise. When a vector or a matrix function satisfies an order relation it is to be understood that the norm satisfies the order relation.

We define two classes of functions, P^∞ and P_S^∞ whose elements will be referred to as “*functions of the periodic two variable type*”. By P^∞ we denote the class of scalar functions χ^* of p , τ and ε for which hold

- (i) $\chi^* \in C^\infty(\mathbb{R} \times I \times [0, \varepsilon_0], \mathbb{R})$,
- (ii) $\chi^*(p + 2\pi, \tau, \varepsilon) = \chi^*(p, \tau, \varepsilon)$ for $(p, \tau, \varepsilon) \in \mathbb{R} \times I \times [0, \varepsilon_0]$.

P^∞ also contains the class of scalar functions which depend on p and τ only and which satisfy conditions (i) and (ii) omitting the ε dependence.

Let S be a scalar function which belongs to the class $C^\infty(I \times [0, \varepsilon_0], \mathbb{R})$ and which has a strictly positive derivative with respect to τ on I ; thus

$$\frac{\partial}{\partial \tau} S(\tau, \varepsilon) \geq \delta_0 > 0 \quad \text{for } (\tau, \varepsilon) \in I \times [0, \varepsilon_0].$$

Then P_S^∞ consists of all functions χ of τ and ε for which hold

- (i) $\chi \in C^\infty(I \times [0, \varepsilon_0], \mathbb{R})$,
- (ii) $\chi(\tau, \varepsilon) = \chi^*(p, \tau, \varepsilon), \quad p = \varepsilon^{-1} S(\tau, \varepsilon)$,

where $\chi^* \in P^\infty$.

A vector or a matrix function is said to belong to P^∞ or P_S^∞ if all its elements belong to P^∞ , respectively P_S^∞ . If a function belongs to P_S^∞ , its “*two variable counterpart*” in P^∞ will be indicated by the same symbol with an asterisk. Let H_ε be a vector function where $H_\varepsilon \in C^0(\mathbb{R}^n \times I, \mathbb{R}^n)$ and consider the first order ordinary differential equation in $\mathbb{R}^n$

$$(2.1a) \quad \varepsilon y'_\varepsilon = H_\varepsilon(y_\varepsilon, \tau), \quad \tau \in I,$$

with initial condition

$$(2.1b) \quad y_\varepsilon(0) = \alpha_\varepsilon$$

DEFINITION 1. Assume that problem (2.1) has a unique solution y_ε on I . A function $\tilde{u}_\varepsilon$ is an N -th order asymptotic solution of (2.1) on I if

- (i) $\tilde{u}_\varepsilon(\tau) \in \mathbb{R}^n$ for $\tau \in I$,
- (ii) $y_\varepsilon - \tilde{u}_\varepsilon = O(\varepsilon^{N+1})$.

DEFINITION 2. A function u_ε is an N -th order formal asymptotic solution of problem (2.1) on I if

- (i) $u_\varepsilon \in C^1(I, \mathbb{R}^n)$,
- (ii) $u_\varepsilon = O(1)$,
- (iii) $\varepsilon u'_\varepsilon(\tau) - H_\varepsilon\{u_\varepsilon(\tau), \tau\} \stackrel{\text{def}}{=} \varepsilon^{N+1} g_\varepsilon(t) = O(\varepsilon^{N+1})$,
- (iv) $\alpha_\varepsilon - u_\varepsilon(0) \stackrel{\text{def}}{=} \varepsilon^{N+1} \tilde{\alpha}_\varepsilon = O(\varepsilon^{N+1})$.

The function g_ε will be called the *residual function* of u_ε . Accordingly we may define asymptotic solutions of the second order problem (1.2).

DEFINITION 3. A scalar function ϕ_N is an N th order formal asymptotic solution of (1.2) on I if

- (i) $\phi_N \in C^2(I, \mathbb{R})$,
- (ii) ϕ_N and $\varepsilon \phi'_N = O(1)$,
- (iii) $\varepsilon^2 \phi''_N + F(\phi_N, \tau) + \varepsilon f(\phi_N, \varepsilon \phi'_N, \tau, \varepsilon) \stackrel{\text{def}}{=} \varepsilon^{N+1} g_N(\tau, \varepsilon) = O(\varepsilon^{N+1})$, $\tau \in I$,
- (iv) $\alpha_1(\varepsilon) - \phi_N(0, \varepsilon) \stackrel{\text{def}}{=} \varepsilon^{N+1} \tilde{\alpha}_1(\varepsilon) = O(\varepsilon^{N+1})$,
 $\alpha_2(\varepsilon) - \varepsilon \phi'_N(0, \varepsilon) \stackrel{\text{def}}{=} \varepsilon^{N+1} \tilde{\alpha}_2(\varepsilon) = O(\varepsilon^{N+1})$.

It is seen that the vector function u_ε , $u_\varepsilon(\tau) = \text{col}(\phi_N(\tau, \varepsilon), \varepsilon \phi'_N(\tau, \varepsilon))$, is an N th order formal asymptotic solution (in the sense of Definition 2) of the system of first order equations in $\mathbb{R}^2$ associated with (1.2) if we take $y_\varepsilon(\tau) = \text{col}(x(\tau, \varepsilon), \varepsilon x'(\tau, \varepsilon))$.

3. Formal asymptotic solutions of the periodic two variable type. In this section we deduce some theorems that eventually lead to conditions on the residual function g_N of an N th order formal asymptotic solution ϕ_N of the periodic two variable type of problem (1.2). We shall repeat a fundamental theorem on a first order system of differential equations in $\mathbb{R}^n$ that has been proved by Hoogstraten and Kaper in [5].

THEOREM 1. Consider the first order initial value problem in $\mathbb{R}^n$

$$(3.1a) \quad \varepsilon y'_\varepsilon = H_\varepsilon(y_\varepsilon, \tau), \quad \tau \in I,$$

$$(3.1b) \quad y_\varepsilon(0) = \alpha_\varepsilon,$$

where the following assumptions are made about $H_\varepsilon(z, \tau) = \text{col}(H_1(z, \tau, \varepsilon), \dots, H_n(z, \tau, \varepsilon))$:

- (i) H_i and the elements $\{\nabla H_\varepsilon(z, \tau)\}_{i,j} = (\partial H_i / \partial z_j)(z, \tau, \varepsilon)$,
 $i, j = 1, \dots$, of the Jacobian ∇H_ε belong to $C^0(\mathbb{R}^n \times I, \mathbb{R})$,

- (ii) for each $R > 0$ there exists a number M , depending on R and L only, such that, for all (u, τ) and $(v, \tau) \in \mathbb{R}^n \times I$ with $|u|, |v| \leq R$,

$$\left| \frac{\partial H_i}{\partial z_j}(u, \tau, \varepsilon) - \frac{\partial H_i}{\partial z_j}(v, \tau, \varepsilon) \right| \leq M|u - v|, \quad i, j = 1, \dots, n.$$

Let u_ε be an N th order formal asymptotic solution of problem (3.1) on I with residual function g_ε . Let Ψ_ε be the fundamental matrix solution of the first variational equation of (3.1) with respect to u_ε ,

$$(3.2) \quad \varepsilon z'_\varepsilon = \nabla H_\varepsilon\{u_\varepsilon(\tau), \tau\} z_\varepsilon, \quad \tau \in I,$$

satisfying $\Psi_\varepsilon(0) = E$.

If the asymptotic order relations

$$(3.3) \quad \Psi_\varepsilon = O(1), \quad \Psi_\varepsilon^{-1} = O(1)$$

and

$$(3.4) \quad \int_0^\tau \Psi_\varepsilon^{-1}(s) g_\varepsilon(s) ds = O(\varepsilon)$$

hold on I , and if $N \geq 1$, then problem (3.1) has a unique solution y_ε on I and, moreover, u_ε is an N -th order asymptotic solution of problem (3.1) on I .

It is clear that in order to verify conditions (3.3) and (3.4) it suffices to have a zeroth order asymptotic approximation of Ψ_ε that satisfies (3.3) and (3.4). We notice that Ψ_ε is the solution of a linear matrix initial value problem in $\mathbb{R}^{n \times n}$ of the following type:

$$(3.5a) \quad \varepsilon Z'_\varepsilon = A_\varepsilon(\tau) Z_\varepsilon, \quad \tau \in I,$$

$$(3.5b) \quad Z_\varepsilon(0) = E,$$

where $A_\varepsilon \in C^0(I, \mathbb{R}^{n \times n})$. Moreover, we make the additional assumption

$$(3.6) \quad \int_0^\tau \text{tr } A_\varepsilon(s) ds = O(\varepsilon).$$

From the theory of linear ordinary differential equations (cf., e.g., Coddington and Levinson [2]) we know that problem (3.5) has a unique solution Ψ_ε which belongs to $C^1(I, \mathbb{R}^{n \times n})$. Assumption (3.6) implies that $\det \Psi_\varepsilon(\tau)$,

$$\det \Psi_\varepsilon(\tau) = \exp \left[\frac{1}{\varepsilon} \int_0^\tau \text{tr } A_\varepsilon(s) ds \right]$$

(cf. [2]), is uniformly bounded away from zero; thus

$$\det \Psi_\varepsilon(\tau) \geq \delta_0 > 0, \quad \tau \in I.$$

This means that if $\psi_\varepsilon = O(1)$, then also $\psi_\varepsilon^{-1} = O(1)$.

The following lemma on the boundedness and asymptotic approximation of Ψ_ε will be needed in the sequel.

LEMMA 1. Let the matrix W_ε be a zeroth order formal asymptotic solution of (3.5a), that is,

- (i) $W_\varepsilon \in C^1(I, \mathbb{R}^{n \times n})$,
- (ii) $W_\varepsilon = O(1)$,
- (iii) $\varepsilon W'_\varepsilon(\tau) - A_\varepsilon(\tau) W_\varepsilon(\tau) \stackrel{\text{def}}{=} \varepsilon G_\varepsilon(\tau) = O(\varepsilon)$,
- (iv) $|\det W_\varepsilon(\tau)| \geq \delta > 0$ for $\tau \in I$.

If there exists a matrix C_ε belonging to $C^1(I, \mathbb{R}^{n \times n})$ which satisfies the asymptotic order relations

$$(3.7a) \quad \int_0^\tau [W_\varepsilon^{-1}(s)G_\varepsilon(s)C_\varepsilon(s) + C'_\varepsilon(s)]ds = O(\varepsilon),$$

$$(3.7b) \quad C_\varepsilon(0) = W_\varepsilon^{-1}(0) + O(\varepsilon),$$

then Ψ_ε (and consequently Ψ_ε^{-1}) is $O(1)$ and, moreover, the matrix function $\tilde{W}_\varepsilon$,

$$\tilde{W}_\varepsilon(\tau) = W_\varepsilon(\tau)C_\varepsilon(\tau),$$

is a zeroth order asymptotic solution of problem (3.5), that is,

$$\Psi_\varepsilon(\tau) = \tilde{W}_\varepsilon(\tau) + O(\varepsilon).$$

Proof. We shall show that Ψ_ε can be written in the form

$$\Psi_\varepsilon = W_\varepsilon(C_\varepsilon + \varepsilon R_\varepsilon),$$

where R_ε and C_ε are both $O(1)$. It is evident that the lemma has been proved then. The matrix R_ε should satisfy the differential equation

$$(3.8a) \quad R'_\varepsilon = -W_\varepsilon^{-1}G_\varepsilon R_\varepsilon - \frac{1}{\varepsilon} [W_\varepsilon^{-1}G_\varepsilon C_\varepsilon + C'_\varepsilon], \quad \tau \in I,$$

with initial condition

$$(3.8b) \quad R_\varepsilon(0) = \frac{1}{\varepsilon} [W_\varepsilon^{-1}(0) - C_\varepsilon(0)].$$

Note that $R_\varepsilon(0) = O(1)$. The initial value problem (3.8) can be transformed into the linear Volterra integral equation

$$(3.9) \quad R_\varepsilon(\tau) = R_\varepsilon(0) - \frac{1}{\varepsilon} \int_0^\tau [W_\varepsilon^{-1}(s)G_\varepsilon(s)C_\varepsilon(s) + C'_\varepsilon(s)] ds \\ - \int_0^\tau W_\varepsilon^{-1}(s)G_\varepsilon(s)R_\varepsilon(s) ds.$$

From the assumed properties of W_ε and condition (3.7a) we obtain the inequality

$$|R_\varepsilon(\tau)| \leq |R_\varepsilon(0)| + M_0 + M_0^2 \int_0^\tau |R_\varepsilon(s)| ds \quad \text{for } \tau \in I.$$

Upon application of Gronwall's lemma [2, Probl. 1, p. 37] we get

$$|R_\varepsilon(\tau)| \leq \{|R_\varepsilon(0)| + M_0\} \exp[M_0^2 L], \quad \tau \in I,$$

from which it follows that $R_\varepsilon = O(1)$. From relation (3.7) we may deduce that a positive number M_0 exists such that

$$|C_\varepsilon(\tau)| \leq M_0(1 + \varepsilon) + M_0^2 \int_0^\tau |C_\varepsilon(s)| ds, \quad \tau \in I.$$

Applying again Gronwall's lemma we get the estimate

$$|C_\varepsilon(\tau)| \leq M_0(1 + \varepsilon) \exp[M_0^2 L].$$

valid for $\tau \in I$. Hence $C_\varepsilon = O(1)$. This completes the proof of Lemma 1.

We now assume that the N th order formal asymptotic solution u_ε of the initial value problem (3.1) belongs to P_S^∞ and that the linear matrix equation associated with the first variational equation (3.2) has a zeroth order formal asymptotic solution W_ε which belongs to P_S^∞ . With the help of Lemma 1 we transform the conditions (3.3) and (3.4) of Theorem 1 for asymptotic correctness of u_ε into a condition which involves W_ε instead of Ψ_ε . This is shown by the following.

THEOREM 2. *Consider problem (3.1). Let u_ε be an N -th order formal asymptotic solution of (3.1) on I which belongs to P_S^∞ and for which*

$$(3.10) \quad \int_0^\tau \text{tr} \nabla H_\varepsilon\{u_\varepsilon(s), s\} ds = O(\varepsilon).$$

Let W_ε be a zeroth order formal asymptotic solution in the sense of Lemma 1 of the first variational matrix equation in $\mathbb{R}^{n \times n}$,

$$(3.11) \quad \varepsilon Z'_\varepsilon = \nabla H_\varepsilon\{u_\varepsilon(\tau), \tau\} Z_\varepsilon, \quad \tau \in I,$$

and let W_ε belong to P_S^∞ also.

If the asymptotic order relation

$$(3.12) \quad \int_0^{2\pi} W_\varepsilon^{*-1}(p, \tau) g_\varepsilon^*(p, \tau) dp = O(\varepsilon)$$

holds on I , where $W_\varepsilon^(p, \tau) = W_\varepsilon(\tau)$, $g_\varepsilon^*(p, \tau) = g_\varepsilon(\tau)$ for $p = \varepsilon^{-1}S(\tau, \varepsilon)$, and if $N \geq 1$, then problem (3.1) has a unique solution y_ε on the interval I . Moreover, u_ε is an N -th order asymptotic solution of the periodic two variable type of problem (3.1) on I .*

Proof. Because of condition (3.10) and the assumed properties of H_ε and u_ε , the solution Ψ_ε of equation (3.11) with $\Psi_\varepsilon(0) = E$ satisfies a matrix initial value problem of the type (3.5). We shall show first that a matrix function $C_\varepsilon \in C^\infty(I \times [0, \varepsilon_0], \mathbb{R}^{n \times n})$ exists such that

$$(3.13) \quad \Psi_\varepsilon = W_\varepsilon C_\varepsilon + O(\varepsilon).$$

Since W_ε and $\nabla H_\varepsilon\{u_\varepsilon(\tau), \tau\}$ both belong to P_S^∞ , we also have that the residual function G_ε , which corresponds to W_ε , belongs to P_S^∞ . Hence we may write

$$G_\varepsilon(\tau) = G_\varepsilon^*(p, \tau), \quad p = \varepsilon^{-1}S(\tau, \varepsilon).$$

Let C_ε be the solution of the matrix initial value problem in $\mathbb{R}^{n \times n}$

$$(3.14a) \quad C'_\varepsilon(\tau) = B_\varepsilon(\tau)C_\varepsilon(\tau), \quad \tau \in I,$$

$$(3.14b) \quad C_\varepsilon(0) = W_\varepsilon^{-1}(0),$$

where

$$B_\varepsilon(\tau) = \frac{-1}{2\pi} \int_0^{2\pi} W_\varepsilon^{*-1}(p, \tau) G_\varepsilon^*(p, \tau) dp.$$

Since $B_\varepsilon \in C^\infty(I \times [0, \varepsilon_0], \mathbb{R}^{n \times n})$ the existence of a unique solution $C_\varepsilon \in C^\infty(I \times [0, \varepsilon_0], \mathbb{R}^{n \times n})$ of problem (3.14) is guaranteed by the theory of linear ordinary differential equations. Because of the properties of W_ε and B_ε we have

$$|\det C_\varepsilon(\tau)| = \left| \det W_\varepsilon^{-1}(0) \right\} \exp \left[\int_0^\tau \operatorname{tr} B_\varepsilon(s) ds \right] \geq \delta_0 > 0, \quad \tau \in I,$$

and hence C_ε^{-1} also belongs to $C^\infty(I \times [0, \varepsilon_0], \mathbb{R}^{n \times n})$.

According to Lemma 2 (cf. Appendix A), the fact that C_ε satisfies equation (3.14a) is just a sufficient condition that the integral

$$\int_0^\tau [W_\varepsilon^{*-1}(p, s) G_\varepsilon^*(p, s) C_\varepsilon(s) + C'_\varepsilon(s)] ds, \quad p = \varepsilon^{-1} S(s, \varepsilon),$$

be $O(\varepsilon)$. Hence it is seen that both conditions (3.7a,b) of Lemma 1 are satisfied. Thus, by virtue of Lemma 1, we have shown relation (3.13). Moreover, we know by Lemma 1 that ψ_ε and ψ_ε^{-1} are both uniformly $O(1)$ for $\tau \in I$. Next we shall show that condition (3.4) is satisfied. Then, by virtue of Theorem 1, we have proved Theorem 2. Using (3.13) it is evident that (3.4) is satisfied if

$$(3.15) \quad \int_0^\tau C_\varepsilon^{-1}(s) W_\varepsilon^{-1}(s) g_\varepsilon(s) ds = O(\varepsilon).$$

Note that the integrand in relation (3.15) belongs to P_S^∞ . Then, using Lemma 2 (cf. Appendix A) we see that a sufficient condition for (3.15) to hold is

$$C_\varepsilon^{-1}(\tau) \int_0^{2\pi} W_\varepsilon^{*-1}(p, \tau) g_\varepsilon^*(p, \tau) dp = O(\varepsilon).$$

This latter condition is satisfied because of condition (3.12). This completes the proof of Theorem 2.

Next we shall apply Theorem 2 to the second order initial value problem (1.2) in order to deduce conditions on the residual function g_N of an N th order formal asymptotic solution ϕ_N . Problem (1.2) may be brought into the vector form (3.1) by setting

$$y_\varepsilon(\tau) = \begin{pmatrix} x(\tau, \varepsilon) \\ \varepsilon x'(\tau, \varepsilon) \end{pmatrix}, \quad \alpha_\varepsilon = \begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix},$$

$$H_\varepsilon(y_\varepsilon, \tau) = \begin{pmatrix} \varepsilon x' \\ -F(x, \tau) - \varepsilon f(x, \varepsilon x', \tau, \varepsilon) \end{pmatrix}.$$

We assume that $F \in C^\infty(\mathbb{R} \times I, \mathbb{R})$, $f \in C^\infty(\mathbb{R}^2 \times I \times [0, \varepsilon_0], \mathbb{R})$ and $\alpha_i \in C^\infty([0, \varepsilon_0])$, $i = 1, 2$. It is clear that H_ε has the properties required in Theorem 1 (and consequently in Theorem 2).

In § 4 we shall develop under certain assumptions a construction technique for an N th order formal asymptotic solution ϕ_N . It will be of the form

$$(3.16a) \quad \phi_N(\tau, \varepsilon) = \phi_N^*(p, \tau, \varepsilon) = U_0(p, \tau) + \varepsilon \bar{U}(p, \tau, \varepsilon), \quad p = \varepsilon^{-1}S(\tau, \varepsilon),$$

$$(3.16b) \quad S(\tau, \varepsilon) = S_{-1}(\tau) + \varepsilon \bar{S}(\tau, \varepsilon),$$

where ϕ_N^* , U_0 and $\bar{U}$ belong to P^∞ , S_{-1} belongs to $C^\infty(I, \mathbb{R})$ and $\bar{S}$ to $C^\infty(I \times [0, \varepsilon_0], \mathbb{R})$. $S_{-1, \tau}$ is a strictly positive function on I , thus

$$S_{-1, \tau} \geq \delta_0 > 0, \quad \tau \in I.$$

Then the residual function g_N of ϕ_N will belong to P_S . The function U_0 is a solution of the ordinary differential equation

$$(3.17) \quad S_{-1, \tau}^2(\tau) U_{0, pp} + F(U_0, \tau) = 0,$$

in which τ should be considered as a fixed parameter. Consider the first variational equation of (3.17) with respect to U_0 ,

$$(3.18) \quad S_{-1, \tau}^2(\tau) z_{pp}^* + F_x\{U_0(p, \tau), \tau\} z^* = 0$$

in which τ should be considered again as a fixed parameter. Let z_i^* , $i = 1, 2$, be two linearly independent solutions of (3.18) which both belong to P^∞ and have a Wronskian D ,

$$D(z_1^*, z_2^*) = z_1^* z_{2, p}^* - z_{1, p}^* z_2^* = 1.$$

A complete description of z_1^* and z_2^* in terms of U_0 is given in Appendix B. From the functions z_1 and z_2 defined by

$$z_i(\tau, \varepsilon) = z_i^*(p, \tau), \quad p = \varepsilon^{-1}S(\tau, \varepsilon), \quad i = 1, 2,$$

we form the matrix function W_ε ,

$$W_\varepsilon(\tau) = \begin{pmatrix} z_1(\tau, \varepsilon) & z_2(\tau, \varepsilon) \\ \varepsilon z_1'(\tau, \varepsilon) & \varepsilon z_2'(\tau, \varepsilon) \end{pmatrix},$$

which obviously belongs to P_S^∞ .

We shall show that W_ε is a zeroth order formal asymptotic solution with residual function G_ε of a linear matrix equation of the type (3.11) where

$$\nabla H_\varepsilon(u_\varepsilon, \tau) = \begin{pmatrix} 0 & 1 \\ -F_x(\phi_N, \tau) & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 & 0 \\ -f_1(\phi_N, \varepsilon \phi_N', \tau, \varepsilon) & -f_2(\phi_N, \varepsilon \phi_N', \tau, \varepsilon) \end{pmatrix}.$$

The subscripts 1 and 2 indicate the partial derivatives with respect to the first, respectively second, independent variable of f . Before calculating G_ε we note that

$$(3.19) \quad W_\varepsilon(\tau) = W_\varepsilon^*(p, \tau) = \begin{pmatrix} z_1^* & z_2^* \\ S_{-1, \tau} z_{1, p}^* & S_{-1, \tau} z_{2, p}^* \end{pmatrix} + O(\varepsilon), \quad p = \varepsilon^{-1}S(\tau, \varepsilon),$$

and

$$\nabla H_\varepsilon\{u_\varepsilon(\tau), \tau\} = \nabla H_\varepsilon\{u_\varepsilon^*(p, \tau), \tau\} = \begin{pmatrix} 0 & 1 \\ -F_x\{U_0(p, \tau), \tau\} & 0 \end{pmatrix} + O(\varepsilon), \quad p = \varepsilon^{-1}S(\tau, \varepsilon).$$

Then we obtain for G_ε ,

$$\begin{aligned} \varepsilon G_\varepsilon(\tau) &= \varepsilon W'_\varepsilon(\tau) - \nabla H_\varepsilon\{u_\varepsilon(\tau), \tau\} W_\varepsilon(\tau) \\ &= S_{-1, \tau} W_{\varepsilon, p}^* + \varepsilon W_{\varepsilon, \tau}^* - \nabla H_\varepsilon\{u_\varepsilon^*(p, \tau), \tau\} W_\varepsilon^* \\ &= S_{-1, \tau} \begin{pmatrix} z_{1, p}^* & z_{2, p}^* \\ S_{-1, \tau} z_{1, pp}^* & S_{-1, \tau} z_{2, pp}^* \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -F_x(U_0, \tau) & 0 \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} z_1^* & z_2^* \\ S_{-1, \tau} z_{1, p}^* & S_{-1, \tau} z_{2, p}^* \end{pmatrix} + O(\varepsilon) \\ &= O(\varepsilon), \quad p = \varepsilon^{-1}S(\tau, \varepsilon). \end{aligned}$$

Moreover, we have

$$\det W_\varepsilon(\tau) = S_{-1, \tau} D(z_1^*, z_2^*) + \varepsilon(z_1^* z_{2, \tau}^* - z_{1, \tau}^* z_2^*) = S_{-1, \tau} + O(\varepsilon).$$

Hence, $\det W_\varepsilon(\tau) \geq \delta_0 > 0$ for $\tau \in I$. Thus we have shown that W_ε is a zeroth order formal asymptotic solution of a linear matrix equation of the type (3.11), which belongs to P_S^∞ . Furthermore, we note that condition (3.10) is satisfied in the present case.

We are now in a position to apply Theorem 2 to the system of first order equations in $\mathbb{R}^2$ associated with the second order problem (1.2). Making use of (3.19) we may rewrite condition (3.12) of Theorem 2 in the following way:

$$\int_0^{2\pi} \begin{pmatrix} S_{-1, \tau} z_{2, p}^* & -z_2^* \\ -S_{-1, \tau} z_{1, p}^* & z_1^* \end{pmatrix} \begin{pmatrix} 0 \\ g_N^* \end{pmatrix} dp = O(\varepsilon),$$

where $g_N^*(p, \tau, \varepsilon) = g_N(\tau, \varepsilon)$ for $p = \varepsilon^{-1}S(\tau, \varepsilon)$. Then Theorem 2 leads to the following theorem which summarizes the above results.

THEOREM 3. Consider problem (1.2) where $F \in C^\infty(\mathbb{R} \times I, \mathbb{R})$, $f \in C^\infty(\mathbb{R}^2 \times I \times [0, \varepsilon_0], \mathbb{R})$ and $\alpha_{1,2} \in C^\infty([0, \varepsilon_0])$. Let ϕ_N be an N -th order formal asymptotic solution of (1.2) on I that belongs to P_S^∞ and that is of the form (3.16). Then the residual function g_N of ϕ_N belongs to P_S^∞ . Let z_1^* and z_2^* be the two linearly independent solutions of the τ -parameter equation (3.18). If

$$(3.20) \quad \int_0^{2\pi} g_N^*(p, \tau, 0) z_i^*(p, \tau) dp = 0, \quad i = 1, 2,$$

and if $N \geq 1$, then problem (1.2) has a unique solution x on the interval I and, moreover, ϕ_N is an N -th order asymptotic solution on I , that is,

$$x - \phi_N = O(\varepsilon^{N+1}) \quad \text{and} \quad x' - \phi'_N = O(\varepsilon^N), \quad \tau \in I.$$

Remark. Theorem 3 does not provide an explicit condition for a zeroth order formal asymptotic solution, ϕ_0 , to be a zeroth order asymptotic solution of (1.2) on

I . Hence, in general, one should obtain first a first order asymptotic solution, $\tilde{\phi}_1$. The asymptotic validity of ϕ_0 may be established by showing that

$$\phi_0 - \tilde{\phi}_1 \quad \text{and} \quad \varepsilon(\phi'_0 - \tilde{\phi}'_1)$$

are $O(\varepsilon)$ on I .

4. A two variable construction technique. We consider the class of nonlinear oscillation problems (1.2),

$$(1.2a) \quad \varepsilon^2 x'' + F(x, \tau) + \varepsilon f(x, \varepsilon x', \tau, \varepsilon) = 0, \quad \tau \geq 0,$$

$$(1.2b) \quad x(0, \varepsilon) = \alpha_1(\varepsilon), \quad \varepsilon x'(0, \varepsilon) = \alpha_2(\varepsilon),$$

where

$$(i) \quad F \in C^\infty(\mathbb{R} \times I, \mathbb{R}), \quad f \in C^\infty(\mathbb{R}^2 \times I \times [0, \varepsilon_0], \mathbb{R}), \quad \alpha_i \in C^\infty([0, \varepsilon_0]), \quad i = 1, 2,$$

$$(ii) \quad xF(x, \tau) > 0 \quad \text{for all } \tau \in I \quad \text{and } x \neq 0.$$

In the first part of this section we develop a two variable construction technique for an N th order formal asymptotic solution ϕ_N of problem (1.2) on I . It is this function for which we deduced in § 3 conditions on the residual function g_N ,

$$(3.20) \quad \int_0^{2\pi} g_N^*(p, \tau, 0) z_i^*(p, \tau) dp = 0, \quad i = 1, 2,$$

in order that ϕ_N be an N th order asymptotic solution. In the second part we shall describe an algorithmic procedure for successively generating asymptotic solutions of increasing order.

The form of ϕ_N is anticipated as a finite generalized asymptotic power series involving uniformly bounded functions of two variables p and τ :

$$(4.1a) \quad \phi_N(\tau, \varepsilon) = \sum_{\nu=0}^N \varepsilon^\nu U_\nu(p, \tau), \quad p = \varepsilon^{-1} S(\tau, \varepsilon; N),$$

$$(4.1b) \quad S(\tau, \varepsilon; N) = \sum_{j=-1}^N \varepsilon^{j+1} S_j(\tau),$$

where we require that

- (i) $U_\nu \in C^\infty(\mathbb{R} \times I, \mathbb{R})$, $\nu = 0, 1, \dots, N$,
- (ii) U_ν and all its partial derivatives with respect to p and τ are uniformly bounded on $\mathbb{R} \times I$,
- (iii) $S_j \in C^\infty(I, \mathbb{R})$, $j = -1, 0, \dots, N$,
- (iv) $S_{-1, \tau} \geq \delta_0 > 0$ for $\tau \in I$,
- (v) $S_{-1}(0) = 0$.

We insert the series (4.1) into the left-hand side of equation (1.2a) and expand (1.2a) in powers of ε in the following way:

$$\begin{aligned}
 & \varepsilon^2 \phi_N'' + F(\phi_N, \tau) + \varepsilon f(\phi_N, \varepsilon \phi_N', \tau, \varepsilon) \\
 &= (S_{-1,\tau} + \varepsilon S_{0,\tau} + \cdots)^2 (U_{0,pp} + \varepsilon U_{1,pp} + \cdots) + \varepsilon (S_{-1,\tau\tau} + \varepsilon S_{0,\tau\tau} + \cdots) \\
 & \quad \cdot (U_{0,p} + \varepsilon U_{1,p} + \cdots) + 2\varepsilon (S_{-1,\tau} + \varepsilon S_{0,\tau} + \cdots) \\
 & \quad \cdot (U_{0,p\tau} + \varepsilon U_{1,p\tau} + \cdots) + \varepsilon^2 (U_{0,\tau\tau} + \varepsilon U_{1,\tau\tau} + \cdots) \\
 (4.2) \quad & + F(U_0, \tau) + \varepsilon F_x(U_0, \tau) U_1 + \cdots + \varepsilon f(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) \\
 & + \varepsilon^2 f_1(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) U_1 + \varepsilon^2 f_2(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) \\
 & \quad \cdot (U_{0,\tau} + S_{-1,\tau} U_{1,p} + S_{0,\tau} U_{0,p}) \\
 & + \varepsilon^2 f_4(U_0, S_{-1,\tau} U_{0,p}, \tau, 0) + \cdots,
 \end{aligned}$$

where f_i denotes the partial derivative of f with respect to its i th argument ($i = 1, 2, 4$). According to the definition of the N th order formal asymptotic solution of problem (1.2), the right-hand side of (4.2) should be $O(\varepsilon^{N+1})$. As a first step, therefore, we equate to zero the $O(1)$ -contribution in (4.2). This yields an ordinary differential equation for U_0 ,

$$(4.3) \quad S_{1,\tau}^2 U_{0,pp} + F(U_0, \tau) = 0, \quad (p, \tau) \in \mathbb{R} \times I,$$

in which τ is to be considered as a fixed parameter.

Let us examine the existence of periodic solutions of equation (4.3). With respect to the independent variable p , (4.3) describes a conservative system with one degree of freedom. Because of assumption (ii) on F , the potential energy

$$S_{-1,\tau}^{-2}(\tau) V_0(U_0, \tau),$$

where

$$V_0(U_0, \tau) = \int^{U_0} F(\xi, \tau) d\xi$$

has an absolute minimum at $U_0 = 0$ for all $\tau \in I$. The total mechanical energy E_0 of the system,

$$E_0 = \frac{1}{2} U_{0,p}^2 + S_{-1,\tau}^{-2} V_0(U_0, \tau),$$

is said to be noncritical if E_0 is different from any of the values of the potential energy at its critical points¹ uniformly for τ on I , that is,

$$|S_{-1,\tau}^{-2} V_0(U_0, \tau)| \geq \delta_0 > 0 \quad \text{for all } \tau \text{ on } I$$

at any point U_0 where $E_0 = S_{-1,\tau}^{-2} V_0(U_0, \tau)$. Arnold [1] proved a theorem on the existence of periodic solutions of equations of the above type (4.3) with respect to p for a noncritical value of the energy E_0 . On a compact interval $[U_0^{(1)}, U_0^{(2)}]$ of the

¹ A critical point of a function is a point where the derivative of that function vanishes.

U_0 -axis for which

$$E_0 = S_{-1,\tau}^{-2} V_0(U_0^{(1)}, \tau) = S_{-1,\tau}^{-2} V_0(U_0^{(2)}, \tau)$$

and

$$S_{-1,\tau}^{-2} V_0(U_0, \tau) < E_0 \quad \text{for } U_0^{(1)} < U_0 < U_0^{(2)},$$

the solution of (4.3) is a periodic function oscillating between $U_0^{(1)}$ and $U_0^{(2)}$. The values of E_0 follow from the initial values of U_0 and $U_{0,p}$ with respect to p which among other things depend on the initial conditions $\alpha_{1,2}(\varepsilon)$ of the perturbed system. In view of assumption (ii) on F there are noncritical values of the energy E_0 which imply a compact interval of the above type. At this moment we assume that the formal solution ϕ_N leads to such a value of E_0 for τ on I , e.g., by a suitable choice of the initial conditions $\alpha_{1,2}(\varepsilon)$. Then (4.3) has periodic solutions which moreover oscillate between a negative and a positive value.

Equation (4.3) can be solved by quadratures. The “constants of integration” may be functions of τ and ε . However, because of the required uniform boundedness of U_0 and its partial derivatives, the period of U_0 as a function of p should be independent of τ and ε . For instance, if the period P depends on τ , then we derive from

$$U_0\{p + nP(\tau), \tau\} = U_0(p, \tau), \quad n \text{ an integer,}$$

that

$$U_{0,\tau}(p, \tau) = nP_\tau U_{0,p}\{p + nP(\tau), \tau\} + U_{0,\tau}\{p + nP(\tau), \tau\}$$

which is not uniformly bounded for $(p, \tau) \in \mathbb{R} \times I$ when n tends to infinity. In solving equation (4.3) we shall show that this condition is satisfied for a special choice of S_{-1} .

We now set

$$(4.4) \quad U_0(p, \tau) = \eta(\tau) + A_0(\tau)\Phi_0(p, \tau),$$

where A_0 figures as a constant of integration and where Φ_0 is a periodic function of its first argument which oscillates between the values -1 and $+1$. The function A_0 has the meaning of an amplitude function and η is the algebraic average of the extreme values of U_0 . Note that η is not equal to the averaged values of U_0 over one period in p in general. Note furthermore that the second constant of integration which corresponds to a shift in the phase p has been absorbed in $p = \varepsilon^{-1}S$. This may be done without loss of generality since S itself has not yet been determined at this stage of the construction of ϕ_N .

Substituting (4.4) into (4.3), multiplying by $A_0\Phi_{0,p}$ and integrating once with respect to p , we get

$$(4.5) \quad \frac{1}{2}S_{-1,\tau}^2 A_0^2 \Phi_{0,p}^2 + V_0(\eta + A_0\Phi_0, \tau) = V_0(\eta + A_0, \tau),$$

The functional dependence of η on A_0 and τ follows from the equality

$$(4.6a) \quad V_0(\eta + A_0, \tau) = V_0(\eta - A_0, \tau)$$

and will be indicated by

$$(4.6b) \quad \eta(\tau) = \tilde{\eta}\{A_0(\tau), \tau\}.$$

Normalizing the period of Φ_0 to 2π we get the dispersion relation² between $S_{-1,\tau}$ and A_0 which is obtained by integration of (4.5)

$$(4.7a) \quad S_{-1,\tau}(\tau) = \tilde{\omega}\{A_0(\tau), \tau\} = \omega(\tau),$$

where

$$(4.7b) \quad \tilde{\omega}(A_0, \tau) = \pi\sqrt{2} \left(\int_{-1}^{+1} A_0 d\xi / [V_0\{\tilde{\eta}(A_0, \tau) + A_0, \tau\} - V_0\{\tilde{\eta}(A_0, \tau) + A_0\xi, \tau\}]^{1/2} \right)^{-1}.$$

The function Φ_0 is of the form

$$(4.8a) \quad \Phi_0(p, \tau) = \tilde{\Phi}_0\{p, A_0(\tau), \tau\},$$

where $\tilde{\Phi}_0(p, A_0, \tau)$ is an even 2π -periodic function of p for which the following implicit representation may be obtained by integration of (4.5):

$$(4.8b) \quad p = 2^{-1/2} \tilde{\omega}(A_0, \tau) \int_{\tilde{\Phi}_0}^1 A_0 d\xi / [V_0\{\tilde{\eta}(A_0, \tau) + A_0, \tau\} - V_0\{\tilde{\eta}(A_0, \tau) + A_0\xi, \tau\}]^{1/2}, \quad p \in [0, \pi].$$

We require that A_0 belongs to the class $C^\infty(I, \mathbb{R})$. Then it is seen that U_0 belongs to P^∞ because of the smoothness of all functions involved.

Summarizing, we have up to now the following expression for ϕ_N :

$$(4.9a) \quad \phi_N(\tau, \varepsilon) = \eta(\tau) + A_0(\tau)\Phi_0(p, \tau) + \sum_{\nu=1}^N \varepsilon^\nu U_\nu(p, \tau), \quad p = \varepsilon^{-1}S(\tau, \varepsilon; N),$$

$$(4.9b) \quad S(\tau, \varepsilon; N) = \int_0^\tau \omega(\sigma) d\sigma + \varepsilon \sum_{j=0}^N \varepsilon^j S_j(\tau)$$

Using (4.9), we may replace (4.2) by

$$(4.10) \quad \begin{aligned} &\varepsilon^2 \phi_N'' + F(\phi_N, \tau) + \varepsilon f(\phi_N, \varepsilon \phi_N', \tau, \varepsilon) \\ &= \sum_{\nu=1}^N \varepsilon^\nu [\omega^2 U_{\nu,pp} + F_x(\eta + A_0 \Phi_0, \tau) U_\nu + \gamma_\nu] + \varepsilon^{N+1} g_{N+1}^*, \end{aligned}$$

where

$$\begin{aligned} \gamma_1(p, \tau) &= 2\omega S_{0,\tau} A_0 \Phi_{0,pp} + \omega_\tau A_0 \Phi_{0,p} + 2\omega (A_0 \Phi_{0,p})_\tau + f^{(0)}(p, \tau), \\ \gamma_2(p, \tau) &= (2\omega S_{1,\tau} + S_{0,\tau}^2) A_0 \Phi_{0,pp} + 2\omega S_{0,\tau} U_{1,pp} + S_{0,\tau\tau} A_0 \Phi_{0,p} \\ &\quad + 2S_{0,\tau} (A_0 \Phi_{0,p})_\tau + \omega_\tau U_{1,p} + 2\omega U_{1,p\tau} + \eta_{\tau\tau} + (A_0 \Phi_0)_{\tau\tau} \\ &\quad + f^{(1)}(p, \tau) + F^{(2)}(p, \tau), \end{aligned}$$

² In the theory of wave propagation (governed by partial differential equations), a dispersion relation is a relation between typical quantities of a uniform wave, like frequency, wavenumber(s), amplitude, etc. In the present case this relation only involves the frequency and the amplitude.

$$\begin{aligned} \gamma_\nu(p, \tau) = & (2\omega S_{\nu-1, \tau} + \sum_{i+j=\nu-2} S_{i, \tau} S_{j, \tau}) A_0 \Phi_{0, pp} \\ & + \sum_{\substack{i+j=\nu \\ 1 \leq j < \nu}} (2\omega S_{i-1, \tau} + \sum_{\substack{k+1=i-2 \\ k, 1 > 0}} S_{k, \tau} S_{1, \tau}) U_{j, pp} \\ & + S_{\nu-2, \tau\tau} A_0 \Phi_{0, p} + 2S_{\nu-2, \tau} (A_0 \Phi_{0, p})_\tau \\ & + \omega_\tau U_{\nu-1, p} + 2\omega U_{\nu-1, p\tau} + \sum_{\substack{i+j=\nu-2 \\ i \geq 0, j \geq 1}} (S_{i, \tau\tau} U_{j, p} + 2S_{i, \tau} U_{j, p\tau}) \\ & + U_{\nu-2, \tau\tau} + f^{(\nu-1)}(p, \tau) + F^{(\nu)}(p, \tau), \quad \nu = 3, \dots, N+1, \end{aligned}$$

$$g_{N+1}^*(p, \tau, \varepsilon) = \gamma_{N+1}(p, \tau) + O(\varepsilon), \quad \text{uniformly for } (p, \tau) \in \mathbb{R} \times I.$$

The functions $f^{(\nu)}$, $\nu = 0, 1, \dots, N$, denote the coefficients in the expansion of f

$$f(\phi_N, \varepsilon \phi'_N, \tau, \varepsilon) = \sum_{\nu=0}^N \varepsilon^\nu f^{(\nu)}(p, \tau) + \varepsilon^{N+1} r_N^*(p, \tau, \varepsilon)$$

with remainder $\varepsilon^{N+1} r_N^*$ for which $r_N^* \{\varepsilon^{-1} S(\tau, \varepsilon; N), \tau, \varepsilon\} = r_N(\tau, \varepsilon) = O(1)$ holds.

The expansion of F is as follows:

$$\begin{aligned} F(\phi_N, \tau) = & F(\eta + A_0 \Phi_0, \tau) + \varepsilon F_x(\eta + A_0 \Phi_0, \tau) U_1 \\ & + \sum_{\nu=2}^N \varepsilon^\nu [F_x(\eta + A_0 \Phi_0, \tau) U_\nu + F^{(\nu)}(p, \tau)] + \varepsilon^{N+1} F^{(N+1)}(p, \tau) \\ & + \varepsilon^{N+2} R_{N+1}^*(p, \tau, \varepsilon), \end{aligned}$$

with remainder $\varepsilon^{N+2} R_{N+1}^*$ for which $R_{N+1}^* \{\varepsilon^{-1} S(\tau, \varepsilon; N), \tau, \varepsilon\} = R_{N+1}(\tau, \varepsilon) = O(1)$ holds. It should be noted that, although p represents a function depending also on ε , the functions Φ_0 , U_ν ($\nu = 1, \dots, N$) and their partial derivatives with respect to p and τ have not been expanded in powers of ε . This is allowed because of the generalized asymptotic power series expansions used in this construction procedure. In view of the conditions on f and F and the requirements made for all quantities in (4.9) it follows that $f^{(\nu)}$, $\nu = 0, 1, 2, \dots, N$, and $F^{(\nu)}$, $\nu = 2, 3, 4, \dots, N+1$, satisfy assumptions (i) and (ii) for U_ν .

Since

$$f^{(0)}(p, \tau) = f(\eta + A_0 \Phi_0, \omega A_0 \Phi_{0, p}, \tau, 0),$$

it follows that $f^{(0)}$ belongs to P^∞ . In view of (4.6), (4.7) and (4.8) we may write

$$f^{(0)}(p, \tau) = \tilde{f}^{(0)}\{p, A_0(\tau), \tau\}.$$

Proceeding with the construction, we equate to zero successively the contributions of $O(\varepsilon)$, $\dots$, $O(\varepsilon^N)$ to the right-hand side of (4.10). This yields a set of recurrent equations for $U_1, \dots, U_N$. If these equations have solutions $U_1, \dots, U_N$ satisfying requirements (i) and (ii), then the function g_N ,

$$g_N(\tau, \varepsilon) = g_N^*(p, \tau, \varepsilon), \quad p = \varepsilon^{-1} \int_0^\tau \omega(\sigma) d\sigma + \sum_{j=0}^N \varepsilon^j S_j(\tau),$$

is uniformly bounded on I . This is a necessary condition for ϕ_N to be an N th order formal asymptotic solution of problem (1.2) on I .

The required uniform boundedness of U_ν , $\nu = 1, \dots, N$, leads to equations for $A_0, S_0, S_1, \dots, S_{N-1}$. However, this set of functions does not suffice to achieve the uniform boundedness of U_ν , $\nu = 1, \dots, N$. In the construction procedure other unknown functions $A_1, \dots, A_N$ appear which should belong to $C^\infty(I, \mathbb{R})$. They provide a sufficient additional degree of freedom. We will illustrate this explicitly by the determination of U_1 .

The function U_1 satisfies the linear ordinary differential equation

$$(4.11) \quad \omega^2 U_{1,pp} + F_x(\eta + A_0 \Phi_0, \tau) U_1 = -\gamma_1(p, \tau).$$

In consequence of the results obtained above γ_1 belongs to the class P^∞ . The homogeneous equation corresponding to (4.11) is the first variational equation of equation (4.3) with respect to U_0 . In the Appendix B two linearly independent solutions z_1^* and z_2^* of the homogeneous equation are determined in terms of U_0 . The functions z_1^* and z_2^* are odd, respectively even, 2π -periodic functions of p with a Wronskian equal to 1. Both z_1^* and z_2^* belong to P^∞ and have the alternative representation

$$z_i^*(p, \tau) = \tilde{z}_i^*\{p, A_0(\tau), \tau\}, \quad i = 1, 2.$$

The requirement that U_1 should satisfy the uniform boundedness condition (ii) leads to the condition on γ_1 ,

$$\int_0^{2\pi} \gamma_1(p, \tau) z_i^*(p, \tau) dp = 0, \quad i = 1, 2.$$

This condition, known as the suppression of “secular terms” in the determination of U_1 , gives first order differential equations from which A_0 and S_0 may be determined. We define

$$\int_0^{2\pi} \tilde{f}^{(0)}(p, A_0, \tau) \tilde{z}_i^*(p, A_0, \tau) dp = \tilde{\lambda}_i^{(1)}(A_0, \tau), \quad i = 1, 2,$$

$$\frac{1}{A_0} \int_0^{2\pi} \{\tilde{z}_i^*(p, A_0, \tau)\}^2 dp = \tilde{L}(A_0, \tau),$$

and we note that

$$\begin{aligned} A_0 \int_0^{2\pi} \tilde{\Phi}_{0,pp}(p, A_0, \tau) \tilde{z}_2^*(p, A_0, \tau) dp &= \int_0^{2\pi} z_{1,p}^* z_2^* dp \\ &= - \int_0^{2\pi} z_1^* z_{2,p}^* dp = - \int_0^{2\pi} (1 + z_{1,p}^* z_2^*) dp, \end{aligned}$$

so that

$$A_0 \int_0^{2\pi} \tilde{\Phi}_{0,pp}(p, A_0, \tau) \tilde{z}_2^*(p, A_0, \tau) dp = -\pi.$$

Then, taking into account that the function γ_1 consists of even terms in p as well as odd terms, the suppression of secular terms reduces to the equations

$$(4.12a) \quad \frac{d}{d\tau} \{A_0 \tilde{\omega}(A_0, \tau) \tilde{L}(A_0, \tau)\} = -\tilde{\lambda}_1^{(1)}(A_0, \tau),$$

$$(4.12b) \quad 2\pi \tilde{\omega}(A_0, \tau) S_{0,\tau} = \tilde{\lambda}_2^{(1)}(A_0, \tau).$$

Equation (4.12a) is a nonlinear equation for A_0 whereas (4.12b) is a linear one for S_0 . It is essential for the construction procedure that equation (4.12a) possess a solution on I which belongs to $C^\infty(I, \mathbb{R})$.

In order to determine a particular solution Φ_1 of (4.11) we define functions $a_i^{(1)}(p, \tau)$ by

$$(4.13) \quad a_{i,p}^{(1)} = \frac{1}{\omega^2} \gamma_1 z_i^*, \quad i = 1, 2.$$

then the method of variation of constants yields

$$\Phi_1(p, \tau) = a_2^{(1)}(p, \tau) z_1^*(p, \tau) - a_1^{(1)}(p, \tau) z_2^*(p, \tau).$$

Because of the suppression of secular terms in γ_1 , Φ_1 is a 2π -periodic function of p . The “constants of integration” in $a_i^{(1)}$ can be chosen in such a way that

$$\int_0^{2\pi} \Phi_1(p, \tau) z_i^*(p, \tau) dp = 0, \quad i = 1, 2,$$

in other words, Φ_1 does not contain secular terms. This is convenient in later stages of the construction procedure.

The solution U_1 of (4.11) is taken to be

$$U_1(p, \tau) = A_1(\tau) z_2^*(p, \tau) + \Phi_1(p, \tau).$$

If we require that A_1 belong to $C^\infty(I, \mathbb{R})$, then U_1 belongs to P^∞ because of the smoothness of all functions involved and the suppression of secular terms in γ_1 .

It should be noted that a homogeneous solution proportional to $z_1^* = A_0 \Phi_{0,p}$ need not be included in U_1 since this has already been taken into account in U_0 because of the assumed expansion (4.1b) of S .

The determination of the functions U_ν , $\nu = 2, 3, \dots, N$, is completely analogous to that of U_1 . Each time we first suppress the secular terms in the right-hand side $-\gamma_\nu$ which depends on $U_0, U_1, \dots, U_{\nu-1}$. This yields equations for $S_{\nu-1}$ and $A_{\nu-1}$. In addition to the particular solution Φ_ν , which should satisfy the conditions

$$\int_0^{2\pi} \Phi_\nu(p, \tau) z_i^*(p, \tau) dp = 0, \quad i = 1, 2,$$

the function U_ν should contain a term proportional to z_2^*

$$U_\nu(p, \tau) = A_\nu(\tau) z_2^*(p, \tau) + \Phi_\nu(p, \tau).$$

If we require that $A_\nu \in C^\infty(I, \mathbb{R})$, then U_ν belongs to P^∞ . Together with S_ν , the function A_ν may serve as a means of suppressing the secular terms in the right-hand side of the equation for $U_{\nu+1}$.

We shall restrict ourselves to an explicit determination of the equations for S_1 and A_1 . These equations follow from the suppression of secular terms in γ_2 . In view of the results obtained above γ_2 is of the form

$$\begin{aligned}\gamma_2(p, \tau) = & (2\omega S_{1,\tau} + S_{0,\tau}^2)A_0\Phi_{0,pp} + 2\omega S_{0,\tau}A_1z_{2,pp}^* + S_{0,\tau\tau}A_0\Phi_{0,p} \\ & + 2S_{0,\tau}(A_0\Phi_{0,p})_\tau + \omega_\tau A_1z_{2,p}^* + 2\omega(A_1z_{2,p}^*)_\tau + \eta_{\tau\tau} + (A_0\Phi_0)_{\tau\tau} \\ & + f^{(1)}(p, \tau) + F^{(2)}(p, \tau) \\ & + \text{functions proportional to } \Phi_1 \text{ and its partial derivatives,}\end{aligned}$$

where

$$\begin{aligned}f^{(1)}(p, \tau) = & f_1(\eta + A_0\Phi_0, \omega A_0\Phi_{0,p}, \tau, 0)(A_1z_2^* + \Phi_1) \\ & + f_2(\eta + A_0\Phi_0, \omega A_0\Phi_{0,p}, \tau, 0) \\ & \cdot \{\eta_\tau + (A_0\Phi_0)_\tau + S_{0,\tau}A_0\Phi_{0,p} + \omega A_1z_{2,p}^* + \omega\Phi_{1,p}\} \\ & + f_4(\eta + A_0\Phi_0, \omega A_0\Phi_{0,p}, \tau, 0), \\ F^{(2)}(p, \tau) = & \frac{1}{2}F_{xx}(\eta + A_0\Phi_0, \tau)(A_1z_2^* + \Phi_1)^2.\end{aligned}$$

The functions proportional to Φ_1 and its partial derivatives which have been left unspecified in the expression for γ_2 do not play a role in the suppression of secular terms in γ_2 . We define

$$\int_0^{2\pi} f^{(1)}(p, \tau) z_i^*(p, \tau) dp = \lambda_i^{(2)}(\tau), \quad i = 1, 2,$$

and

$$\int_0^{2\pi} F^{(2)}(p, \tau) z_i^*(p, \tau) dp = \Lambda_i^{(2)}(\tau), \quad i = 1, 2.$$

Then the suppression of secular terms in γ_2 yields the following equations for S_1 and A_1 :

$$(4.14a) \quad 2\pi\omega A_{1,\tau} + \pi\omega_\tau A_1 + 2\omega A_1 \int_0^{2\pi} z_{2,p\tau}^* z_1^* dp = \frac{d}{d\tau} (A_0 S_{0,\tau} \tilde{L}) - \Lambda_1^{(2)} - \lambda_1^{(2)},$$

$$(4.14b) \quad -2\pi\omega S_{1,\tau} + 2\omega A_1 S_{0,\tau} \int_0^{2\pi} z_{2,pp}^* z_2^* dp = - \int_0^{2\pi} (A_0 \Phi_0)_{\tau\tau} z_2^* dp - \Lambda_2^{(2)} - \lambda_2^{(2)}.$$

Since $F^{(2)}$ and hence, $\Lambda_1^{(2)}$, depend on A_1 in a nonlinear way, (4.14a) for A_1 is a nonlinear equation. On the other hand, (4.14b) for S_1 is linear. It is straightforward to check that the equations for A_ν and S_ν , $\nu = 2, 3, \dots, N-1$, are linear.

When the calculation of $U_1, \dots, U_N$ and the determination of the equations for $A_0, S_0, A_1, S_1, \dots, A_{N-1}, S_{N-1}$ have been performed, we obtain an N th order formal asymptotic solution ϕ_N of the form

$$\begin{aligned}(4.15) \quad \phi_N(\tau, \varepsilon) = & \eta(\tau) + A_0\Phi_0(p, \tau) + \sum_{\nu=1}^N \varepsilon^\nu [A_\nu(\tau) z_2^*(p, \tau) + \Phi_\nu(p, \tau)], \\ p = & \varepsilon^{-1} S(\tau, \varepsilon; N) = \varepsilon^{-1} \int_0^\tau \omega(\sigma) d\sigma + \sum_{j=0}^N \varepsilon^j S_j(\tau),\end{aligned}$$

in which no equations for A_N and S_N have been determined yet. In view of the requirements that A_ν and S_ν , $\nu = 0, \dots, N$, belong to $C^\infty(I, \mathbb{R})$, ϕ_N belongs to P_S^∞ . Consequently, the residual function g_N of ϕ_N belongs to P_S^∞ . From the definition of g_N in the expansion (4.10) the condition (3.20) in order that ϕ_N be an asymptotic solution uniformly valid on I corresponds to the suppression of secular terms in γ_{N+1} :

$$\int_0^{2\pi} \gamma_{N+1}(p, \tau) z_i^*(p, \tau) dp = 0, \quad i = 1, 2.$$

This yields two linear ordinary differential equations for A_N and S_N . In order to obtain the initial values for A_ν and S_ν , $\nu = 0, \dots, N$, we insert the expansion (4.15) into the expressions

$$\alpha_1(\varepsilon) - \phi_N(0, \varepsilon) \quad \text{and} \quad \alpha_2(\varepsilon) - \varepsilon \phi'_N(0, \varepsilon)$$

and expand them completely in powers of ε . Then the required initial conditions follow from equating to zero the contributions of $O(\varepsilon^\nu)$, $\nu = 0, \dots, N$, to these expansions.

If the nonlinear initial value problems for A_0 and A_1 have solutions belonging to $C^\infty(I, \mathbb{R})$, then, by virtue of the theory of linear ordinary differential equations, we know that A_ν , $\nu = 2, \dots, N$, and S_ν , $\nu = 0, \dots, N$, are functions belonging to $C^\infty(I, \mathbb{R})$. Moreover, if, e.g., for a suitable choice of $\alpha_{1,2}(\varepsilon)$, the nonlinear problem (4.12a) for A_0 could be solved in such a way that the total mechanical energy E_0 is noncritical and that E_0 implies the bounded interval $(\eta - A_0, \eta + A_0)$ of the U_0 -axis on which the potential energy is less than E_0 for all τ on I , then the solution U_0 of (4.3) is a periodic function. This completes the construction of an N th order asymptotic solution ϕ_N of (1.2) for $N \geq 1$. According to the remark following Theorem 3 the correctness of $\phi_0 = \eta + A_0 \Phi_0$ as a zeroth order asymptotic solution is established if we are able to show that the equation for A_1 has a solution belonging to $C^\infty(I, \mathbb{R})$.

From the N th order asymptotic solution ϕ_N constructed above we may obtain any lower order asymptotic solution ϕ_k ($k < N$) by deleting from ϕ_N the terms with $\Phi_{k+1}, \dots, \Phi_N, A_{k+1}, \dots, A_N$ and $S_{k+1}, \dots, S_N$. If we compare the expansion for ϕ_k obtained in this way with the expansion of ϕ_k obtained by a calculation starting from the beginning as described for ϕ_N in this section, then we see that both results are exactly the same. This is a result of the fact that all of the functions Φ_ν , A_ν , $\nu = 0, \dots, N$, and S_ν , $\nu = -1, \dots, N$, do not depend on N . There is only a difference in the argument by which the equations for A_k and S_k are determined. In the result obtained by truncating the expansions in ϕ_N the equations follow from the suppression of secular terms in the right-hand side of the equation for U_{k+1} , whereas in the independent calculation of ϕ_k starting from the beginning the equations for A_k and S_k follow from condition (3.20) of Theorem 3 on the residual function. In view of this equality in the determination of asymptotic solutions we may indicate an algorithm which successively generates a sequence of asymptotic solutions $\phi_0, \phi_1, \dots$, etc. This can be done by

putting $N = \infty$ in (4.9):

$$(4.16a) \quad \phi(\tau, \varepsilon) \sim \eta(\tau) + A_0(\tau)\Phi_0(p, \tau) + \sum_{\nu=1}^{\infty} \varepsilon^{\nu} U_{\nu}(p, \tau), \quad p = \varepsilon^{-1} S(\tau, \varepsilon),$$

$$(4.16b) \quad S(\tau, \varepsilon) \sim \int_0^{\tau} \omega(\sigma) d\sigma + \varepsilon \sum_{j=0}^{\infty} \varepsilon^j S_j(\tau),$$

where η , ω and Φ_0 are given by (4.6), (4.7) and (4.8), respectively. The calculation proceeds in the same way as described above for ϕ_N . Successively, we determine the functions A_0 , S_0 , Φ_1 , A_1 , S_1 , Φ_2 , $\dots$.

In order to obtain a k th order asymptotic solution, the algorithm is terminated after the determination of A_k and S_k , and ϕ_k is then given by (4.16) truncated after $\nu = k$ and $j = k$.

5. A perturbed Duffing equation. As an application of the algorithmic procedure of § 4 we determine a zeroth order asymptotic solution of a perturbed Duffing equation with slowly varying coefficients

$$(5.1a) \quad \varepsilon^2 x'' + a(\tau)x + b(\tau)x^3 + \varepsilon c(\tau)x^2 + \varepsilon^2 d(\tau)x' = 0, \quad \tau \in I,$$

with initial conditions

$$(5.1b) \quad x(0, \varepsilon) = \alpha_1(\varepsilon), \quad \varepsilon x'(0, \varepsilon) = 0.$$

We assume that a , b , c and d belong to $C^{\infty}(I, \mathbb{R})$ and that a and b are strictly positive on I . Then, equation (5.1) is an example of the class of oscillation problems (1.2) as considered in § 4. The cases in which a and b have signs on I which differ from the above could be treated in a similar way, cf. Kaper [6]. In [7] Kuzmak constructed a first order formal asymptotic solution of the Duffing equation with slowly varying coefficients ($c = d = 0$ in (5.1)). He succeeded in the suppression of secular terms in the equation for U_1 with the introduction of only one degree of freedom. This is due to the special form of his example and without a reference to the special choice of the initial conditions. This may be verified below in the equation for S_0 when $c \equiv d \equiv 0$ and $S_0(0) = 0$ which implies $S_0 \equiv 0$.

Because of the symmetry of V_0 ,

$$V_0(x, \tau) = \frac{1}{2}a(\tau)x^2 + \frac{1}{4}b(\tau)x^4,$$

with respect to $x = 0$, the function η in the infinite series (4.16) is equal to zero. The even, 2π -periodic solution Φ_0 of the τ -parameter equation

$$(5.2) \quad \omega^2(\tau)A_0(\tau)\Phi_{0,pp} + a(\tau)A_0(\tau)\Phi_0 + b(\tau)A_0^3(\tau)\Phi_0^3 = 0$$

can be expressed in terms of the Jacobian elliptic function cn with modulus λ , $0 < \lambda^2 < 1$. The differential equation satisfied by cn is

$$(\text{cn}')^2 = (1 - \text{cn}^2)(1 - \lambda^2 + \lambda^2 \text{cn}^2), \quad 0 < \lambda^2 < 1.$$

Differentiation of both sides yields

$$\text{cn}'' + (1 - 2\lambda^2)\text{cn} + 2\lambda^2 \text{cn}^3 = 0.$$

If we put

$$\Phi_0(p, \tau) = \operatorname{cn} \left[\frac{2K(\lambda)}{\pi} p; \lambda \right],$$

then Φ_0 is the even, 2π -periodic solution of (5.2) if

$$(5.3a) \quad a - (1 - 2\lambda^2) \omega^2 \frac{4K^2}{\pi^2} = 0,$$

$$(5.3b) \quad bA_0^2 - 2\lambda^2 \omega^2 \frac{4K^2}{\pi^2} = 0,$$

where K denotes the complete elliptic integral of the first kind with modulus λ ,

$$(5.4) \quad K(\lambda) = \int_0^1 \{(1 - \zeta^2)(1 - \lambda^2 \zeta^2)\}^{-1/2} d\zeta.$$

Eliminating λ from (5.3) we get the dispersion relation between ω and A_0

$$(5.5a) \quad \omega(\tau) = \tilde{\omega}\{A_0(\tau), \tau\},$$

where

$$(5.5b) \quad \begin{aligned} \tilde{\omega}(A_0, \tau) &= \frac{\pi}{2K(\lambda)} \{a(\tau) + b(\tau)A_0^2\}^{1/2}, \\ \lambda^2 &= \frac{1}{2}b(\tau)A_0^2 \{a(\tau) + b(\tau)A_0^2\}^{-1}. \end{aligned}$$

As a and b are positive functions it follows that $0 < \lambda^2 < \frac{1}{2}$. This implies that Φ_0 is a 2π -periodic function of p indeed. The odd and even 2π -periodic solutions z_1^* , respectively z_2^* , of the first variational equation of (5.2) with respect to $A_0\Phi_0$,

$$\omega^2(\tau)z_{pp}^* + \left\{ a(\tau) + 3b(\tau)A_0^2(\tau) \operatorname{cn}^2 \left[\frac{2K(\lambda)}{\pi} p; \lambda \right] \right\} z^* = 0,$$

can also be expressed in terms of the Jacobian elliptic functions:

$$z_1^*(p, \tau) = A_0(\tau) \frac{2K(\lambda)}{\pi} \operatorname{cn}' \left[\frac{2K(\lambda)}{\pi} p; \lambda \right],$$

and

$$z_2^*(p, \tau) = H \left\{ A_0(\tau) \operatorname{cn} \left[\frac{2K(\lambda)}{\pi} p; \lambda \right], A_0(\tau), \tau \right\},$$

where (cf. Appendix B)

$$H(\zeta, A_0, \tau) = \frac{1}{2} \tilde{\omega}^2(A_0, \tau) \left\{ \frac{1}{2}a(\tau)(A_0^2 - \zeta^2) + \frac{1}{4}b(\tau)(A_0^4 - \zeta^4) \right\}^{1/2}.$$

$$\cdot \int_0^\zeta \left\{ \frac{1}{2}a(\tau)(A_0^2 - \xi^2) + \frac{1}{4}b(\tau)(A_0^4 - \xi^4) \right\}^{-3/2} d\xi.$$

We substitute (4.16) in the left-hand side of (5.1a), expand the expression considered as a function of p, τ and ε in an asymptotic power series with respect to

ε and equate to zero the $O(\varepsilon)$ -contribution. This leads to the equation for U_1 :

$$\omega^2(\tau)U_{1,pp} + \left\{ a(\tau) + 3b(\tau)A_0^2(\tau) \operatorname{cn}^2\left[\frac{2K}{\pi}p; \lambda\right] \right\} U_1 = -\gamma_1(p, \tau),$$

where

$$\begin{aligned} \gamma_1 = & 2\omega S_{0,\tau} A_0 \frac{4K^2}{\pi^2} \operatorname{cn}''\left[\frac{2K}{\pi}p; \lambda\right] + \omega_\tau A_0 \frac{2K}{\pi} \operatorname{cn}'\left[\frac{2K}{\pi}p; \lambda\right] \\ & + 2\omega \left(A_0 \frac{2K}{\pi} \operatorname{cn}'\left[\frac{2K}{\pi}p; \lambda\right] \right)_\tau + cA^2 \operatorname{cn}^2\left[\frac{2K}{\pi}p; \lambda\right] \\ & + d\omega A_0 \frac{2K}{\pi} \operatorname{cn}'\left[\frac{2K}{\pi}p; \lambda\right]. \end{aligned}$$

The secular terms in the equation for U_1 are suppressed if A_0 and S_0 satisfy the first order differential equations

$$(5.6a) \quad \frac{d}{d\tau} [A_0(\tau) \tilde{\omega}\{A_0(\tau), \tau\} \tilde{L}\{A_0(\tau), \tau\}] = -d(\tau) A_0(\tau) \tilde{\omega}\{A_0(\tau), \tau\} \tilde{L}\{A_0(\tau), \tau\}$$

and

$$(5.6b) \quad \begin{aligned} 2\pi \tilde{\omega}\{A_0(\tau), \tau\} S_{0,\tau}(\tau) = & 4c(\tau) A_0^2(\tau) \frac{\pi}{2K(\lambda)} \int_0^{K(\lambda)} \operatorname{cn}^2[u; \lambda] \\ & \cdot H\{A_0(\tau) \operatorname{cn}[u; \lambda], A_0(\tau), \tau\} du, \end{aligned}$$

where

$$\tilde{L}(A_0, \tau) = 4 \frac{2K(\lambda)}{\pi} A_0 \int_0^{K(\lambda)} \{\operatorname{cn}'[u; \lambda]\}^2 du.$$

The initial values for A_0 and S_0 follow from the asymptotic power series expansion of the expressions

$$\alpha - \phi(0, \varepsilon) \quad \text{and} \quad \varepsilon \phi'(0, \varepsilon)$$

where ϕ is given by (4.16). At this stage these expansions are known up to $O(\varepsilon)$. Equating to zero the $O(1)$ -contribution we get

$$A_0(0) = \alpha \quad \text{and} \quad S_0(0) = 0.$$

Solving the nonlinear equation (5.6a) we arrive at the nonlinear expression for A_0 :

$$(5.7a) \quad A_0^2 \tilde{\omega}(A_0, \tau) K(\lambda) L_I(\lambda) = \alpha^2 \tilde{\omega}(\alpha, 0) K(\lambda_0) L_I(\lambda_0) \exp \left[- \int_0^\tau d(\sigma) d\sigma \right],$$

where

$$\lambda_0^2 = \frac{1}{2} b(0) \alpha^2 \{a(0) + b(0) \alpha^2\}^{-1}$$

and

$$L_I(\lambda) = \int_0^{K(\lambda)} \{\operatorname{cn}'[u; \lambda]\}^2 du.$$

Integrating equation (5.6b) we get

(5.7b)

$$S_0(\tau) = \int_0^\tau \frac{c(\sigma)A_0^2(\sigma)d\sigma}{\tilde{\omega}\{A_0(\sigma), \sigma\}K(\lambda)} \int_0^{K(\lambda)} \text{cn}^2[u; \lambda] H\{A_0(\sigma)\text{cn}[u; \lambda], A_0(\sigma), \sigma\} du.$$

For a zeroth order asymptotic solution ϕ_0 the procedure should be terminated after the determination of A_0 and S_0 . Then, ϕ_0 is obtained from the formal series (4.16) by truncating after A_0 and S_0 :

$$\phi_0(\tau, \varepsilon) = A_0(\tau)\text{cn}\left[\frac{2K(\lambda)}{\pi}p; \lambda\right], \quad p = \frac{1}{\varepsilon}S(\tau, \varepsilon),$$

$$S(\tau, \varepsilon) = \int_0^\tau \omega(\sigma) d\sigma + \varepsilon S_0(\tau),$$

with $K, \omega, \lambda, A_0, S_0$ are given by (5.4), (5.5) and (5.7) respectively.

In order to examine the solvability of expression (5.7a) for A_0 we rewrite it as an expression involving λ^2 . With the help of (5.5) we find after some simple transformations

$$(5.8) \quad p_I(\lambda) = \beta(\tau)p_I(\lambda_0), \quad \tau \in I,$$

where

$$p_I(\lambda) = \frac{\lambda^2 L_I(\lambda)}{(1 - 2\lambda^2)^{3/2}}$$

and

$$\beta(\tau) = \frac{b(\tau)}{b(0)} \left\{ \frac{a(0)}{a(\tau)} \right\}^{3/2} \exp \left[- \int_0^\tau d(\sigma) d\sigma \right].$$

For $a(\tau) > 0$ and $b(\tau) > 0$, $\tau \in I$, we have $0 < \lambda^2 < 1/2$ for all values of the amplitude $A_0(\tau)$. Hence, we need to examine the solvability of (5.8) on $(0, \frac{1}{2})$ as τ takes on values on I .

In order to calculate the derivative of p_I with respect to λ we use the equality

$$L_I(\lambda) = \int_0^{K(\lambda)} (\text{cn}[u; \lambda])^2 du = \int_0^1 t^2 \frac{(1 - \lambda^2 t^2)^{1/2}}{(1 - t^2)^{1/2}} dt.$$

Then,

$$\frac{dp_I}{d\lambda} = \frac{\lambda^2}{(1 - 2\lambda^2)^{5/2}} \int_0^1 \frac{t^2}{(1 - t^2)^{1/2}} [2(1 + \lambda^2) - t^2(3\lambda^2 - \lambda^4)] \frac{dt}{\sqrt{1 - \lambda^2 t^2}}.$$

The expression between square brackets could be estimated on $0 \leq t \leq 1$ in the following way:

$$2(1 + \lambda^2) - t^2(3\lambda^2 - \lambda^4) \geq \lambda^4 - \lambda^2 + 2 \geq 1.$$

It follows that

$$\frac{d}{d\lambda} p_I > 0 \quad \text{for} \quad 0 < \lambda^2 < 1/2.$$

Thus, p_I is a monotonically increasing function from zero to infinity as λ^2 increases from 0 to $1/2$.

Note that $\beta(0) = 1$ and that $\beta(\tau) > 0$ for $\tau \in I$. Then, for each positive value of $A_0(0) = \alpha$, λ^2 could be solved as a function of τ on I from equation (5.8) with

$$0 < \lambda^2 < 1/2.$$

From (5.5b) we have

$$A_0^2(\tau) = \frac{2a(\tau)\lambda^2(\tau)}{b(\tau)}\{1 - 2\lambda^2(\tau)\}^{-1}, \quad \tau \in I.$$

Since the asymptotic correctness of a zeroth order asymptotic solution ϕ_0 is obtained by comparison with a first order asymptotic solution ϕ_1 we ought to show the existence of ϕ_1 first. As indicated in § 4 this implies the examination of the solvability of a first order ordinary differential equation for A_1 , which is given by (4.14a) for the general nonlinear case. For problem (5.1) the equation for A_1 is given by

$$2\pi\omega A_{1,\tau} + \left(\pi\omega_\tau + 2\omega \int_0^{2\pi} z_{2,p\tau}^* z_1^* dp\right) A_1 = -\frac{d}{d\tau}(A_0 S_{0,\tau} \tilde{L}) - \Lambda_1^{(2)} - \lambda_1^{(2)},$$

where

$$\Lambda_1^{(2)} = \int_0^{2\pi} 3bA_0\Phi_0(A_1 z_2^* + \Phi_1)^2 z_1^* dp,$$

and

$$\begin{aligned} \lambda_1^{(2)} = & \int_0^{2\pi} 2cA_0\Phi_0(A_1 z_2^* + \Phi_1) z_1^* dp \\ & + d \int_0^{2\pi} \{(A_0\Phi_0)_\tau + S_{0,\tau}A_0\Phi_{0,p} + \omega A_1 z_{2,p}^* + \omega\Phi_{1,p}\} z_1^* dp. \end{aligned}$$

Φ_0 is the even, 2π -periodic Jacobian elliptic function cn ; z_1^* and z_2^* represent the odd, respectively even, homogeneous solutions of the first variational equation of (5.2) with respect to $A_0\Phi_0$. As the odd contributions to the integrands of $\Lambda_1^{(2)}$ and $\lambda_1^{(2)}$ vanish when we integrate over one period 2π , the quadratic dependence of $\Lambda_1^{(2)}$ on A_1 vanishes. Hence, the equation for A_1 is a linear first order differential equation with smooth coefficient functions on I . The existence of a solution A_1 on I implies the existence of ϕ_1 .

Appendix A.

LEMMA 2. Let J be a function of τ and ε which belongs to P_S^∞ . If

$$(A.1) \quad j(\tau, \varepsilon) = \int_0^{2\pi} J^*(p, \tau, \varepsilon) dp = O(\varepsilon),$$

then

$$(A.2) \quad \int_0^\tau J(s, \varepsilon) ds = O(\varepsilon).$$

Proof. The following identity holds:

$$(A.3) \quad \int_0^\tau J^*\{\varepsilon^{-1}S(s, \varepsilon), s, \varepsilon\} ds = \int_0^\tau J^*\{\varepsilon^{-1}S(s, \varepsilon), \tau, \varepsilon\} ds \\ - \int_0^\tau d\xi \int_0^\xi J_\xi^*\{\varepsilon^{-1}S(s, \varepsilon), \xi, \varepsilon\} ds.$$

We introduce the new integration variable $\eta = \varepsilon^{-1}S(s, \varepsilon)$ which has an inverse transform for small $\varepsilon > 0$ by virtue of the properties of S :

$$s = \zeta(\varepsilon\eta, \varepsilon), \quad \zeta \in C^\infty(\mathbb{R} \times [0, \varepsilon_0], \mathbb{R}).$$

For the first integral in the right-hand side of (A.3) we find

$$\int_0^\tau J^*\{\varepsilon^{-1}S(s, \varepsilon), \tau, \varepsilon\} ds = \varepsilon \int_0^{\varepsilon^{-1}S(\tau, \varepsilon)} J^*(\eta, \tau, \varepsilon) \zeta'(\varepsilon\eta, \varepsilon) d\eta \\ = \varepsilon \zeta'\{S(\tau, \varepsilon), \varepsilon\} \int_0^{\varepsilon^{-1}S(\tau, \varepsilon)} J^*(\eta, \tau, \varepsilon) d\eta \\ - \varepsilon^2 \int_0^{\varepsilon^{-1}S(\tau, \varepsilon)} \zeta''(\varepsilon\xi, \varepsilon) d\xi \int_0^\xi J^*(\eta, \tau, \varepsilon) d\eta,$$

where the prime indicates differentiation with respect to the first argument. By virtue of the given property of j and the periodicity of J^* as a function of p , we know that

$$\int_0^\xi J^*(\eta, \tau, \varepsilon) d\eta = O(1) \quad \text{uniformly for } (\xi, \tau) \in [0, \varepsilon^{-1}S(\tau, \varepsilon)] \times I.$$

Since $\zeta \in C^\infty(\mathbb{R} \times [0, \varepsilon_0])$ we may conclude that the first integral in the right-hand side of (A.3) is $O(\varepsilon)$.

Similarly,

$$\int_0^\xi J_\xi^*\{\varepsilon^{-1}S(s, \varepsilon), \xi, \varepsilon\} ds = \varepsilon \int_0^{\varepsilon^{-1}S(\xi, \varepsilon)} J_\xi^*(\eta, \xi, \varepsilon) \zeta'(\varepsilon\eta, \varepsilon) d\eta$$

is uniformly $O(\varepsilon)$ for $\xi \in I$ if

$$(A.4) \quad j'(\tau, \varepsilon) = \int_0^{2\pi} J_\tau^*(p, \tau, \varepsilon) dp = O(\varepsilon).$$

To show (A.4) we note that j is infinitely differentiable for $(\tau, \varepsilon) \in I \times [0, \varepsilon_0]$ and $j = O(\varepsilon)$. Then $j(\tau, 0) = 0$ for $\tau \in I$ and (A.4) follows from the fact that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \frac{\partial j}{\partial \tau}(\tau, \varepsilon) = \left[\frac{\partial}{\partial \varepsilon} \left(\frac{\partial j}{\partial \tau} \right) \right]_{\varepsilon=0}$$

is bounded on I . Hence both integrals with respect to s in the right-hand side of (A.3) are uniformly $O(\varepsilon)$ for $\tau \in I$, respectively $\xi \in I$, from which (A.2) follows.

Appendix B. In this Appendix we determine the functions z_1^* and z_2^* which occur in §§ 3 and 4 as solutions of the differential equation

$$(B.1) \quad S_{-1,\tau}^2 z_{pp}^* + F_{yy}\{U_0(p, \tau), \tau\} z^* = 0, \quad (p, \tau) \in \mathbb{R} \times I,$$

where $F(y, \tau)$ has an absolute minimum at $y = 0$ and where the total mechanical energy E_0 of the system (4.3) is assumed to be noncritical uniformly for $\tau \in I$. Both z_1^* and z_2^* should belong to P^∞ and their Wronskian $D(z_1^*, z_2^*) = z_1^* z_{2,p}^* - z_{1,p}^* z_2^*$ should be equal to 1.

The first solution z_1^* of (B.1) is obtained by noting that $U_{0,p}$ is a solution of (B.1):

$$(B.2) \quad \begin{aligned} z_1^*(p, \tau) &= U_{0,p}(p, \tau) = A_0(\tau) \Phi_{0,p}(p, \tau) = A_0(\tau) \tilde{\Phi}_{0,p}\{p, A_0(\tau), \tau\} \\ &= \tilde{z}_1^*\{p, A_0(\tau), \tau\}, \end{aligned}$$

where we used the representations (4.4) for U_0 and (4.8a) for Φ_0 . Clearly, z_1^* belongs to P^∞ and since U_0 is an even, 2π -periodic function of p , z_1^* is an odd, 2π -periodic function of p .

The second solution z_2^* of (B.1) is obtained by putting $z_2^* = w U_{0,p}$. Substituting this in equation (B.1), integrating once with respect to p and choosing the constant of integration equal to 1, we get $w_p = U_{0,p}^{-2}$. This holds for any p -interval where $U_{0,p} \neq 0$. So we have

$$(B.3) \quad z_2^* = U_{0,p} \int_{p_0}^p U_{0,p}^{-2} dp, \quad 0 < p < \pi,$$

where p_0 denotes the zero of U_0 on the interval $(0, \pi)$. The choice of p_0 as the lower bound of integration in (B.3) is a matter of convenience. With the help of results from § 4 on Φ_0 we deduce

$$U_{0,p} = -\tilde{\omega}^{-1}(A_0, \tau) \sqrt{2\{F(\tilde{\eta} + A_0, \tau) - F(U_0, \tau)\}^{1/2}}, \quad 0 < p < \pi.$$

We see that U_0 is a monotonic function of p on $[0, \pi]$. Then, introducing U_0 as a new integration variable in (B.3), we get for all $(p, \tau) \in (0, \pi) \times I$,

$$(B.4) \quad \begin{aligned} z_2^*(p, \tau) &= U_{0,p} \int_0^{U_0} U_{0,p}^{-3} dU_0 \\ &= \frac{1}{2} \tilde{\omega}^2(A_0, \tau) [F\{\tilde{\eta}(A_0, \tau) + A_0, \tau\} - F(U_0, \tau)]^{1/2} \\ &\quad - \int_0^{U_0} [F\{\tilde{\eta}(A_0, \tau) + A_0, \tau\} - F(\xi, \tau)]^{-3/2} d\xi \\ &= H\{U_0(p, \tau), A_0(\tau), \tau\} = \tilde{z}_2^*\{p, A_0(\tau), \tau\}. \end{aligned}$$

As the total mechanical energy E_0 is noncritical, that is,

$$F_y(\tilde{\eta} \pm A_0, \tau) \neq 0 \quad \text{for all } \tau \in I,$$

the function $F(\tilde{\eta} + A_0, \tau) - F(U_0, \tau)$ has simple zeros at $U_0 = \eta \pm A_0$ and hence the limit values of H at $U_0 = \eta \pm A_0$ exist. Therefore (B.4) defines a continuous function z_2^* on $[0, \pi] \times I$. We now extend z_2^* onto the next half period π by using symmetry considerations

$$z_2^*(\pi + p, \tau) = z_2^*(\pi - p, \tau), \quad 0 \leq p \leq \pi.$$

and further on by the periodicity condition

$$z_2^*(p + 2\pi, \tau) = z_2^*(p, \tau).$$

The resulting function, defined on the whole p -axis satisfies equation (B.1) everywhere. From similar considerations for the derivative of z_2^* with respect to p it is seen that z_2^* is an even, 2π -periodic function which belongs to P^∞ . A simple calculation shows that the Wronskian of z_1^* and z_2^* is equal to 1.

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