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More on massive 3D gravity

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We explore the space of static solutions of the recently discovered three-dimensional "new massive gravity" (NMG), allowing for either sign of the Einstein-Hilbert term and a cosmological term parametrized by a dimensionless constant λ. For $\lambda = -1$ we find black hole solutions asymptotic (but not isometric) to the unique (anti) de Sitter [(A)dS] vacuum, including extremal black holes that interpolate between this vacuum and (A)dS$_2 \times S^1$. We also investigate unitarity of linearized NMG in (A)dS vacua. We find unitary theories for some dS vacua, but (bulk) unitarity in AdS implies negative central charge of the dual conformal field theories (CFT), except for $\lambda = 3$ where the central charge vanishes and the bulk gravitons are replaced by "massive photons." A similar phenomenon is found in the massless limit of NMG, for which the linearized equations become equivalent to Maxwell's equations.

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I. INTRODUCTION

We recently found a novel three-dimensional (3D) gravity model that propagates massive positive-energy spin 2 modes, of both helicities ± 2, in a Minkowski vacuum [1]; in other words, an interacting, and generally covariant, extension of the Pauli-Fierz (PF) theory for massive spin 2 in 3D. The action for this "new massive gravity" (NMG) is the sum of a "wrong-sign" Einstein-Hilbert (EH) term and a particular higher-derivative term, which introduces a mass parameter m. Models of this type are known to be renormalizable in four dimensions [2], and this implies power-counting super-renormalizability in three dimensions. Unitarity has since been confirmed [3,4], as has super-renormalizability [5], so that NMG must now be considered a promising candidate for a fully consistent theory of quantum gravity, albeit in three dimensions and with massive gravitons.

We also found a parity-violating extension of NMG to a "general massive gravity" (GMG) in which the ± 2 helicity modes propagate with different masses $m_\pm$; NMG is recovered by setting $m_+ = m_- = m$ while the limit $m_- \to \infty$ yields the well-known "topologically massive gravity" (TMG) [6], which propagates only a single mode of helicity 2. Also considered briefly in [1] was the extension of GMG to allow for a cosmological constant; this can be chosen to be proportional to λm^2, where λ is a new *dimensionless* parameter. Studies of "cosmological TMG" (CTMG) [7–9] suggest that one should also allow for either sign of the EH term in "cosmological" GMG (CGMG), so it is convenient to introduce a sign σ such that $\sigma = 1$ for the "right-sign" EH term and $\sigma = -1$ for the "wrong-sign" EH term.[1] For the "mostly plus" signature convention that we will use here, the CGMG action is

$$S_{\rm CGMG}[g] = \frac{1}{\kappa^2}\int d^3x\left\{\sqrt{|g|}\left[\sigma R + \frac{1}{m^2}K - 2\lambda m^2\right] + \frac{1}{\mu}\mathcal{L}_{\rm LCS}\right\}, \tag{1.1}$$

where

$$|g| = -\det g, \qquad K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2, \tag{1.2}$$

and the Lorentz-Chern-Simons (LCS) term is

$$\mathcal{L}_{\rm LCS} = \frac{1}{2}\varepsilon^{\mu\nu\rho}\left[\Gamma^\alpha_{\mu\beta}\partial_\nu\Gamma^\beta_{\rho\alpha} + \frac{2}{3}\Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta}\Gamma^\beta_{\rho\alpha}\right], \tag{1.3}$$

where Γ is the usual Levi-Civita connection for the spacetime metric g. The Riemann curvature tensor is determined, in 3D, by the Ricci tensor, which is

$$R_{\mu\nu} \equiv R_{\rho\mu}{}^{\rho}{}_{\nu} = \partial_\rho\Gamma^\rho_{\mu\nu} - \partial_\mu\Gamma^\rho_{\rho\nu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu}. \tag{1.4}$$

It would be possible to remove the higher-derivative term K by a local redefinition of the metric, but this would just introduce an infinite series of yet higher-derivative terms.[2] To remove *all* higher-derivative terms (other than the LCS term) would require an inadmissible nonlocal redefinition.

In this paper we focus on the cosmological NMG (CNMG) model obtained as the $\mu \to \infty$ limit of the CGMG model, although all static solutions of CNMG are

[1]We use the terms "right-sign" and "wrong-sign" because the actual signs are convention dependent.

[2]Linearization would then yield, for $\sigma = -1$ and $\lambda = 0$, another higher-derivative model equivalent to the PF theory, so this redefinition is perhaps not without interest.

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also solutions of the CGMG for any value of μ because the LCS term plays no role for static solutions. We shall make use of the following variant of the CNMG action, involving an auxiliary symmetric tensor field f [1]:

$$S_{\mathrm{CNMG}}[g,f] = \frac{1}{\kappa^2}\int d^3x\sqrt{|g|}\Big[-2\lambda m^2 + \sigma R + f^{\mu\nu}G_{\mu\nu} - \frac{1}{4}m^2(f^{\mu\nu}f_{\mu\nu} - f^2)\Big]. \tag{1.5}$$

The $\mu \to \infty$ limit of (1.1) is recovered after elimination of f by its field equation

$$f_{\mu\nu} = \frac{2}{m^2}S_{\mu\nu}, \qquad S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}. \tag{1.6}$$

The tensor S appears in the conformal approach to supergravity as the gauge field corresponding to conformal boosts [10,11], and has been called the "Schouten" tensor [12].

As an aside, we remark here that the action (1.5) has the feature that it allows one to take the $m \to 0$ limit by defining $\tilde{f}_{\mu\nu} = m^2 f_{\mu\nu}$ and keeping $m\kappa \equiv \beta$ fixed. This yields the action

$$S_{\mathrm{massless}}[g,\tilde{f}] = \frac{1}{\beta^2}\int d^3x\sqrt{|g|} \times \Big[\tilde{f}^{\mu\nu}G_{\mu\nu} - \frac{1}{4}(\tilde{f}^{\mu\nu}\tilde{f}_{\mu\nu} - \tilde{f}^2)\Big], \tag{1.7}$$

and elimination of $\tilde{f}$ now yields

$$S[g] = \frac{1}{\beta^2}\int d^3x\sqrt{|g|}K. \tag{1.8}$$

The quadratic approximation to this "pure" higher-derivative model was recently studied by Deser [4], who shows that there is a single, physical, massless propagating mode. We confirm this result here by showing that the linearized action in this case is equivalent to a Maxwell action for a vector field, which is itself on-shell equivalent to the action for a massless scalar. The "Weyl invariance" discussed in [4] is realized as a Maxwell gauge invariance in this context. However, this gauge invariance is broken by the interactions, so the consistency of the nonlinear "pure-K" model is problematic. We investigate this issue here (in an appendix) because it turns out to be relevant to the $\lambda = 3$ case of the CNMG model. We also investigate the "cosmological" extension of the pure-K model obtained by the addition of a cosmological term.

By definition, any maximally symmetric vacuum of CGMG is such that

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{1.9}$$

where G is the Einstein tensor, and Λ is a constant (the sign is chosen such that $\Lambda > 0$ for de Sitter (dS) spacetime and $\Lambda < 0$ for anti–de Sitter (AdS) spacetime). It was shown in [1] that such configurations solve the CGMG field equations if, and only if, Λ solves the quadratic equation

$$\Lambda^2 + 4m^2\sigma\Lambda - 4\lambda m^4 = 0. \tag{1.10}$$

In other words, using $\sigma^2 = 1$,

$$\Lambda = -2m^2[\sigma \pm \sqrt{1+\lambda}]. \tag{1.11}$$

We see that there is a Minkowski vacuum when $\lambda = 0$, but also a dS ($\sigma = -1$) or AdS ($\sigma = 1$) vacuum.[3] For $\lambda > -1$, there are two solutions and hence two possible maximally symmetric vacua. These two vacua merge at $\lambda = -1$ to give a unique vacuum that is AdS if $\sigma = 1$ and dS if $\sigma = -1$. For $\lambda < -1$ there is no maximally symmetric solution.

Maximally symmetric vacua constitute a special subclass of the more general class of static solutions, which we investigate here. Any static metric can be put in the form

$$ds^2 = -ab^2dt^2 + a^{-1}dr^2 + \rho^2 d\theta^2, \tag{1.12}$$

for some functions (a, b, ρ) of the one independent variable r. The special case in which $b^2 = \rho^2/a$ yields the "domain-wall" metrics

$$ds^2 = \rho^2(-dt^2 + d\theta^2) + a^{-1}dr^2. \tag{1.13}$$

These have an additional 2D "worldvolume" Poincaré isometry. We shall show that all static solutions of this special type are locally equivalent to an AdS vacuum. This is perhaps disappointing but one interesting feature emerges from this analysis: for $\lambda = 3$ the domain-wall "energy" (the negative of the Lagrangian) is a perfect square so that the AdS vacuum for this case saturates a Bogomolon'yi-type bound, suggesting a possible connection with supersymmetry.

The general static metric includes static black hole spacetimes, and whenever there exists an AdS vacuum we know that we must also find Banados-Teitelboim-Zanelli (BTZ) black holes [14] because these are locally isometric to the AdS vacuum. An interesting question is whether there exist static black holes that are *not* locally isometric to the (A)dS vacuum. There are two aspects to this problem: first one must find static solutions that are not locally (A)dS, and then one must determine which of these is nonsingular on and outside an event horizon. We reduce the first part of this problem to the solution of a pair of coupled ordinary differential equations (ODEs) for the functions (a, b) and we find the general solution of these equations for which $b \equiv 1$. The only black hole solutions that we find in this way for generic λ are the BTZ black holes but we find a new class of black holes for $\lambda = -1$. For $\sigma = -1$ they are analogous to dS black holes in that there is both a black hole horizon and a cosmological horizon. For $\sigma = 1$ they are asymptotically AdS black

[3]That higher-derivative gravity theories generically permit cosmological solutions even without an explicit cosmological constant in the action is well-known [13].

holes, generically with nonzero surface gravity but there is an "extremal" limit that yields a black hole with zero surface gravity (and hence zero Hawking temperature). In this case the black hole interpolates between the AdS_3 vacuum at infinity and, near the horizon, an $AdS_2 \times S^1$ Kaluza-Klein (KK) vacuum found previously by Clement [15]. One can also take an extremal limit for $\sigma = -1$; in this case the black hole and cosmological singularities coincide. In appropriate new coordinates, the result is a time-dependent cosmological solution that is asymptotic to a $dS_2 \times S^1$ KK vacuum in the far past. We verify directly that these KK vacua are indeed solutions of the CGMG equations at $\lambda = -1$. An interesting corollary is that the dimensional reduction of the $\lambda = -1$ CGMG theory yields a 2D gravitational theory with either a dS_2 vacuum (if $\sigma = -1$) or an AdS_2 vacuum (if $\sigma = 1$).

We should mention here that stationary non-BTZ black hole solutions of CNMG have been found previously [15,16], but these do not include any new *static* black hole solutions. A class of "AdS wave" solutions has been found in [17]. While the original version of this paper was in the stages of completion, we were informed by R. Troncoso that he and collaborators had independently found new solutions at the special point $\lambda = -1$, and their paper has now appeared [18]. We have also been informed by A. Maloney that he and A. Wissanji have found new stationary black hole solutions for arbitrary λ.

Another purpose of this paper is to examine the stability of the (A)dS vacua of CNMG, and to determine which of the linearized field theories are unitary. Here we recall one of the main results of [1]: for $\sigma = -1$ and $\lambda = 0$, the linearization about the Minkowski vacuum yields a unitary field theory propagating two massive, and nontachyonic, modes of spin 2, with helicities 2 and -2. The question we address is whether these nice features persist in other maximally-symmetric vacua.

For AdS backgrounds we will also discuss aspects of the dual conformal field theory (CFT) on the boundary; in particular, we determine the regions in parameter space where the central charges of the asymptotic Virasoro algebra are positive, leading to unitary CFTs, and we will compare with our findings for the bulk modes.

II. THE REDUCED ENERGY FUNCTIONAL

Our main tool for finding static solutions is the principle of symmetric criticality [19]; the application to general relativity (GR) has been discussed in [20,21]. This principle sanctions (under conditions that are satisfied in our examples) the substitution of an ansatz into the action provided that the ansatz is the most general one (up to diffeomorphisms) permitted by some group of isometries. We shall use this principle to obtain a reduced action for the functions (a, b, ρ) that appear in the general static metric ansatz (1.12). A special feature of static metrics is that the LCS term does not contribute to the reduced action, so we can work with the simpler cosmological NMG model without losing generality. This simplification fails for the more general class of stationary metrics, so an investigation of stationary solutions using this method will be considerably more complicated.

The reduced action may be computed easily by using the fact that the Einstein tensor for the metric (1.12) is

$$G = -\frac{ab^2}{2\rho}[\rho' a' + 2a\rho'']dt^2 + \frac{\rho'(ba' + 2ab')}{2ab\rho}dr^2 + \frac{\rho^2(ba' + 2ab')'}{2b}d\theta^2, \tag{2.1}$$

where the prime indicates differentiation with respect to r. Let us record here that this takes the following form for the subclass of domain-wall metrics with $b^2 = \rho^2/a$:

$$G = \rho\sqrt{a}(\rho\sqrt{a})'[-dt^2 + d\theta^2] + \left(\frac{\rho'}{\rho}\right)^2 dr^2. \tag{2.2}$$

We shall work with the action (1.5), so we need to extend the static ansatz (1.12) to the auxiliary tensor field; we do this by taking it to be time independent, without further restriction, but its equation of motion (1.6) implies that it is also diagonal. It is then convenient to also eliminate from the action the diagonal components f^{rr} and f^{tt}; one is left with $f^{\theta\theta}$, which one can trade for the new function

$$c(r) = m^2 r^3 f^{\theta\theta}(r). \tag{2.3}$$

Proceeding in this way, and dropping an overall negative factor, we arrive at the energy functional

$$E[a, b, c, \rho] = \int dr\Big\{2\lambda m^2 b\rho - \sigma a(\rho' b' - \rho'' b) + \frac{1}{2m^2\rho}[(ba' + 2ab')([a(\rho')^2]' + \rho c') + bc^2 + 2(ab\rho')'c]\Big\}. \tag{2.4}$$

This is invariant under the following gauge transformations:

$$\delta_\xi \rho = \xi\rho', \qquad \delta_\xi b = (\xi b)', \qquad \delta_\xi a = \xi a' - 2\xi' a, \qquad \delta c = \xi c', \tag{2.5}$$

from which we derive the identity

$$\rho'\frac{\delta E}{\delta\rho} + c'\frac{\delta E}{\delta c} + 3a'\frac{\delta E}{\delta a} + 2a\left(\frac{\delta E}{\delta a}\right)' - b\left(\frac{\delta E}{\delta b}\right)' \equiv 0. \tag{2.6}$$

Domain walls

One can further "reduce" the energy functional by the substitution

$$b^2 = \rho^2/a, \tag{2.7}$$

because this yields the generic domain-wall metrics of (1.13) compatible with worldvolume boost invariance. After eliminating the variable c, and discarding a boundary term, we find the new energy functional

$$E[a,\rho] = \int dr \frac{1}{\sqrt{a}} \times \left\{\frac{1}{6}\left[\frac{a(\rho')^2 - 6m^2\sigma}{m\rho}\right]^2 + 2(\lambda-3)m^2\rho^2\right\}, \tag{2.8}$$

which is manifestly positive for $\lambda > 3$. For $\lambda = 3$ the integrand of E is a perfect square, which is evidently minimized when $\sigma = 1$ by solutions of the first-order equation

$$\rho' = \pm\frac{\sqrt{6}m}{\sqrt{a}}. \tag{2.9}$$

This yields the AdS vacuum of the model, and the way that we have found it suggests that it might be a supersymmetric vacuum in the context of the supersymmetric extension of CGMG.

A remarkable feature of (2.8) is that the equation of motion for a is purely algebraic. Variation with respect to a yields a quadratic equation for ρ' with the solution

$$a(\rho')^2 = \alpha^2\rho^2, \qquad \alpha^2 = 2m^2[\sigma \pm \sqrt{1+\lambda}]. \tag{2.10}$$

Because of gauge invariance, this implies the equation of motion for ρ unless $\rho' = 0$, but $\rho' = 0$ is possible only for $\lambda = 0$ and it leads only to the Minkowski vacuum (or identifications of it). Excluding this trivial case, (2.10) is the only equation that we need to consider, and it has no solution for $\lambda < -1$. There is also no solution unless $[\sigma \pm \sqrt{1+\lambda}]$ is positive. When this condition is satisfied, there is a solution that depends on the function a, a choice of which amounts to a choice of gauge. For $a = 1$ we find the metric

$$ds^2 = e^{2\alpha r}(-dt^2 + d\theta^2) + dr^2, \tag{2.11}$$

which is just an AdS vacuum (in the absence of any identifications). Thus all "domain-wall solutions of CGMG (of the specified type) are actually locally equivalent to an AdS vacuum.

III. BLACK HOLES

We may fix the gauge invariance of the energy functional (2.4) by setting

$$\rho(r) = r. \tag{3.1}$$

From (2.6) we see that the ρ equation of motion is implied by the other equations of motion when $\rho = r$, so that this is a permissible gauge choice. The gauge-fixed energy functional is

$$E = \int dr\Big\{2\lambda m^2 br - \sigma ab' + \frac{1}{2m^2 r} \times [(ba' + 2ab')(rc' + a') + bc^2 + 2(ab)'c]\Big\}. \tag{3.2}$$

Note that this involves only the variables and their first derivatives, which was made possible by the introduction of the "auxiliary" variable c. The equation of motion for c is

$$c = \frac{r}{2b}(ba' + 2ab')' - \frac{(ab)'}{b}. \tag{3.3}$$

We can use this to simplify the a equation of motion to

$$r(2m^2\sigma b' + bc'' - b'c') + ba'' - b'a' + 2bc' = 0. \tag{3.4}$$

Finally, the b equation of motion is

$$0 = -4\lambda m^4 r^3 + (-2\sigma m^2 a' + a'c' + 2ac'')r^2 + [(a')^2 + 2a(a'' + c') - c^2]r - 2a(a' + c). \tag{3.5}$$

We shall seek solutions of these equations with

$$b \equiv 1. \tag{3.6}$$

This means that the metric has the form

$$ds^2 = -a dt^2 + a^{-1}dr^2 + r^2 d\theta^2, \tag{3.7}$$

and that the Einstein tensor is

$$G = \frac{aa'}{2r}dt^2 + \frac{a'}{2ar}dr^2 + \frac{r^2 a''}{2}d\theta^2. \tag{3.8}$$

It follows that all solutions with $G_{\mu\nu} = -\Lambda g_{\mu\nu}$ have $a' = -2\Lambda r$. More generally, any solution with

$$a = a_0 - \Lambda r^2 \tag{3.9}$$

is locally isometric to dS_3 if $\Lambda > 0$ and to AdS_3 if $\Lambda < 0$. Any solution not of this form will not be locally isometric to (A)dS.

Given that $b = 1$, one may show that the Eqs. (3.3) and (3.4) imply that

$$a = a_0 + a_1 r + \frac{1}{2}a_2 r^2 - \alpha(r\log r - r), \qquad c = \frac{1}{2}ra'' - a', \tag{3.10}$$

where (a_0, a_1, a_2) and α are constants. Substitution into (3.5) yields

$$\alpha = 0, \qquad a_1(a_2 - 4m^2\sigma) = 0, \qquad a_2 = 4m^2[\sigma \pm \sqrt{1+\lambda}]. \tag{3.11}$$

We see that there is no solution when $\lambda < -1$. For $\lambda > -1$ we also have $a_1 = 0$, so a takes the form (3.9) with $\Lambda = -2m^2[\sigma \pm \sqrt{1+\lambda}]$. For $a_0 = 1$ we recover the (A)dS vacua. In an AdS vacuum we may choose $a_0 < 0$ to get a static BTZ black hole with mass parameter $M = -a_0$. We do not expect the energy to coincide with M but we do expect the energy to be positive when the entropy is positive. We return to this point in Sec. VI.

A. New black holes at $\lambda = -1$

When $\lambda = -1$ we do not need to set $a_1 = 0$, so in this case the metric has the form (3.7) but with

$$a = a_0 + a_1 r + 2\sigma m^2 r^2. \tag{3.12}$$

For $a_1 \neq 0$ these solutions are not locally (A)dS, and for $a_1^2 \geq 8\sigma m^2 a_0$, we have

$$\begin{aligned} a &= 2m^2\sigma(r - r_+)(r - r_-), \\ r_\pm &= \frac{1}{4m^2}[\sigma a_1 \pm \sqrt{a_1^2 - 8\sigma m^2 a_0}], \end{aligned} \tag{3.13}$$

for real $r_\pm$.

For $\sigma = 1$ the spacetime is asymptotically AdS with $\Lambda = -2m^2$. When $r_\pm$ are real and $r_+ > 0$ we have an asymptotically AdS black hole with horizon at $r = r_+$. There will also be an "inner" horizon at $r = r_-$ if $r_- > 0$. As long as $r_- \neq r_+$ the black hole will have a nonzero surface gravity but the limiting case $r_- = r_+ = r_0$ yields an extremal black hole with zero surface gravity. The extremal black hole metric is

$$ds^2 = -2m^2(r - r_0)^2 dt^2 + \frac{dr^2}{2m^2(r - r_0)^2} + r^2 d\theta^2. \tag{3.14}$$

Near the horizon at $r = r_0$ this metric becomes the metric of $\mathrm{AdS}_2 \times S^1$. In the following subsection, we verify directly that this near-horizon spacetime is a solution of the CGMG field equations at $\lambda = -1$ and $\sigma = 1$.

For $\sigma = -1$, we need real $r_\pm$ in order to have a static region in which $a > 0$. When $r_+ > 0$ the surface $r = r_+$ is analogous to the cosmological horizon of dS space. If $r_- > 0$ too then we have a black hole in this "cosmological" spacetime with horizon at $r = r_-$. As for $\sigma = 1$, we can again consider the limiting case in which $r_+ = r_- = r_0$ but the static region shrinks to the now unique horizon at $r = r_0$. In terms of the new coordinates $(\tilde{t}, \tilde{r})$ defined by

$$e^{\sqrt{2}m\tilde{t}} = r - r_0, \qquad \tilde{r} = \sqrt{2}mt, \tag{3.15}$$

one finds that the metric is

$$ds^2 = -d\tilde{t}^2 + e^{2\sqrt{2}m\tilde{t}}d\tilde{r}^2 + (r_0 + e^{\sqrt{2}m\tilde{t}})d\theta^2. \tag{3.16}$$

This is not a static metric but in the limit that $\tilde{t} \to -\infty$ (which is a near-horizon limit in which $r \to r_0$), it reduces to

$$ds^2 = -d\tilde{t}^2 + e^{2\sqrt{2}m\tilde{t}}d\tilde{r}^2 + r_0^2 d\theta^2. \tag{3.17}$$

This is locally isometric to the Kaluza-Klein metric $\mathrm{dS}_2 \times S^1$. We shall verify below that this is a solution of the CNMG equations for $\lambda = -1$ and $\sigma = -1$.

B. Kaluza-Klein vacua at $\lambda = -1$

The gauge choice $\rho = r$ is inconvenient if we wish to seek solutions for which the metric has an S^1 factor, since these are most easily investigated by considering $\rho' = 0$. We stress that this is a matter of convenience; any solution that we find with $\rho' = 0$ must correspond to a solution in the $\rho = r$ gauge, but it will be a solution with $b \neq 1$ and will be expressed in different coordinates. To avoid this we return to the gauge-invariant energy function (2.4) and make the alternative gauge choice $b = 1$. Strictly speaking, this is not an admissible gauge choice because only the derivative of the b equation of motion is implied by the other equations of motion when $b = 1$, as one can see from (2.6). This means that substitution of $b = 1$ into (2.4) is not permitted. We therefore substitute both $b = 1$ and $\rho = 1$ into the equations of motion of (2.4). The c and a equations imply that

$$2c = a'', \qquad c'' = 0. \tag{3.18}$$

Using this to simplify the b and ρ equations we deduce that

$$(a'')^2 = 3a'a''' + 4m^2\sigma a'', \qquad (a'')^2 = 2a'a''' - 16\lambda m^4. \tag{3.19}$$

The Eqs. (3.18) imply that a is a cubic polynomial in r. Using this information in (3.19) we deduce (i) that a is actually a quadratic polynomial in r with $a'' = -4m^2\lambda\sigma$, and (ii) that $\lambda(1 + \lambda) = 0$.

When $\lambda = 0$, the quadratic polynomial a is actually linear. This means that the metric is flat, as can be seen from the fact that

$$dx^\mu dx^\nu R_{\mu\nu}|_{\rho=b=1} = -\frac{1}{2}a''[-adt^2 + a^{-1}dr^2]. \tag{3.20}$$

So the only nontrivial case is $\lambda = -1$. We then have the metric

$$\begin{aligned} ds^2 &= -adt^2 + a^{-1}dr^2 + d\theta^2, \\ a &= a_0 + a_1 r + 2m^2\sigma r^2. \end{aligned} \tag{3.21}$$

Its Ricci tensor is

$$dx^\mu dx^\nu R_{\mu\nu} = -2m^2\sigma[-adt^2 + a^{-1}dr^2]. \tag{3.22}$$

The metric is therefore locally isometric to $\mathrm{AdS}_2 \times S^1$ when $\sigma = 1$, which is the KK solution found previously in [15], and to $\mathrm{dS}_2 \times S^1$ when $\sigma = -1$.

IV. LINEARIZATION

We will now find the field theories obtained by linearization about an arbitrary maximally symmetric vacuum of the CNMG model defined by the action (1.5). We write the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \tag{4.1}$$

and define

$$h = \bar{g}^{\mu\nu} h_{\mu\nu}, \qquad h_\mu = \nabla^\nu h_{\mu\nu}, \tag{4.2}$$

where ∇ is the covariant derivative constructed with the background metric, which we take to satisfy

$$\bar{R}_{\mu\nu} \equiv R_{\mu\nu}(\bar{g}) = 2\Lambda \bar{g}_{\mu\nu} \qquad (\Rightarrow \bar{R} \equiv \bar{g}^{\mu\nu}\bar{R}_{\mu\nu} = 6\Lambda). \tag{4.3}$$

One finds that the full Ricci tensor has the expansion

$$R_{\mu\nu} = 2\Lambda \bar{g}_{\mu\nu} + \kappa R^{(1)}_{\mu\nu} + \kappa^2 R^{(2)}_{\mu\nu} + \mathcal{O}(\kappa^3), \tag{4.4}$$

where

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}(\nabla^2 h_{\mu\nu} - 2\nabla_{(\mu}h_{\nu)} + \nabla_\mu \nabla_\nu h - 6\Lambda h_{\mu\nu} + 2\Lambda \bar{g}_{\mu\nu} h). \tag{4.5}$$

The explicit form of $R^{(2)}_{\mu\nu}$ is not required for our analysis, but only its trace, which is given by

$$\bar{g}^{\mu\nu} R^{(2)}_{\mu\nu} = \frac{1}{2} h^{\mu\nu}\left(R^{(1)}_{\mu\nu} - \tfrac{1}{2}R^{(1)}\bar{g}_{\mu\nu}\right) + \text{total derivative}. \tag{4.6}$$

Here we denote by $R^{(1)}$ the trace of $R^{(1)}_{\mu\nu}$ in the background metric. The Einstein tensor is given by

$$G_{\mu\nu} = -\Lambda \bar{g}_{\mu\nu} + \kappa G^{(1)}_{\mu\nu} + \kappa^2 G^{(2)}_{\mu\nu} + \mathcal{O}(\kappa^3), \tag{4.7}$$

where

$$\begin{aligned} G^{(1)}_{\mu\nu} &= R^{(1)}_{\mu\nu} - \tfrac{1}{2}R^{(1)}\bar{g}_{\mu\nu} - 3\Lambda h_{\mu\nu} + \Lambda h \bar{g}_{\mu\nu}, \\ G^{(2)}_{\mu\nu} &= R^{(2)}_{\mu\nu} - \frac{1}{2}R^{(1)} h_{\mu\nu} - \frac{1}{2}(R^{(2)} - h^{\rho\sigma}R^{(1)}_{\rho\sigma})\bar{g}_{\mu\nu} \\ &\quad - \Lambda h^{\rho\sigma}h_{\rho\sigma}\bar{g}_{\mu\nu} + \Lambda h h_{\mu\nu}. \end{aligned} \tag{4.8}$$

We also have to expand the auxiliary field $f_{\mu\nu}$. For nonflat backgrounds, there will be a nonvanishing background value for $f_{\mu\nu}$ determined by (1.6). It turns out to be convenient to expand the auxiliary tensor field $f_{\mu\nu}$ as

$$f_{\mu\nu} = \frac{1}{m^2}\{\Lambda[\bar{g}_{\mu\nu} + \kappa h_{\mu\nu}] - \kappa k_{\mu\nu}\} + \mathcal{O}(\kappa^2), \tag{4.9}$$

where $k_{\mu\nu}$ is an independent symmetric tensor fluctuation field.

At this stage it is worthwhile to discuss the presence of gauge symmetries in this analysis. The full nonlinear theory is invariant under

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= D_\mu \xi_\nu + D_\nu \xi_\mu, \\ \delta_\xi f_{\mu\nu} &= \xi^\rho \partial_\rho f_{\mu\nu} + \partial_\mu \xi^\rho f_{\nu\rho} + \partial_\nu \xi^\rho f_{\mu\rho}, \end{aligned} \tag{4.10}$$

where D_μ denotes the full covariant derivative. Next, we expand the diffeomorphism parameter according to

$$\xi_\mu = \bar{\xi}_\mu + \kappa \epsilon_\mu. \tag{4.11}$$

Since the background metric $\bar{g}_{\mu\nu}$ is nondynamical we are restricted to diffeomorphisms that are also background isometries, which restricts $\bar{\xi}_\mu$ to be a background Killing vector field. For the metric fluctuations one finds the standard gauge transformations

$$\delta_\epsilon h_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu, \tag{4.12}$$

while the fluctuations of the auxiliary field in (4.9) have been defined such that $k_{\mu\nu}$ is gauge invariant. In contrast to the expansion about flat space, the linearized Einstein tensor is not invariant by itself but only in a combination that appears in the linearized field equation. Specifically, expansion of the EH action plus cosmological term yields the action

$$S = -\frac{\sigma}{2}\int d^3x \sqrt{\bar{g}}\, h^{\mu\nu}\mathcal{G}_{\mu\nu}(h), \tag{4.13}$$

where $\mathcal{G}$ is the self-adjoint tensor-valued linear differential operator defined by

$$\begin{aligned} \mathcal{G}_{\mu\nu}(h) &\equiv G^{(1)}_{\mu\nu}(h) + \Lambda h_{\mu\nu} \\ &= R^{(1)}_{\mu\nu} - \frac{1}{2}R^{(1)}\bar{g}_{\mu\nu} - 2\Lambda h_{\mu\nu} + \Lambda h \bar{g}_{\mu\nu}. \end{aligned} \tag{4.14}$$

The invariance of $\mathcal{G}(h)$ under (4.12) may be verified using the relation

$$[\nabla_\mu, \nabla_\nu]V_\rho = \Lambda(\bar{g}_{\mu\rho}V_\nu - \bar{g}_{\nu\rho}V_\mu). \tag{4.15}$$

The action (4.13) is therefore gauge-invariant despite the presence, for $\Lambda \neq 0$, of a masslike term quadratic in h.

We are now ready to linearize (1.5) around a maximally symmetric vacuum. The terms linear in $1/\kappa$ cancel as a consequence of (1.10). The quadratic, κ-independent, terms in the Lagrangian are

$$\mathcal{L}_2 = \frac{(\Lambda - 2m^2\sigma)}{4m^2} h^{\mu\nu}\mathcal{G}_{\mu\nu}(h) - \frac{1}{m^2}k^{\mu\nu}\mathcal{G}_{\mu\nu}(h) - \frac{1}{4m^2}(k^{\mu\nu}k_{\mu\nu} - k^2). \tag{4.16}$$

When $\Lambda = 2m^2\sigma$, which can happen only if $\lambda = 3$, the first term is absent and we may use the self-adjointness of the tensor operator $\mathcal{G}$ to rewrite the second term so that

$$m^2\mathcal{L}_2 = -h^{\mu\nu}\mathcal{G}_{\mu\nu}(k) - \frac{1}{4}(k^{\mu\nu}k_{\mu\nu} - k^2). \tag{4.17}$$

Below we will show that this propagates massive photons rather than massive gravitons.

We now proceed on the assumption that $\Lambda \neq 2m^2\sigma$. The fields h and k can then be decoupled by the field redefinition,

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{2}{(\Lambda - 2m^2\sigma)}k_{\mu\nu}, \tag{4.18}$$

which yields

$$\mathcal{L}_2 = \frac{(\Lambda - 2m^2\sigma)}{4m^2}\bar{h}^{\mu\nu}\mathcal{G}_{\mu\nu}(\bar{h}) - \frac{1}{m^2(\Lambda - 2m^2\sigma)}k^{\mu\nu}\mathcal{G}_{\mu\nu}(k) - \frac{1}{4m^2}(k^{\mu\nu}k_{\mu\nu} - k^2). \tag{4.19}$$

The first term involving $\bar{h}_{\mu\nu}$ is the Einstein-Hilbert action linearized about the vacuum, and therefore it does not propagate physical degrees of freedom. The relevant part of the quadratic Lagrangian is therefore

$$\mathcal{L}_2(k) = \frac{1}{m^2(\Lambda - 2m^2\sigma)}\left\{-\frac{1}{2}k^{\mu\nu}\mathcal{G}_{\mu\nu}(k) - \frac{1}{4}M^2(k^{\mu\nu}k_{\mu\nu} - k^2)\right\}, \tag{4.20}$$

where

$$M^2 = -\sigma m^2 + \frac{1}{2}\Lambda. \tag{4.21}$$

Here we recall that $\mathcal{G}_{\mu\nu}(k)$ is defined as in (4.14), with $h_{\mu\nu}$ replaced by $k_{\mu\nu}$.

V. UNITARITY AND STABILITY

We turn now to the analysis of unitarity for the cases $\sigma = \pm 1$. We have implicitly supposed that $m^2 > 0$ up to now but the linearization actually applies also to $m^2 < 0$ and so in the following we inspect this case as well. Leaving aside the special case that leads to (4.17), which we discuss below, we see from (4.20) that ghosts are avoided if

$$m^2(\Lambda - 2m^2\sigma) > 0. \tag{5.1}$$

In addition M^2 has to satisfy a bound in order to avoid tachyons. About flat space ($\Lambda = 0$) we require $M^2 \geq 0$, so both ghosts and tachyons are avoided for $\sigma = -1$ (and $m^2 > 0$). From (1.10) we see that $\Lambda = 0$ is possible only when $\lambda = 0$, so we have recovered the result of [1] that (wrong-sign) NMG with $\lambda = 0$ is equivalent at the linearized level, when expanded about its Minkowski vacuum, to the (3D) Pauli-Fierz theory for massive spin 2, and is therefore a unitary interacting and generally covariant extension of that field theory.

On dS space, not only may M^2 not be negative but small positive values below a critical value are also forbidden [22]. To be specific, one must have

$$M^2 \geq \Lambda. \tag{5.2}$$

From (4.21) one sees that this bound is equivalent to

$$\Lambda \leq -2m^2\sigma. \tag{5.3}$$

This cannot be satisfied for dS vacua with $\sigma = 1$ but is satisfied for $\sigma = -1$ if $\lambda < 0$ and if one chooses the dS vacuum with the smaller value of Λ. When $\lambda = -1$, the bound may be saturated; this is another special case that will be discussed below.

For AdS vacua it is well-known that unitarity allows scalar fields to have a negative mass squared, provided that the Breitenlohner-Freedman (BF) bound is satisfied [23]. For $D = 3$ this bound is just (5.2) [24]. It has been argued that the same bound applies to spin 2 (see e.g. [9,12]). Although we are not aware of any direct proof from unitarity of this spin 2 mass bound, we will allow negative M^2 in what follows in order to examine the possible consequences.

A. The special cases $\lambda = 3$ and $\lambda = -1$

For $\lambda = 3$ the cosmological constant can take the value $\Lambda = 2m^2\sigma$, for which the linearized action is given by (4.18). It is clear from this result that the metric perturbation h is now a Lagrange multiplier for the constraint $\mathcal{G}(k) = 0$. The solution of this constraint is

$$k_{\mu\nu} = 2\nabla_{(\mu}A_{\nu)}. \tag{5.4}$$

If we now substitute for k in (4.18) we arrive at the linearized Lagrangian

$$\mathcal{L}_2 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + 4m^2\sigma A^\mu A_\mu, \qquad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}. \tag{5.5}$$

This is just the Proca Lagrangian for a massive spin 1 field A, propagating the two helicities ± 1 with mass squared $-8m^2\sigma$. Unitarity evidently requires $m^2 > 0$, so the spin 1 modes are tachyons for $\sigma = 1$ but physical for $\sigma = -1$. Thus, linearization about the AdS vacuum of the $\lambda = 3$ and $\sigma = -1$ model yields a unitary stable theory for massive spin 1 in the AdS background.

We next discuss $\lambda = -1$, which on dS corresponds to saturating the bound (5.2). This point is special in that there is an enhancement of gauge symmetry, first discussed by Deser and Nepomechie [25]. To be precise, when $M^2 = \Lambda$ the three-dimensional Pauli-Fierz Lagrangian (4.20) has the following gauge invariance with infinitesimal scalar parameter ζ [26]:

$$\delta_\zeta k_{\mu\nu} = \nabla_\mu\nabla_\nu\zeta + \Lambda\bar{g}_{\mu\nu}\zeta. \tag{5.6}$$

This phenomenon is usually discussed for dS space since the condition $M^2 = \Lambda$ implies that $M^2 < 0$ on AdS (although the conjectured spin 2 version of the BF-type bound would still be satisfied; in fact, saturated). For dS space, the enhancement of gauge symmetry is related to the possibility of "partially massless" fields [27,28]. It implies that the number of propagating modes is one less than the generic massive case, and hence that the theory propagates just one mode rather than two. Moreover, this one mode does not have a well-defined helicity in 3D and thus it resembles more a massless than a massive mode.[4] An interesting question with regard to the consistency of the theory at $\lambda = -1$ is whether this additional gauge symmetry descends from a nonlinear symmetry of the full theory. We leave this for future work.

Summarizing, for generic values of λ, CNMG propagates two massive graviton modes about the maximally symmetric vacua, with the exception of $\lambda = -1$ and $\lambda = 3$, for which there is a single partially massless mode and a massive vector, respectively.

B. AdS vacua

We are now ready to discuss unitarity for the various cases: Let us first start with the AdS case, i.e., with $\Lambda < 0$. We distinguish four cases depending on the signs of σ and m^2. In the cases for which $m^2 < 0$ we use the new mass parameter $\tilde{m}$ defined by

$$\tilde{m}^2 = -m^2. \tag{5.7}$$

Our results, both for AdS and dS, are summarized in Figs. 1–4, together with the results for the central charges to be discussed in the following subsection.

(i) $m^2 > 0$, $\sigma = -1$.
This choice corresponds to the original one of [1]. The condition (5.1) requires

$$\Lambda > -2m^2. \tag{5.8}$$

From the explicit relation between Λ and m^2 as implied by (1.10) it follows that $\lambda < 3$. In total we find that there are AdS vacua around which modes with helicity ± 2 propagate unitarily for the following range of the cosmological parameter in the action

$$0 < \lambda < 3. \tag{5.9}$$

Moreover, since the mass term is manifestly positive, there are no tachyonic modes and so the vacua are always stable.
As we saw in subsection VA, the limiting case

$$\lambda = 3 \tag{5.10}$$

also yields a unitary theory without tachyons but the propagating massive modes have spin 1 rather than spin 2.

(ii) $m^2 > 0$, $\sigma = 1$.
In the case of a negative value of Λ it is not possible to satisfy (5.1), and therefore this theory necessarily contains ghosts.

(iii) $m^2 < 0$, $\sigma = 1$.
The bound (5.1) implies

$$2\tilde{m}^2 + \Lambda < 0. \tag{5.11}$$

With the explicit form of Λ this condition amounts

[4]We thank Andrew Waldron for discussions on this point.

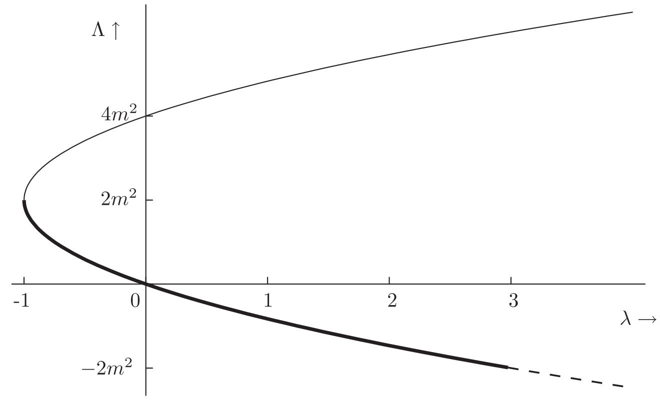

FIG. 1. The case $\sigma = -1$, $m^2 > 0$, where the thick line indicates the unitary region, and the dashed line the region with positive central charge.

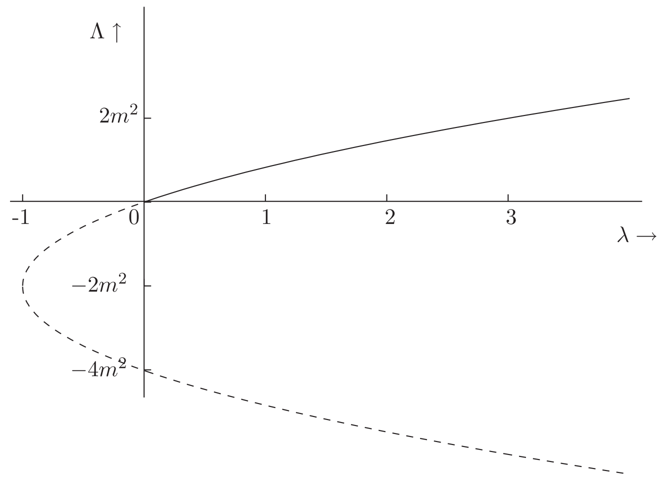

FIG. 2. The case $\sigma = 1$, $m^2 > 0$, with the dashed line indicating the region with positive central charge. There is no region with unitary gravitons.

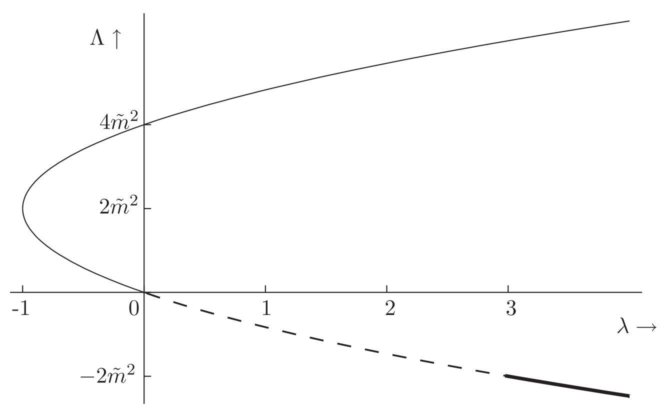

FIG. 3. The case $\sigma = 1$, $m^2 < 0$, with the thick line indicating the region consistent with the BF-type bound, and the dashed line the region with positive central charge.

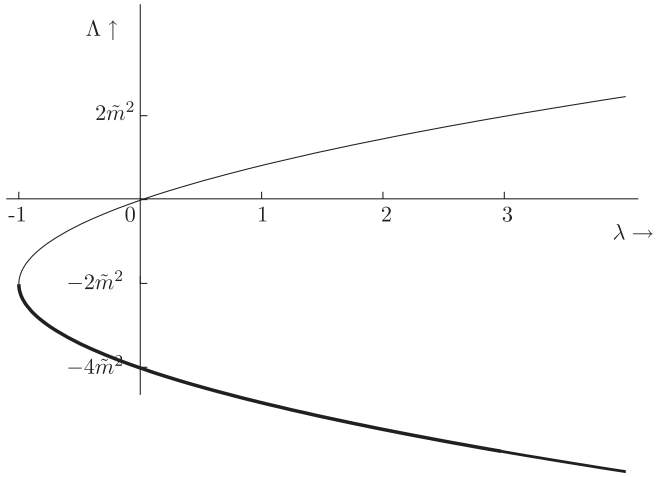

FIG. 4. The case $\sigma = -1$, $m^2 < 0$, where the thick line indicates the region consistent with the BF-type bound, while there is no region with positive central charge.

to $2 < \mp\sqrt{1+\lambda}$. Since the lower sign gives rise to the AdS vacua, this can be satisfied if we set

$$\lambda > 3. \tag{5.12}$$

We also have to check the BF-type bound as the mass parameter m^2 is now negative. However, by (4.21) the bound becomes $2\tilde{m}^2 \geq \Lambda$, which is always satisfied for negative Λ.

We should mention that the equations of motion corresponding to the action for this choice of signs were analyzed in [29]. Although the field equations appear perfectly physical for $\lambda < 3$ (even without invoking a spin 2 BF-type bound) the required overall sign for the action implies that the propagated modes are ghosts. This conclusion agrees with the finding of [29], which we confirm in the following section, that gravitons have negative energy whenever the central charge is positive.

(iv) $m^2 < 0$, $\sigma = -1$.
The condition (5.1) amounts to $-2\tilde{m}^2 + \Lambda < 0$, which is always satisfied. The BF-type bound, on the other hand, reads $-2\tilde{m}^2 \geq \Lambda$. Using the explicit form of Λ this means $0 \geq \pm\sqrt{1+\lambda}$, which is only satisfied for the lower sign. This corresponds to the lower branch AdS solutions and so in this region the theory is stable.

In summary, we conclude that for either sign of the Einstein-Hilbert term there are regions of λ-space for which linearization about an AdS vacuum yields a unitary theory, provided both sign choices of m^2 (consistent with the BF-type bound) are allowed.

C. dS vacua

Let us next analyze the de Sitter case with $\Lambda > 0$. Here we have to choose $m^2 > 0$ in order to avoid tachyons.

(i) $m^2 > 0$, $\sigma = -1$:
The bound (5.2) implies together with (4.21)

$$m^2 \geq \frac{\Lambda}{2}. \tag{5.13}$$

One infers that this can only be satisfied for the smaller solutions of (1.10), which implies that precisely the lower-branch dS solutions in the range

$$-1 \leq \lambda \leq 0 \tag{5.14}$$

are unitary. The special point $\lambda = -1$ corresponds to the gauge symmetry enhancement, giving rise to a single ‘‘partially massless’’ degree of freedom.

(ii) $m^2 > 0$, $\sigma = 1$:
Here the bound (5.2) reads $-2m^2 \geq \Lambda$, which cannot be satisfied for positive Λ.
The point $\lambda = 3$ is again special in that the linear modes propagating about the dS vacuum have spin 1 rather than spin 2, but these modes are tachyonic. In other words, the dS vacuum is unstable.

In summary, we conclude that for dS backgrounds only the original sign choice of [1] leads to unitary nontachyonic modes, which have spin 2.

VI. BOUNDARY CFT AND CENTRAL CHARGES

Now we discuss some aspects of the boundary degrees of freedom governed by a CFT and compare them with the bulk analysis of the previous section. It has been shown by Brown and Henneaux [30] that gravity theories with an AdS_3 vacuum generally admit an asymptotic symmetry group at the boundary which consists of two copies of the Virasoro algebra, corresponding to the conformal symmetry of a two-dimensional dual field theory. The central charges of the left- and right-moving copy of the Virasoro algebra (i.e. the holographic Weyl anomaly [31]) can be expressed in terms of the 3D Newton constant G_3 (related to our parameter κ by $\kappa^2 = 16\pi G_3$) and the AdS length ℓ (related to our parameter Λ by $\Lambda = -1/\ell^2$). In the case of pure AdS Einstein gravity, it reads

$$c = \frac{3\ell}{2G_3}. \tag{6.1}$$

The central charges also encode the entropy of black holes via Cardy's formula. Explicitly, the entropy of the BTZ black hole is given by [32]

$$S = \frac{A_{\rm BTZ}}{4G_3}\Omega, \tag{6.2}$$

where $A_{\rm BTZ}$ is the standard area of the BTZ black hole and $\Omega = \frac{2G_3}{3\ell}c$, which is proportional to the central charge c and equal to 1 for pure AdS gravity.

Next, we are going to discuss the central charges for NMG with AdS background. The systematic way to determine the central charges would be to follow the Brown-Henneaux argument [30] (see also [31,33,34]). Luckily, it has been shown how for any parity-preserving higher-derivative gravity theory admitting AdS_3 vacua the central charge can be derived from the general formula [32,35,36]

$$c = \frac{\ell}{2G_3} g_{\mu\nu} \frac{\partial \mathcal{L}_3}{\partial R_{\mu\nu}}. \tag{6.3}$$

Here $\mathcal{L}_3$ denotes the (higher-derivative) Lagrangian (without the $1/\kappa^2$ prefactor). The factors in (6.3) are determined such that Eq. (6.3) reproduces the Brown-Henneaux result (6.1) for pure AdS gravity. Moreover, it has been shown that for a generic higher-derivative theory admitting BTZ black hole solutions, the entropy computed from the Wald formula coincides with the entropy derived from the central charge (6.3) according to (6.2). Applying (6.3) to CNMG we find

$$c = \frac{3\ell}{2G_3}\left(\sigma - \frac{\Lambda}{2m^2}\right) = \frac{3\ell}{2G_3}\left(\sigma + \frac{1}{2m^2\ell^2}\right). \tag{6.4}$$

In order for the boundary CFT to be unitary the central charges need to be positive. As the central charges encode the entropy, this also amounts to positive entropy. Here we do not attempt to compute other physical parameters of the BTZ black hole like mass or angular momentum from first principles, but it is reasonable to expect that also their values will be "physical" if and only if the central charges are positive.[5] In the following we analyze the central charges for the possible choices of parameters.

(i) $m^2 > 0$, $\sigma = -1$.
Here one finds that positivity of the central charge requires $2m^2\ell^2 < 1$, or equivalently $\Lambda < -2m^2$. The critical value $\Lambda = -2m^2$ is precisely the one found from the unitarity analysis above [see (5.8)] corresponding to the special value $\lambda = 3$, but the allowed range is the opposite. Thus, there is no region in which both the bulk gravitons and the BTZ black holes are well-behaved, except possibly at $\lambda = 3$ where the central charge vanishes and the bulk gravitons are replaced by bulk photons.

(ii) $m^2 > 0$, $\sigma = 1$.
In this case the central charges are manifestly positive. However, there was no region with unitarily propagating gravitons.

(iii) $m^2 < 0$, $\sigma = 1$.
The positivity condition implies by use of the explicit expression for Λ

$$2 > \sqrt{1 + \lambda}, \tag{6.5}$$

which is satisfied for

$$0 < \lambda < 3. \tag{6.6}$$

This is again the opposite of the corresponding bound (5.12) found for unitarity in the bulk.

(iv) $m^2 < 0$, $\sigma = -1$.
In this case the central charges are manifestly negative.

To summarize, the unitary regions of the dual CFT corresponding to AdS/BTZ backgrounds never coincide with regions in parameter space corresponding to unitary positive-energy massive spin 2 modes in the bulk. The extreme case is $\lambda = 3$ at which the bulk modes are given by unitary massive spin 1 excitations and where the central charges vanish. In other words, as in TMG there is a conflict on AdS backgrounds between having "well-behaved" graviton modes or BTZ black holes.

Potentially, this conflict might be resolved by following the same route as "chiral gravity" [8]. Here it has been conjectured for TMG with right-sign Einstein-Hilbert term that there are sufficiently strong boundary conditions such that the massive bulk modes disappear at a chiral point, where one of the central charges vanishes; see e.g. [37] for a very recent discussion. If we consider the most general CGMG model (with LCS term) something similar might happen. In this case the parity-violating Chern-Simons term leads to different left- and right-moving sectors [38]. One finds

$$\begin{aligned} c_L &= \frac{3\ell}{2G_3}\left(\sigma + \frac{1}{2\ell^2 m^2} + \frac{1}{\mu\ell}\right), \\ c_R &= \frac{3\ell}{2G_3}\left(\sigma + \frac{1}{2\ell^2 m^2} - \frac{1}{\mu\ell}\right), \end{aligned} \tag{6.7}$$

which allows for chiral choices with, say, $c_R = 0$ and $c_L > 0$. The question is whether there are consistent boundary conditions which indeed lead to chiral and finite charges at the boundary and which eliminate all problematic modes in the bulk. First attempts into this direction appeared already in [39]. Concerning the special point $\lambda = 3$ there might be yet another way to establish a chiral theory of gravity: if the spin content is not modified by the addition of the LCS term then there might be a choice of parameters for which precisely one central charge vanishes while there is no bulk graviton anyway, but only a spin 1 excitation.

VII. CONCLUSIONS AND OUTLOOK

In this paper we have further investigated the new massive gravity theory proposed in [1] as a generally covariant interacting extension of the three-dimensional Pauli-Fierz theory for massive spin 2, and to a lesser extent its generalization to general massive gravity. We have allowed for a cosmological term, parametrized by the dimensionless constant λ already introduced in [1], and we have focused on new features of the various (A)dS vacua. With regard to both static solutions and unitarity properties, we have found that the values $\lambda = -1$ and $\lambda = 3$ are special.

In particular, we have found a new class of asymptotically AdS black hole solutions for $\lambda = -1$, including an

[5] See also [15].

extremal black hole solution that interpolates between the AdS_3 vacuum and an $adS_2 \times S^1$ solution. There is also an enhanced gauge-invariance of the linearized theory at $\lambda = -1$, which implies that one of the bulk modes can be gauged away, leaving a single "partially massless" degree of freedom without spin.

The existence of $(A)dS_2 \times S^1$ KK vacua when $\lambda = -1$ implies that the dimensional reduction of the $\lambda = -1$ models leads to a 2D "gravitational" theory that has either a dS vacuum (if $\sigma = -1$) or an AdS vacuum (if $\sigma = 1$). It would be of interest to see what this 2D theory is, and to determine its properties. In this context the recent work of [40,41] may be relevant.

The models with $\lambda = 3$ are also special. First, $\lambda = 3$ is the border of the unitarity region, as shown in Figs. 1 and 3. Second, at this point there are no spin 2 gravitons in the bulk but rather massive spin 1 modes. We also found that $\lambda = 3$ is special in our attempt to find static domain-wall solutions. The domain-wall energy is a perfect square for $\lambda = 3$, so its minimization leads to a first-order Bogomol'nyi-type equation, solutions of which yield the AdS vacuum at $\lambda = 3$. This suggests that this vacuum might be supersymmetric in the supergravity context. It was shown in [1] that the Minkowski vacuum for $\lambda = 0$ is supersymmetric, but not enough is known about the nonlinearities of the supergravity theory to decide the issue for other vacua. The supergravity extension is therefore still an outstanding open problem. It also seems likely that supergravity extensions will have improved ultraviolet behavior; it is not unreasonable to expect an ultraviolet finite theory for a sufficient number of supersymmetries.

As in the expansion about flat space analyzed in [1] we found that in order to have unitarily propagating gravitons we have to choose the wrong-sign Einstein-Hilbert term for flat space and dS vacua. Assuming a proposed spin 2 version of the BF-type bound, we found that it was possible to have unitarily propagating gravitons in an AdS vacuum for either sign of the EH term, but in all cases bulk unitarity corresponds to a negative central charge for the boundary CFT, with the possible exception of the AdS vacuum of the wrong-sign model with $\lambda = 3$. Thus, as in cosmological TMG, there is an incompatibility between bulk gravitons and unitarity of the boundary CFT. As the sign of the central charge is related to the sign of the entropy of BTZ black holes, there is an apparent conflict between physical bulk gravitons and BTZ black holes. We mentioned that one resolution of the conflict might be to consider a generalization of chiral gravity, which would eliminate the bulk gravitons. Another point of view might be to keep the unitary bulk gravitons but to discard the BTZ black holes; it has been argued for TMG that there is a superselection argument that would exclude them [9] but for NMG this option, if it is an option, has to be considered in the light of the fact that there are other types of black hole.

There are clearly many further features of NMG and GMG that deserve study. To mention just one: a connection to Hořava-Lifshitz gravity has recently been proposed [42].

ACKNOWLEDGMENTS

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APPENDIX: *K* MODELS

In our linearization analysis, all results were obtained for an arbitrary sign σ but they also apply if $\sigma = 0$. By setting $\sigma = 0$ we effectively have a model with a cosmological and (higher-derivative) K term but no EH term. We shall call these the K models; in them, the cosmological constant is related to our parameter λ by

$$\Lambda^2 = 4\lambda m^2. \tag{A1}$$

We will consider both signs of m^2. For either sign one can choose λ to get any desired value of $\Lambda^2 \geq 0$. The choice $\lambda = 0$ implies $\Lambda = 0$ and hence a unique Minkowski vacuum. This special case yields the massless NMG model mentioned in the introduction. We shall first discuss this special case in some detail, and then comment on its cosmological extension to $\Lambda^2 > 0$.

1. $\Lambda = 0$

We noted in subsection VA that when $\Lambda = 2m^2\sigma$ the metric perturbation becomes a Lagrange multiplier in the linearized theory, imposing a constraint of vanishing linearized curvature for a PF tensor field. This observation still applies when $\sigma = 0$, in which case the metric perturbation becomes a Lagrange multiplier in the linearized theory at $\Lambda = 0$, so our previous analysis of NMG at $\Lambda = 2m^2\sigma$ implies that massless NMG is equivalent, for $m^2 > 0$, to a Maxwell action for a 3-vector field. This is on-shell equivalent to the action for a massless scalar field, so there is one propagating mode, in agreement with [4]. We should point out here that spin is not defined for a massless particle in 3D, so one cannot really assign a spin to this one massless particle.

We will now verify these conclusions, taking as our starting point the pure-K action (1.8) in its form (1.7) with auxiliary PF tensor field $\tilde{f}$. We linearize about the Minkowski vacuum by setting

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{A2}$$

where η is the Minkowski metric and h is now a *dimen-*

sionless metric perturbation. Omitting boundary terms, we may write the resulting quadratic action as

$$S_{\rm lin}[h,f]=\frac{1}{\beta^2}\int d^3x\Big\{h^{\mu\nu}[\mathcal{G}(\tilde f)]_{\mu\nu}-\frac{1}{4}(\tilde f^{\mu\nu}\tilde f_{\mu\nu}-\tilde f^2)\Big\}, \tag{A3}$$

where indices are now raised and lowered with the Minkowski metric, and $\mathcal{G}$ is the self-adjoint "Einstein operator" of (4.14) for $\Lambda=0$; it can be written as [1]

$$\mathcal{G}_{\mu\nu}{}^{\rho\sigma}=-\frac{1}{2}\varepsilon_{(\mu}{}^{\eta\rho}\varepsilon_{\nu)}{}^{\tau\sigma}\partial_\eta\partial_\tau. \tag{A4}$$

On the one hand, elimination of $\tilde f$ yields the quadratic approximation to the action (1.8). On the other hand, we may view h as a Lagrange multiplier for the constraint $\mathcal{G}(\tilde f)=0$, which has the general solution

$$\tilde f_{\mu\nu}=\partial_\mu A_\nu+\partial_\nu A_\mu\equiv 2\partial_{(\mu}A_{\nu)}, \tag{A5}$$

where A is a vector field. Remarkably, the action then reduces to

$$S_{\rm lin}[A]=-\frac{1}{4\beta^2}\int d^3x F^{\mu\nu}F_{\mu\nu},\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\equiv 2\partial_{[\mu}A_{\nu]}. \tag{A6}$$

This is the Maxwell action in 3D with coupling constant β, which is real when $m^2>0$.

There is a precedent for this construction [43]. In *any* dimension, the PF mass term becomes a Maxwell Lagrangian on solving the constraint of zero linearized Riemann tensor for the spin 2 field. In 3D the Riemann tensor is zero if the Einstein tensor is zero, so the relevant action is (A3). In 4D one needs a Lagrange multiplier that is a 4th rank tensor field with the algebraic symmetries of the Riemann tensor, and elimination of the PF tensor field now yields a pure 4th-order Lagrangian for this 4th rank tensor field. A canonical analysis of this "higher-rank representation" of spin 1 confirms its spin 1 content [43]. The linearized pure-K theory is just the 3D analog of this 4D model. One significant difference is that it was not clear in the 4D case how nontrivial interactions could be introduced, whereas in 3D the fully nonlinear pure-K theory was our starting point. However, there are potential difficulties with interactions even in 3D.

As stressed in [4], the quadratic approximation to K is invariant under a linearized Weyl invariance. In this context we note the following convenient rewriting of the (nonlinear) K term,

$$S=\int d^3x\sqrt{|g|}K=-\int\varepsilon_{abc}e^a\wedge f^b\wedge f^c, \tag{A7}$$

where $f^a=dx^\mu(R_\mu{}^a-\frac{1}{4}Re_\mu{}^a)$ is the "conformal boost" gauge field expressed in terms of the curvature. In the first-order form (A3) the Weyl invariance is given by

$$\delta_\zeta h_{\mu\nu}=2\zeta\eta_{\mu\nu},\qquad \delta_\zeta\tilde f_{\mu\nu}=-2\partial_\mu\partial_\nu\zeta. \tag{A8}$$

This linearized Weyl symmetry is a consequence of the conformal covariance property of K noted in [1] ($\delta K\propto K$), which also implies that this "extra" gauge invariance of the quadratic approximation is an artifact of this approximation; it does not extend to the interacting theory. In fact, the linearized gauge invariance (A8) just amounts to the Maxwell gauge invariance of the potential introduced in (A5). This gauge invariance does not survive the introduction of interactions, so there is a constraint in the full theory that is missed on linearization. The implications of this constraint are not clear to us at present, so the consistency of massless NMG is still in doubt. We note, however, that the situation in this respect is much better for the $\lambda=3$ CNMG model since in that case there is no gauge symmetry at the quadratic level that could be broken by interactions.

2. $\Lambda\neq 0$

We now turn to the cosmological K model with $\lambda>0$. We take as our starting point the model obtained from (1.5) by setting $\sigma=0$. This has one dS vacuum and one AdS vacuum, both with the same (arbitrary) value of $|\Lambda|$. The linearization analysis of Sec. IV can be taken over to this case by setting $\sigma=0$ in (4.20). We observe that even though the original action does not have an Einstein-Hilbert term, a linearized Einstein-Hilbert term is generated in the linearization about nonflat backgrounds. Consequently, the linearized excitations are actually massive spin 2 modes, in contrast to the $\lambda=0$ model.

Let us now examine unitarity of the linearized field theory on (A)dS. From (5.1) we infer that unitarity requires $m^2\Lambda>0$. In the dS vacuum this means that we must have $m^2>0$, but since $M^2=\Lambda/2$ the unitarity bound (5.2) is violated. In the AdS vacuum unitarity requires $m^2<0$. The graviton modes have $M^2=\Lambda/2<0$ but this satisfies the spin 2 BF-type bound, so the bulk theory appears physical at the linearized level. However, the central charge (6.4) of the dual CFT is now

$$c=-\frac{3\ell\Lambda}{4G_3m^2}, \tag{A9}$$

which is negative when $m^2<0$. Thus, as in case of CNMG, we cannot simultaneously have unitary bulk gravitons and a unitary boundary CFT.

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