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Brief paper

Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems[☆]

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ABSTRACT

This paper studies some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. First, basic theoretical analysis is carried out for the case where for each agent the second-order dynamics are governed by the position and velocity terms and the asymptotic velocity is constant. A necessary and sufficient condition is given to ensure second-order consensus and it is found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix of the corresponding network play key roles in reaching consensus. Based on this result, a second-order consensus algorithm is derived for the multi-agent system facing communication delays. A necessary and sufficient condition is provided, which shows that consensus can be achieved in a multi-agent system whose network topology contains a directed spanning tree if and only if the time delay is less than a critical value. Finally, simulation examples are given to verify the theoretical analysis.

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1. Introduction

The model of multi-agent dynamical systems has been utilized more and more widely in recent years in the study of biological, social and engineering systems, such as animal groups, sensor networks, robotic teams, and so on. Researchers are especially interested in how coordinated group behavior arises in such multi-agent systems and in particular significant effort has been made to study reaching consensus (Hong, Chen, & Bushnell, 2008; Hong, Hu, & Gao, 2006; Jadbabaie, Lin, & Morse, 2003; Olfati-Saber, 2004; Pecora & Carroll, 1990; Ren, 2007, 2008; Ren & Atkins, 2005; Ren & Beard, 2005, 2008; Yu, Chen, Cao, & Kurths, in press; Yu, Chen, Wang, & Yang, 2009), synchronization (Lü & Chen, 2005; Wu & Chua, 1995; Yu et al., 2009; Yu, Cao, & Lü, 2008; Yu, Chen, & Lü, 2009; Zhou, Lu, & Lü, 2006, 2008), and swarming and flocking (Olfati-Saber, 2006; Reynolds, 1987; Vicsek, Cziok, Jacob, Cohen, & Shochet, 1995). The consensus problem has attracted much attention partly because tools from algebraic graph theory (Fiedler, 1973) have been introduced successfully to construct conditions under which agents can reach an agreement on certain global criteria of common interest by sharing information locally with their neighbors. It has been shown that consensus in a network with a dynamically changing topology can be reached if and only if the time-varying network topology contains a spanning tree frequently enough as the network evolves with time (Cao, Morse, & Anderson, 2008; Jadbabaie et al., 2003; Olfati-Saber, 2004; Ren & Beard, 2005).

In the literature related to the consensus problem, agents are usually considered to be governed by first-order dynamics (Bliman & Ferrari-Trecate, 2008; Jadbabaie et al., 2003; Lü & Chen, 2005; Olfati-Saber, 2004; Ren & Beard, 2005; Tian & Liu, 2008; Wu & Chua, 1995; Yu et al., 2009, 2008, 2009). In the meanwhile, there is a growing interest in consensus algorithms where all agents are governed by second-order dynamics (Hong et al., 2008, 2006; Olfati-Saber, 2006; Ren, 2007, 2008; Ren & Atkins, 2005; Yu et al., in press). Here, the second-order consensus problem is concerned with how to reach an agreement among a group of autonomous agents governed by second-order dynamics. The insight into the second-order consensus problem may lead to introducing more realistic dynamics into the model of each individual agent based on the general framework of multi-agent systems, which is especially meaningful for the implementation of

cooperative control strategies in engineering networked systems. It has been shown that, in sharp contrast to the first-order consensus problem, consensus may fail to be achieved for agents with second-order dynamics even if the network topology has a directed spanning tree (Ren & Atkins, 2005). Although some sufficient conditions have been derived for reaching second-order consensus (Ren, 2007, 2008; Ren & Atkins, 2005; Yu et al., in press), it is still a challenging problem to identify necessary conditions in a general setting. One contribution of this paper is that a necessary and sufficient condition is obtained for ensuring second-order consensus in a network containing a directed spanning tree. It is found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix of the network play key roles in reaching second-order consensus.

On the other hand, time delay is ubiquitous in biological, physical, chemical, and electrical systems (Bliman & Ferrari-Trecate, 2008; Tian & Liu, 2008). In biological and communication networks, time delays are usually inevitable due to the possible slow process of interactions among agents. It has been observed from numerical experiments that consensus algorithms without considering time delays may lead to unexpected instability. In Bliman and Ferrari-Trecate (2008) and Tian and Liu (2008), some sufficient conditions are derived for the first-order consensus in delayed multi-agent systems. This paper also considers explicitly the effect of delays for second-order consensus. In this regard, another contribution of this paper is to obtain a necessary and sufficient condition that a second-order consensus can be achieved in a delayed multi-agent system with a directed spanning tree if and only if the time delay is less than a certain critical value.

The rest of the paper is organized as follows. In Section 2, some preliminaries on graph theory and model formulation are given. Second-order consensus algorithms for multi-agent dynamical systems in directed networks and delayed directed networks are discussed in Sections 3 and 4, respectively. In Section 5, numerical examples are simulated to verify the theoretical analysis. Conclusions are finally drawn in Section 6.

2. Preliminaries

In this section, some basic concepts and results about algebraic graph theory are introduced. For more details about algebraic graph theory, please refer to Godsil and Royle (2001).

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, G)$ be a weighted directed graph of order N, with the set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $G = (G_{ij})_{N \times N}$. A directed edge $\mathcal{E}_{ij}$ in network $\mathcal{G}$ is denoted by the ordered pair of nodes (v_i, v_j), where v_i and v_j are called the child and parent nodes, respectively, which means that node v_i can receive information from node v_j. In this paper, only positively weighted directed graphs are considered, i.e., $G_{ij} > 0$ if and only if there is a directed edge (v_i, v_j) in $\mathcal{G}$.

A directed path from node v_j to v_i in $\mathcal{G}$ is a sequence of edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_l}, v_j)$ in the directed network with distinct nodes $v_{i_k}, k = 1, 2, \ldots, l$ (Godsil & Royle, 2001; Horn & Johnson, 1985). A root r is a node having the property that for each node v different from r, there is a directed path from r to v. A directed tree is a directed graph, in which there is exactly one root and every node except for this root has exactly one parent. A directed spanning tree is a directed tree, which consists of all the nodes and some edges in $\mathcal{G}$.

The following notations are used throughout the paper for simplicity. Let I_N (O_N) be the N-dimensional identity (zero) matrix, $1_N \in R^N$ ($0_N \in R^N$) be the vector with all entries being 1 (0), and $\mathcal{R}(u)$ and $\mathcal{I}(u)$ be the real and imaginary parts of a complex number u.

The first-order consensus protocol has been widely studied for networks consisting of N nodes with linearly diffusive coupling (Jadbabaie et al., 2003; Lü & Chen, 2005; Olfati-Saber, 2004; Ren & Beard, 2005; Wu & Chua, 1995; Yu et al., 2009, 2008, 2009; Zhou et al., 2006):

$$\dot{x}_i(t) = \widetilde{c} \sum_{j=1, j \neq i}^{N} G_{ij}(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, N, \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ is the state vector of the ith node, $\widetilde{c}$ is the coupling strength, $G = (G_{ij})_{N \times N}$ is the coupling configuration matrix representing the topological structure of the network and thus is the weighted adjacency matrix of the network. The Laplacian matrix $L = (L_{ij})_{N \times N}$ is defined by

$$L_{ii} = -\sum_{j=1, j \neq i}^{N} L_{ij}, \qquad L_{ij} = -G_{ij}, \quad i \neq j, \quad (2)$$

which ensures the diffusion property $\sum_{j=1}^{N} L_{ij} = 0$.

As to the second-order dynamics, the second-order consensus protocol is Hong et al. (2008, 2006), Ren (2007, 2008) and Ren and Atkins (2005):

$$\dot{x}_i(t) = v_i,$$

$$\dot{v}_i(t) = \alpha \sum_{j=1, j \neq i}^{N} G_{ij}(x_j(t) - x_i(t)) + \beta \sum_{j=1, j \neq i}^{N} G_{ij}(v_j(t) - v_i(t)),$$
$$i = 1, 2, \ldots, N, \quad (3)$$

where $x_i \in R^n$ and $v_i \in R^n$ are the position and velocity states of the ith agent, respectively, and $\alpha > 0$ and $\beta > 0$ are the coupling strengths.

Equivalently, system (3) can be rewritten as follows:

$$\dot{x}_i(t) = v_i,$$

$$\dot{v}_i(t) = -\alpha \sum_{j=1}^{N} L_{ij}x_j(t) - \beta \sum_{j=1}^{N} L_{ij}v_j(t), \quad i = 1, 2, \ldots, N. \quad (4)$$

Let $x = (x_1^T, x_2^T, \ldots, x_N^T)^T$, $v = (v_1^T, v_2^T, \ldots, v_N^T)^T$, and $y = (x^T, v^T)^T$. Then, network (4) can be rewritten in a compact matrix form as

$$\dot{y}(t) = (\widetilde{L} \otimes I_n)y, \quad (5)$$

where $\widetilde{L} = \begin{pmatrix} O_N & I_N \\ -\alpha L & -\beta L \end{pmatrix}$ and $\otimes$ is the Kronecker product (Horn & Johnson, 1991).

Definition 1. Second-order consensus in the multi-agent system (5) is said to be achieved if for any initial conditions,

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0,$$
$$\lim_{t \to \infty} \|v_i(t) - v_j(t)\| = 0, \quad \forall i, j = 1, 2, \ldots, N.$$

3. Second-order consensus in directed networks

In this section, some second-order consensus algorithms for the multi-agent system (5) with directed topologies are developed.

Lemma 1 (*Ren and Beard (2005)*). *The Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree.*

For the linear model (5), eigenvalues of the matrix $\widetilde{L}$ are very important in convergence analysis. Suppose that λ_{ij} ($i = 1, 2, \ldots, N$, $j = 1, 2$) and μ_i ($i = 1, 2, \ldots, N$) are eigenvalues of

$\widetilde{L}$ and the Laplacian matrix L, respectively. First, some relationships between the eigenvalues of $\widetilde{L}$ and L are reviewed (Ren, 2008; Ren & Atkins, 2005).

Let λ be an eigenvalue of matrix $\widetilde{L}$. Then, one has $\det(\lambda I_{2N} - \widetilde{L}) = 0$. Note that

$$
\det(\lambda I_{2N} - \widetilde{L}) = \det \begin{pmatrix} \lambda I_N & -I_N \\ \alpha L & \lambda I_N + \beta L \end{pmatrix}
$$
$$
= \det \left(\lambda^2 I_N + (\alpha + \beta\lambda)L \right)
$$
$$
= \prod_{i=1}^{N} \left(\lambda^2 + (\alpha + \beta\lambda)\mu_i \right) = 0.
$$

Hence,

$$
\lambda_{i1} = \frac{-\beta\mu_i + \sqrt{\beta^2\mu_i^2 - 4\alpha\mu_i}}{2},
$$
$$
\lambda_{i2} = \frac{-\beta\mu_i - \sqrt{\beta^2\mu_i^2 - 4\alpha\mu_i}}{2}, \quad i = 1, 2, \ldots, N. \tag{6}
$$

From (6), it is easy to see that L has a zero eigenvalue of algebraic multiplicity m if and only if $\widetilde{L}$ has a zero eigenvalue of algebraic multiplicity $2m$. In the sequel, for the sake of simplicity, we simply write algebraic multiplicity as multiplicity.

Lemma 2. *Second-order consensus in multi-agent system* (5) *can be achieved if and only if matrix $\widetilde{L}$ has exactly a zero eigenvalue of multiplicity two and all the other eigenvalues have negative real parts. In addition, if second-order consensus is reached, $\|v_i(t) - \sum_{j=1}^{N} \xi_j v_j(0)\| \to 0$ and $\|x_i(t) - \sum_{j=1}^{N} \xi_j x_j(0) - \sum_{j=1}^{N} \xi_j v_j(0)t\| \to 0$ as $t \to \infty$, where ξ is the unique nonnegative left eigenvector of L associated with eigenvalue 0 satisfying $\xi^T 1_N = 1$.*

Proof. See Appendix. □

Although a necessary and sufficient condition is given in Lemma 2 to ensure the second-order consensus in multi-agent system (5), it does not show any relationship between the eigenvalues of matrix $\widetilde{L}$ and the Laplacian matrix L. A natural question is: on what kind of networks can second-order consensus be reached? In Ren and Atkins (2005), an example is given where second-order consensus can be achieved in a network whose topology is a directed spanning tree but cannot be achieved after adding only one extra edge into the directed spanning tree. This is a bit surprising as it is inconsistent with the intuition that connections are helpful for reaching consensus. The following result addresses this issue.

Theorem 1. *Second-order consensus in multi-agent system* (5) *can be achieved if and only if the network contains a directed spanning tree and*

$$
\frac{\beta^2}{\alpha} > \max_{2 \le i \le N} \frac{\mathit{I}^2(\mu_i)}{\mathcal{R}(\mu_i)[\mathcal{R}^2(\mu_i) + \mathit{I}^2(\mu_i)]}, \tag{7}
$$

where μ_i are the nonzero eigenvalues of the Laplacian matrix L, $i = 2, 3, \ldots, N$. In addition, if second-order consensus is reached, $\|v_i(t) - \sum_{j=1}^{N} \xi_j v_j(0)\| \to 0$ and $\|x_i(t) - \sum_{j=1}^{N} \xi_j x_j(0) - \sum_{j=1}^{N} \xi_j v_j(0)t\| \to 0$ as $t \to \infty$, where ξ is the unique nonnegative left eigenvector of L associated with eigenvalue 0 satisfying $\xi^T 1_N = 1$.

Proof. From Lemma 1, one knows that the Laplacian matrix L has a simple eigenvalue 0 and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree. By Lemma 2, one only needs to prove that both $\mathcal{R}(\mu_i) > 0$ ($i = 2, 3, \ldots, N$) and (7) hold if and only if $\mathcal{R}(\lambda_{ij}) < 0$ ($i = 2, 3, \ldots, N; j = 1, 2$).

Let $\sqrt{\beta^2\mu_i^2 - 4\alpha\mu_i} = c + \mathbf{i}d$, where c and d are real, and $\mathbf{i} = \sqrt{-1}$. From (6), $\mathcal{R}(\lambda_{ij}) < 0$ ($i = 2, 3, \ldots, N; j = 1, 2$) if and only if $-\beta\mathcal{R}(\mu_i) < c < \beta\mathcal{R}(\mu_i)$, which is equivalent to $\mathcal{R}(\mu_i) > 0$ and $c^2 < \beta^2\mathcal{R}^2(\mu_i)$ ($i = 2, 3, \ldots, N$). Then, it suffices to prove that (7) holds if and only if $c^2 < \beta^2\mathcal{R}^2(\mu_i)$ ($i = 2, 3, \ldots, N$). It is easy to see that

$$
\beta^2\mu_i^2 - 4\alpha\mu_i = (c + \mathbf{i}d)^2.
$$

Separating the real and imaginary parts, one has

$$
c^2 - d^2 = \beta^2[\mathcal{R}^2(\mu_i) - \mathit{I}^2(\mu_i)] - 4\alpha\mathcal{R}(\mu_i),
$$
$$
cd = \beta^2\mathcal{R}(\mu_i)\mathit{I}(\mu_i) - 2\alpha\mathit{I}(\mu_i).
$$

By simple calculations, one obtains

$$
c^4 - \{\beta^2[\mathcal{R}^2(\mu_i) - \mathit{I}^2(\mu_i)] - 4\alpha\mathcal{R}(\mu_i)\}c^2
$$
$$
- \mathit{I}^2(\mu_i)[\beta^2\mathcal{R}(\mu_i) - 2\alpha]^2 = 0. \tag{8}
$$

It is easy to check that $c^2 < \beta^2\mathcal{R}^2(\mu_i)$ if and only if (7) holds. □

Remark 1. In Theorem 1, in addition to the condition that the network has a directed spanning tree, (7) should also be satisfied. It is easy to verify that if all the other eigenvalues of the Laplacian matrix L are real, then (7) holds. From (7), it is found that both real and imaginary parts of the eigenvalues of the Laplacian matrix play important roles in reaching second-order consensus. Let $\frac{\mathit{I}^2(\mu_k)}{\mathcal{R}(\mu_k)[\mathcal{R}^2(\mu_k) + \mathit{I}^2(\mu_k)]} = \max_{2 \le i \le N} \frac{\mathit{I}^2(\mu_i)}{\mathcal{R}(\mu_i)[\mathcal{R}^2(\mu_i) + \mathit{I}^2(\mu_i)]}$, where $2 \le k \le N$. Then, one can see that in order to reach consensus, the critical value β^2/α increases as $|\mathit{I}(\mu_k)|$ increases and decreases as $\mathcal{R}(\mu_k)$ increases.

Remark 2. If $\frac{\beta^2}{\alpha} > \max_{2 \le i \le N} \frac{1}{\mathcal{R}(\mu_i)}$ holds, (7) is satisfied for sure. So, the sufficient condition for reaching consensus, $\frac{\beta^2}{\alpha} > \max_{2 \le i \le N} \frac{2}{\|\mu_i\| \cos\left(\frac{\pi}{2} - \tan^{-1}\frac{\mathcal{R}(\mu_i)}{\mathit{I}(\mu_i)}\right)} = \max_{2 \le i \le N} \frac{2}{\|\mu_i\| \sin\left(\tan^{-1}\frac{\mathcal{R}(\mu_i)}{\mathit{I}(\mu_i)}\right)} = \max_{2 \le i \le N} \frac{2}{\mathcal{R}(\mu_i)}$, given in Ren and Atkins (2005), is more conservative. Here, the sufficient condition depends only on the real parts of the eigenvalues of the Laplacian matrix L, but are independent of their imaginary parts. Moreover, when β^2/α is very small, consensus may still be achieved even if $\frac{\beta^2}{\alpha} > \max_{2 \le i \le N} \frac{1}{\mathcal{R}(\mu_i)}$ is not satisfied.

4. Second-order consensus in delayed directed networks

In this section, the following second-order consensus protocol with time delays is considered:

$$
\dot{x}_i(t) = v_i,
$$
$$
\dot{v}_i(t) = -\alpha \sum_{j=1}^{N} L_{ij}x_j(t - \tau) - \beta \sum_{j=1}^{N} L_{ij}v_j(t - \tau), \tag{9}
$$
$$
i = 1, 2, \ldots, N,
$$

where $\tau > 0$ is the time-delay constant.

Let $x = (x_1^T, x_2^T, \ldots, x_N^T)^T$, $v = (v_1^T, v_2^T, \ldots, v_N^T)^T$, and $y = (x^T, v^T)^T$. Then, network (9) can be rewritten in a compact matrix form, as follows:

$$
\dot{y}(t) = (\widetilde{L}_1 \otimes I_n)y + (\widetilde{L}_2 \otimes I_n)y(t - \tau), \tag{10}
$$

where $\widetilde{L}_1 = \begin{pmatrix} 0_N & I_N \\ 0_N & 0_N \end{pmatrix}$ and $\widetilde{L}_2 = \begin{pmatrix} 0_N & 0_N \\ -\alpha L & -\beta L \end{pmatrix}$.

In Yu and Cao (2006, 2007) and Yu, Cao, and Chen (2008), stability and Hopf bifurcation of delayed networks were studied, where the time delays are regarded as bifurcation parameters. It was found that Hopf bifurcation occurs when time delays

pass through some critical values where the conditions for local asymptotical stability of the equilibrium are not satisfied. Similarly, this section aims to find the maximum time delay with which the consensus can be achieved in the multi-agent system (10).

The characteristic equation of system (10) is $\det(\lambda I_{2N} - \widetilde{L}_1 - e^{-\lambda\tau}\widetilde{L}_2) = 0$, i.e.,

$$
\begin{aligned}
\det(\lambda I_{2N} - \widetilde{L}_1 - e^{-\lambda\tau}\widetilde{L}_2) &= \det\begin{pmatrix} \lambda I_N & -I_N \\ \alpha e^{-\lambda\tau}L & \lambda I_N + \beta e^{-\lambda\tau}L \end{pmatrix} \\
&= \det\left(\lambda^2 I_N + (\alpha + \beta\lambda)e^{-\lambda\tau}L\right) \\
&= \prod_{i=1}^{N}\left(\lambda^2 + (\alpha + \beta\lambda)e^{-\lambda\tau}\mu_i\right) = 0. \quad (11)
\end{aligned}
$$

Let $g_i(\lambda) = \lambda^2 + (\alpha + \beta\lambda)e^{-\lambda\tau}\mu_i$ and $g(\lambda) = \prod_{i=1}^{N} g_i(\lambda)$. From (11), it is easy to see that L has a zero eigenvalue of multiplicity m if and only if $g(\lambda) = 0$ has a zero root of multiplicity $2m$.

Lemma 3. *Suppose that the network contains a directed spanning tree. Then, $g(\lambda) = 0$ has a purely imaginary root if and only if*

$$
\tau \in \Psi = \left\{\frac{1}{\omega_{i1}}(2k\pi + \theta_{i1}) \mid i = 2, \ldots, N; k = 0, 1, \ldots\right\}, \quad (12)
$$

where $0 \leq \theta_{i1} < 2\pi$, which satisfies $\cos\theta_{i1} = [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_{i1}\beta]/\omega_{i1}^2$ and $\sin\theta_{i1} = [\mathcal{R}(\mu_i)\omega_{i1}\beta + \mathcal{I}(\mu_i)\alpha]/\omega_{i1}^2$, and $\omega_{i1} = \sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}}, i = 2, \ldots, N$.

Proof. See Appendix. □

Lemma 4 (*Yu and Cao (2006)*)**.** *Consider the exponential polynomial*

$$
\begin{aligned}
P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m}) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\
&+ [p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} \\
&+ \cdots + [p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m},
\end{aligned}
$$

where $\tau_i \geq 0 (i = 1, 2, \ldots, m)$ and $p_j^{(i)} (i = 0, 1, \ldots, m; j = 1, 2, \ldots, n)$ are constants. As $(\tau_1, \tau_2, \ldots, \tau_m)$ vary, the sum of the orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \ldots, e^{-\lambda\tau_m})$ on the open right-half plane can change only if a zero appears on or crosses the imaginary axis.

Lemma 5. *Suppose that the network contains a directed spanning tree. Let λ be the solution of $g_i(\lambda) = 0, 2 \leq i \leq N$. Then, $d\lambda/d\tau$ exists at the point $\tau \in \Psi$ and*

$$
\mathcal{R}\left(\frac{d\lambda}{d\tau}\right)\bigg|_{\tau\in\Psi} > 0. \quad (13)
$$

Proof. See Appendix. □

Theorem 2. *Suppose that the network contains a directed spanning tree and (7) is satisfied. Then, second-order consensus in system (10) is achieved if and only if*

$$
\tau < \tau_0 = \min_{2\leq i\leq N}\left\{\frac{\theta_{i1}}{\omega_{i1}}\right\}, \quad (14)
$$

where $0 \leq \theta_{i1} < 2\pi$, which satisfies $\cos\theta_{i1} = [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_{i1}\beta]/\omega_{i1}^2$ and $\sin\theta_{i1} = [\mathcal{R}(\mu_i)\omega_{i1}\beta + \mathcal{I}(\mu_i)\alpha]/\omega_{i1}^2$, $\omega_{i1} = \sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}}$, and μ_i are the nonzero eigenvalues of the Laplacian matrix $L, i = 2, 3, \ldots, N$.
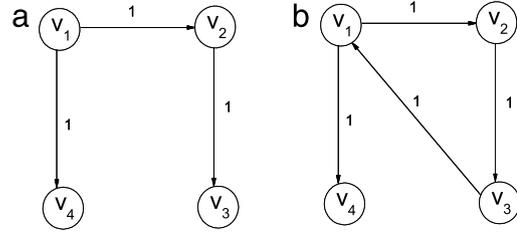

Fig. 1. Network structure of a network with a directed spanning tree.

Proof. Since the network contains a directed spanning tree and (7) is satisfied, from Theorem 1 it follows that the second-order consensus can be achieved in system (10) when $\tau = 0$, where $g(\lambda) = 0$ has exactly a zero root of multiplicity two and all the other roots have negative real parts. When τ varies from 0 to τ_0, by Lemma 3, a purely imaginary root emerges. From Lemmas 4 and 5, one knows that $g(\lambda) = 0$ has exactly a zero root of multiplicity two and all the other roots have negative real parts when $0 \leq \tau < \tau_0$, and there is at least one root with positive real part $\tau > \tau_0$. Therefore, second-order consensus cannot be achieved when $\tau \geq \tau_0$. The proof is completed. □

Remark 3. In Yu and Cao (2006, 2007) and Yu et al. (2008), stability and Hopf bifurcation were studied for delayed networks, where the time delays are regarded as bifurcation parameters. Similar ideas are used here. The result in Theorem 2 is important in that a necessary and sufficient condition is established by computing the critical value τ_0 for the maximum allowable time delay.

5. Simulation examples

In this section, two simulation examples are given to verify the theoretical analysis.

5.1. Second-order consensus in directed networks

Consider the network (4) with the topology shown in Fig. 1. In Fig. 1(a), the Laplacian matrix L has a simple zero eigenvalue and all the other eigenvalues are real. Consensus in (4) can be reached for any $\alpha > 0$ and $\beta > 0$. In Fig. 1(b), the Laplacian matrix L is $\begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ and its four eigenvalues are $0, 1, 1.5 + 0.866i, 1.5 - 0.866i$. Let $\alpha = 1$ and apply Theorem 1. Then, second-order consensus in the multi-agent system (4) can be achieved if and only if $\beta > 0.4082$. The position and velocity states of all the agents are shown in Fig. 2(a) and (b), where consensus cannot be achieved when $\beta = 0.4$ but it can be reached if $\beta = 0.415$. It is easy to see that by appropriately choosing some $\alpha > 0$ and $\beta > 0$, consensus can be achieved but then may fail if a connection between two agents is added.

5.2. Second-order consensus in delayed directed networks

Consider the network (9) with a structure shown in Fig. 1(b) where $\alpha = \beta = 1$. When $\tau = 0$, from Theorem 1, one knows that second-order consensus can be achieved in the network. By simple calculations using Theorem 2, the second-order consensus can be reached if and only if $\tau < 0.29415$. The position and velocity states of all the agents are shown in Fig. 3(a) and (b), where consensus is achieved when $\tau = 0.29$ but it cannot be reached if $\tau = 0.30$.
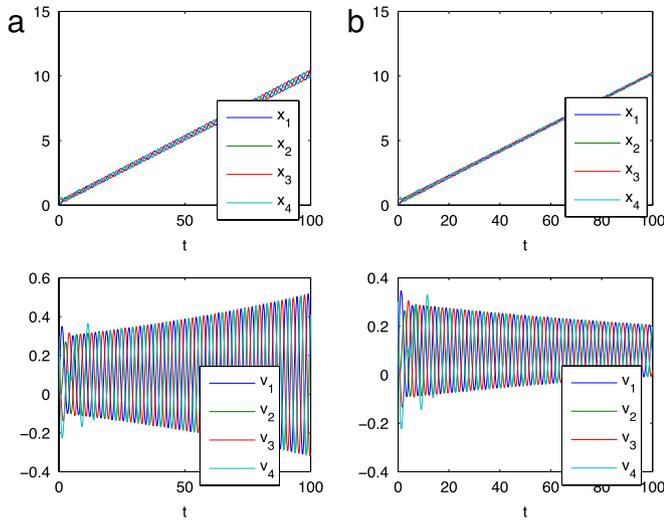
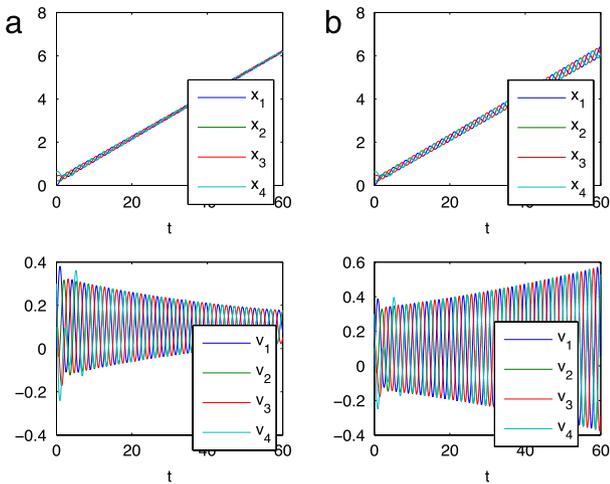**Fig. 2.** Position and velocity states of agents in a network, where $\beta = 0.4$ (a) and $\beta = 0.415$ (b).

Fig. 3. Position and velocity states of agents in a delayed network, where $\tau = 0.29$ (a) and $\tau = 0.30$ (b).

6. Conclusions

In this paper, some second-order consensus algorithms for multi-agent dynamical systems with directed topologies have been studied. Detailed analysis has been performed for the case where the second-order dynamics of each agent are determined by both position and velocity terms. A necessary and sufficient condition has been derived to ensure second-order consensus in multi-agent systems where the network has a directed spanning tree. It was found that both the real and imaginary parts of the eigenvalues of the Laplacian matrix play key roles in reaching consensus. Moreover, the scenario when communication delays are presented in the network has been investigated. A necessary and sufficient condition has also been established, and it was shown that, in this case, the second-order consensus can be achieved in the multi-agent systems with a directed spanning tree if and only if the time delay is less than a critical value.

We are now working on introducing more complicated and realistic agent dynamics to groups of mobile agents. Moreover, the effects of more complicated inter-agent couplings on group behaviors are being investigated. For example, it is of great interest to generalize the results of this paper to the case when the network topology evolves with time, or has certain hierarchical features.

Appendix

A.1. Proof of Lemma 2

Proof. For $n = 1$ and $\alpha = 1$, a proof of this lemma was given in Ren and Atkins (2005). Now, this lemma is proved for any integer $n \geq 1$ and $\alpha > 0$.

(Sufficiency.) Note that 0 is an eigenvalue of matrix $\widetilde{L}$ with multiplicity 2. From calculation of $\widetilde{L}\varphi = 0$, where φ is a unit right eigenvector of matrix $\widetilde{L}$ associated with eigenvalue 0, one can easily obtain that $\varphi = (1_N^T, 0_N^T)^T / \sqrt{N}$, which is unique. So, matrix $\widetilde{L}$ cannot be diagonal since there is only one unit eigenvector of matrix $\widetilde{L}$ associated with eigenvalue 0. Therefore, a Jordan form is used here. If $\widetilde{L}$ has exactly a zero eigenvalue of multiplicity two and all the other eigenvalues have negative real parts, then there exists a nonsingular matrix $P \in R^{2N \times 2N}$, such that $P^{-1}\widetilde{L}P = J$, where J is the Jordan canonical form associated with $\widetilde{L}$. Thus, one has

$$\widetilde{L} = PJP^{-1} = (\zeta_1, \ldots, \zeta_{2N})$$

$$\times \begin{pmatrix} 0 & 1 & 0_{1 \times (2N-2)} \\ 0 & 0 & 0_{1 \times (2N-2)} \\ 0_{(2N-2) \times 1} & 0_{(2N-2) \times 1} & \widetilde{J} \end{pmatrix} \begin{pmatrix} \eta_1^T \\ \vdots \\ \eta_{2N}^T \end{pmatrix}, \qquad (15)$$

where ζ_j and η_j $(j = 1, 2, \ldots, 2N)$ are the right and left eigenvectors or generalized eigenvectors of $\widetilde{L}$, respectively, and $\widetilde{J}$ is the upper diagonal Jordan block matrix associated with the nonzero eigenvalues λ_{ij}, $i = 2, \ldots, N$; $j = 1, 2$. It follows that $e^{\widetilde{J}t} \rightarrow 0_{(2N-2) \times (2N-2)}$ as $t \rightarrow \infty$.

From $\widetilde{L}P = PJ$, one obtains

$$\widetilde{L}(\zeta_1, \zeta_2) = (\zeta_1, \zeta_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0_{2N}, \zeta_1),$$

giving that

$$\widetilde{L}\zeta_1 = 0_{2N}, \qquad \widetilde{L}\zeta_2 = \zeta_1.$$

Therefore, ζ_2 is the generalized right eigenvector of matrix $\widetilde{L}$ associated with eigenvalue 0 and can be computed by

$$\widetilde{L}^2 \zeta_2 = \widetilde{L}\zeta_1 = 0\zeta_2 = 0_{2N}.$$

One can easily obtain a right eigenvector $\zeta_1 = (1_N^T, 0_N^T)^T$ and a generalized right eigenvector $\zeta_2 = (0_N^T, 1_N^T)^T$ of matrix $\widetilde{L}$ associated with eigenvalue 0. Accordingly, $\eta_1 = (\xi^T, 0_N^T)^T$ and $\eta_2 = (0_N^T, \xi^T)^T$ are the generalized left eigenvector and the left eigenvector of matrix $\widetilde{L}$ associated with eigenvalue 0, respectively (Yu et al., in press). Also, in view of the properties of the Kronecker product (Horn & Johnson, 1991), one has

$$\begin{aligned} e^{(\widetilde{L} \otimes I_n)t} &= e^{(P \otimes I_n)(Jt \otimes I_n)(P^{-1} \otimes I_n)} \\ &= (P \otimes I_n)e^{(Jt \otimes I_n)}(P^{-1} \otimes I_n) \\ &= (P \otimes I_n)(e^{Jt} \otimes I_n)(P^{-1} \otimes I_n) \\ &= (Pe^{Jt}P^{-1}) \otimes I_n \\ &= \left\{ P \begin{pmatrix} 1 & t & 0_{1 \times (2N-2)} \\ 0 & 1 & 0_{1 \times (2N-2)} \\ 0_{(2N-2) \times 1} & 0_{(2N-2) \times 1} & e^{\widetilde{J}t} \end{pmatrix} P^{-1} \right\} \otimes I_n. \end{aligned}$$

Therefore, one obtains

$$\lim_{t \to \infty} \left\| \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} - 1_N \otimes \begin{pmatrix} \sum_{j=1}^N \xi_j(x_j(0) + v_j(0)t) \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \right\|$$

$$= \lim_{t \to \infty} \left\| e^{(\widetilde{L} \otimes I_n)t} \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} - 1_N \otimes \begin{pmatrix} \sum_{j=1}^N \xi_j(x_j(0) + v_j(0)t) \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \right\|$$

$$= \lim_{t \to \infty} \left\| \left[\begin{pmatrix} 1_N \xi^T & t 1_N \xi^T \\ O_N & 1_N \xi^T \end{pmatrix} \otimes I_n \right] \begin{pmatrix} x(0) \\ v(0) \end{pmatrix} \right.$$
$$\left. - 1_N \otimes \begin{pmatrix} \sum_{j=1}^N \xi_j(x_j(0) + v_j(0)t) \\ \sum_{j=1}^N \xi_j v_j(0) \end{pmatrix} \right\| = 0, \qquad (16)$$

which indicates that second-order consensus is achieved in system (5).

(Necessity.) If the condition that matrix $\widetilde{L}$ has exactly one zero eigenvalue of multiplicity two and all the other eigenvalues have negative real parts is not satisfied, then $\lim_{t \to \infty} e^{\widetilde{L}t}$ has a rank greater than 2, which contradicts the assumption that second-order consensus is reached. (See Ren and Atkins (2005) for a similar argument.) □

A.2. Proof of Lemma 3

Proof (*Necessity*). Let $\lambda = \mathbf{i}\omega_i$ ($\omega_i \neq 0$). From $g_i(\lambda) = 0$, one has

$$\omega_i^2 = (\alpha + \mathbf{i}\beta\omega_i)e^{-\mathbf{i}\omega_i\tau}\mu_i. \qquad (17)$$

Taking modulus on both sides of (17), one obtains

$$\omega_i^4 - [\mathcal{R}(\mu_i)^2 + \mathcal{I}(\mu_i)^2]\beta^2\omega_i^2 - [\mathcal{R}(\mu_i)^2 + \mathcal{I}(\mu_i)^2]\alpha^2 = 0. \qquad (18)$$

Then,

$$\omega_i^2 = \frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}. \qquad (19)$$

Separating the real and imaginary parts of (17) yields

$$\begin{aligned} \omega_i^2 &= [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]\cos(\omega_i\tau) \\ &\quad + [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]\sin(\omega_i\tau), \\ 0 &= [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]\cos(\omega_i\tau) \\ &\quad - [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]\sin(\omega_i\tau). \end{aligned} \qquad (20)$$

By simple calculations, one obtains

$$\cos(\omega_i\tau) = \frac{\omega_i^2[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]}{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]^2 + [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]^2},$$
$$\sin(\omega_i\tau)$$
$$= \frac{\omega_i^2[\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]}{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]^2 + [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]^2}. \qquad (21)$$

From (18), it follows that $\omega_i^4 = [\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]^2 + [\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]^2$. Thus, (21) can be written as

$$\cos(\omega_i\tau) = \frac{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_i\beta]}{\omega_i^2},$$
$$\sin(\omega_i\tau) = \frac{[\mathcal{R}(\mu_i)\omega_i\beta + \mathcal{I}(\mu_i)\alpha]}{\omega_i^2}. \qquad (22)$$

Let

$$\omega_{i1} = \sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}},$$

$$\omega_{i2} = -\sqrt{\frac{\|\mu_i\|^2\beta^2 + \sqrt{\|\mu_i\|^4\beta^4 + 4\|\mu_i\|^2\alpha^2}}{2}},$$

and $0 \leq \theta_{ij} < 2\pi$, which satisfies $\cos\theta_{ij} = \frac{[\mathcal{R}(\mu_i)\alpha - \mathcal{I}(\mu_i)\omega_{ij}\beta]}{\omega_{ij}^2}$ and $\sin\theta_{ij} = \frac{[\mathcal{R}(\mu_i)\omega_{ij}\beta + \mathcal{I}(\mu_i)\alpha]}{\omega_{ij}^2}, j = 1, 2$. Since $\mathcal{R}(\mu_i) \pm \mathbf{i}\mathcal{I}(\mu_i)$ are two eigenvalues of L, if $\mathcal{I}(\mu_i) \neq 0$, then there exists an integer s, $2 \leq s \leq N$, such that

$$\cos(\omega_{i1}\tau) = \cos(\omega_{s2}\tau), \qquad \sin(\omega_{i1}\tau) = -\sin(\omega_{s2}\tau),$$
$$\theta_{i1} = 2\pi - \theta_{s2}.$$

If $\mathcal{I}(\mu_i) = 0$, then

$$\cos(\omega_{i1}\tau) = \cos(\omega_{i2}\tau), \qquad \sin(\omega_{i1}\tau) = -\sin(\omega_{i2}\tau),$$
$$\theta_{i1} = 2\pi - \theta_{i2}.$$

It follows that

$$\begin{aligned} \tau \in \Psi &= \left\{ \frac{1}{\omega_{i1}}(2k\pi + \theta_{i1}) \mid i = 2, \ldots, N; k = 0, 1, \ldots \right\} \\ &\quad \bigcup \left\{ \frac{1}{\omega_{i2}}(2k\pi + \theta_{i2}) \mid i = 2, \ldots, N; k = -1, -2, \ldots \right\} \\ &= \left\{ \frac{1}{\omega_{i1}}(2k\pi + \theta_{i1}) \mid i = 2, \ldots, N; k = 0, 1, \ldots \right\} \\ &\quad \bigcup \left\{ \frac{1}{\omega_{i1}}(2k\pi - \theta_{i2}) \mid i = 2, \ldots, N; k = 1, 2, \ldots \right\} \\ &= \left\{ \frac{1}{\omega_{i1}}(2k\pi + \theta_{i1}) \mid i = 2, \ldots, N; k = 0, 1, \ldots \right\}. \end{aligned}$$

(Sufficiency.) If $\tau = \frac{1}{\omega_{i1}}(2k\pi + \theta_{i1}) \in \Phi$, by the same process, one obtains that $g_i(\lambda) = 0$ when $\lambda = \mathbf{i}\omega_{i1}$ since (17) is satisfied. The proof is completed. □

A.3. Proof of Lemma 5

Proof. Let $\widetilde{g}_i(\lambda, \tau) = \lambda^2 + (\alpha + \beta\lambda)e^{-\lambda\tau}\mu_i$. Since $\widetilde{g}_i(\mathbf{i}\omega_0, \tau_0) = 0$ if $\tau_0 \in \Phi$ and $\mathbf{i}\omega_0$ is the corresponding purely imaginary root, $\widetilde{g}_i(\lambda, \tau)$ is continuous around the point $(\mathbf{i}\omega_0, \tau_0)$, $\frac{\partial\widetilde{g}_i}{\partial\lambda}$ and $\frac{\partial\widetilde{g}_i}{\partial\tau}$ are continuous, and $\frac{\partial\widetilde{g}_i}{\partial\lambda}|_{(\mathbf{i}\omega_0, \tau_0)} \neq 0$, λ is differentiable with respect to τ around the point $(\mathbf{i}\omega_0, \tau_0)$ according to the implicit function theorem (Rudin, 1976).

Taking the derivative of λ with respect to τ in $g_i(\lambda) = 0$, one obtains

$$2\lambda\frac{d\lambda}{d\tau} + e^{-\lambda\tau}\mu_i\left[\beta\frac{d\lambda}{d\tau} - (\alpha + \beta\lambda)\left(\lambda + \tau\frac{d\lambda}{d\tau}\right)\right] = 0. \qquad (23)$$

If $\tau \in \Psi$, then $\lambda = \mathbf{i}\omega_{ij}$ for some i and j, $2 \leq i \leq N$, $1 \leq j \leq 2$. It follows that (24) given in Box I can be obtained. Let

$$\begin{aligned} q &= [-2\omega_{ij}\sin(\omega_{ij}\tau) + \mathcal{R}(\mu_i)(\beta - \alpha\tau) + \mathcal{I}(\mu_i)\beta\omega_{ij}\tau]^2 \\ &\quad + [2\omega_{ij}\cos(\omega_{ij}\tau) - \mathcal{R}(\mu_i)\beta\omega_{ij}\tau + \mathcal{I}(\mu_i)(\beta - \alpha\tau)]^2. \end{aligned}$$

By simple calculations, one obtains

$$\begin{aligned} q\mathcal{R}\left(\frac{d\lambda}{d\tau}\right)\bigg|_{\tau\in\Psi} &= -[\beta\omega_{ij}\mathcal{R}(\mu_i) + \alpha\mathcal{I}(\mu_i)] \\ &\quad \times [-2\omega_{ij}\sin(\omega_{ij}\tau) + \mathcal{R}(\mu_i)(\beta - \alpha\tau) + \mathcal{I}(\mu_i)\beta\omega_{ij}\tau]\omega_{ij} \\ &\quad + [\alpha\mathcal{R}(\mu_i) - \beta\omega_{ij}\mathcal{I}(\mu_i)][2\omega_{ij}\cos(\omega_{ij}\tau) \\ &\quad - \mathcal{R}(\mu_i)\beta\omega_{ij}\tau + \mathcal{I}(\mu_i)(\beta - \alpha\tau)]\omega_{ij} \\ &= 2\omega_{ij}^2\{\sin(\omega_{ij}\tau)[\beta\omega_{ij}\mathcal{R}(\mu_i) + \alpha\mathcal{I}(\mu_i)] \\ &\quad + \cos(\omega_{ij}\tau)[\alpha\mathcal{R}(\mu_i) - \beta\omega_{ij}\mathcal{I}(\mu_i)]\} \\ &\quad - \beta^2\omega_{ij}^2[\mathcal{R}^2(\mu_i) + \mathcal{I}^2(\mu_i)]. \end{aligned} \qquad (25)$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\bigg|_{\lambda=\mathbf{i}\omega_{ij}} = \frac{\mu_i\lambda(\alpha+\beta\lambda)}{2\lambda e^{\lambda\tau}+\mu_i[\beta-(\alpha+\beta\lambda)\tau]}\bigg|_{\lambda=\mathbf{i}\omega_{ij}}$$

$$= \frac{[\mathcal{R}(\mu_i)+\mathbf{i}\mathcal{I}(\mu_i)]\mathbf{i}\omega_{ij}[\alpha+\mathbf{i}\beta\omega_{ij}]}{2[\cos(\omega_{ij}\tau)+\mathbf{i}\sin(\omega_{ij}\tau)]\mathbf{i}\omega_{ij}+[\mathcal{R}(\mu_i)+\mathbf{i}\mathcal{I}(\mu_i)][(\beta-\alpha\tau)-\mathbf{i}\beta\omega_{ij}\tau]}$$

$$= \frac{-[\beta\omega_{ij}\mathcal{R}(\mu_i)+\alpha\mathcal{I}(\mu_i)]\omega_{ij}+\mathbf{i}[\alpha\mathcal{R}(\mu_i)-\beta\omega_{ij}\mathcal{I}(\mu_i)]\omega_{ij}}{[-2\omega_{ij}\sin(\omega_{ij}\tau)+\mathcal{R}(\mu_i)(\beta-\alpha\tau)+\mathcal{I}(\mu_i)\beta\omega_{ij}\tau]+\mathbf{i}[2\omega_{ij}\cos(\omega_{ij}\tau)-\mathcal{R}(\mu_i)\beta\omega_{ij}\tau+\mathcal{I}(\mu_i)(\beta-\alpha\tau)]}$$

$$(24)$$

Box I.

Substituting (18) and (22) into (25), one has

$$q\mathcal{R}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\bigg|_{\tau\in\Psi} = 2\omega_{ij}^4-\beta^2\omega_{ij}^2[\mathcal{R}^2(\mu_i)+\mathcal{I}^2(\mu_i)]$$

$$= 2[\mathcal{R}(\mu_i)^2+\mathcal{I}(\mu_i)^2](\beta^2\omega_i^2+\alpha^2)$$

$$-\beta^2\omega_{ij}^2[\mathcal{R}^2(\mu_i)+\mathcal{I}^2(\mu_i)]$$

$$= [\mathcal{R}(\mu_i)^2+\mathcal{I}(\mu_i)^2](\beta^2\omega_i^2+2\alpha^2) > 0. \quad (26)$$

This completes the proof. □

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