Self-affine roughness effects on the contact area between elastic bodies

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We have calculated the real contact area between elastic bodies with self-affine rough surfaces, which are described in terms of analytical correlation models in Fourier space. It is found that the roughness has a strong influence on the real contact area \( A(\lambda) \) at lateral length scales \( \lambda \) which are comparable with the in-plane roughness correlation length \( \xi \), and for significant applied loads \( \sigma_n \) beyond the linear regime (or \( A \propto \sigma_n \)). The effect of the roughness exponent \( H \) can be rather complex, depending on the relative magnitude of the roughness correlation length \( \xi \) with respect to the lateral length scale \( \lambda \) where the contact area is considered. Finally, we also show that descriptions of the influence of the roughness that is only based on power law approximations of the self-affine roughness spectrum are rather insufficient, especially for large roughness exponents \( H \) (>0.5).

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of the diameter of the macroscopic contact area, the stress distribution $P(\sigma, Y)$ in the contact area under magnification $Y = L/\lambda$ is given by the differential equation\(^4,5\)

$$\frac{\partial P}{\partial Y} = \left( \frac{dG}{dY} \right) \sigma^2 \frac{\partial^2 P}{\partial \sigma^2}$$

with

$$G(Y) = \frac{1}{8} \left[ \frac{E}{(1 - v^2)\sigma_o} \right]^2 \int_0^{2\pi} q^2 C(q) dq,$$  \(1\)

where $C(q)$ is the Fourier transform (roughness spectrum) of the height–height autocorrelation function $C(r) = \langle h(r) h(0) \rangle$ with $h(r)$ the surface roughness height such that $\langle h \rangle = 0$. (... stands as an ensemble average over possible roughness configurations.

Assuming only elastic deformation (infinite yield stress), we can obtain the ratio $P(Y = L/\lambda)$ of the real contact area $A(\lambda)$ at lateral length scale $\lambda$ (if the surface was smooth on all length scales shorter than $\lambda$; or apparent area of contact on the length scale $\lambda$) to that of the macroscopic contact area $A(L)$\(^4,5\) by the equation $P(Y) = \int_0^{\infty} P(\sigma, Y) d\sigma$. The solution of Eq. (1) with the boundary conditions $P(\sigma = 0, Y) = 0$ (absence of adhesion) and $P(\sigma = +\infty, Y) = \infty$ yields for $P(Y)$\(^4,5\)

$$P(Y) = \frac{2}{\pi} \left[ \int_0^{\infty} \frac{\sin x}{x} e^{-x^2 G(Y)} dx = \text{erf} \left( \frac{1}{2 \sqrt{G(Y)}} \right) \right],$$  \(2\)

which is the main parameter under investigation in the following sections. Calculation of $P(Y)$ requires the knowledge of $G(Y)$ and thus of the roughness spectrum $C(q)$.

**III. SELF-AFFINE SURFACE ROUGHNESS MODEL**

A wide variety of surfaces/interfaces are well described by a kind of roughness associated with self-affine fractal scaling.\(^9,10\) For self-affine surface roughness $C(q)$ scales as a power-law $C(q) \propto q^{-2 - 2H}$ if $q \xi > 1$, and $C(q) \propto \text{const}$ if $q \xi \ll 1$.\(^9,10\) The roughness exponent $H$ is a measure of the degree of surface irregularity,\(^9,10\) such that small values of $H$ characterize more jagged or irregular surfaces at short length scales ($\xi$). This scaling behavior is satisfied by the simple Lorentzian form\(^11\)

$$C(q) = \frac{1}{2\pi} \frac{w^2 \xi^2}{(1 + a q^2 \xi^2)^{1 + H}},$$  \(3\)

with $a = (1/2H)[1 - (1 + aQ_e^2 \xi^2)^{-H}]$ if $0 < H < 1$ (power-law roughness), and $a = (1/2)\ln[1 + aQ_e^2 \xi^2]$ if $H = 0$ (logarithmic roughness).\(^11\) Moreover, we have $Q_e = \pi/a_o$ with $a_o$ on the order of atomic dimensions, while the parameter $w$ is the rms roughness amplitude. For other correlation models see also Refs. 10 and 12.

**IV. RESULTS AND DISCUSSION**

Substitution of Eq. (3) into Eq. (1) for the factor $G(Y)$ yields the simple analytic expression

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**FIG. 1.** $P(\lambda)$ vs lateral length scale $\lambda$ for roughness exponent $H = 0.3$, relatively high applied stress $\sigma_o$ such that $E/\sigma_o = 5$, and various correlation lengths $\xi$.

**FIG. 2.** (a) $P(\lambda)$ vs roughness correlation length $\xi$ for lateral length scale $\lambda = 100$ nm, various roughness exponents $H$, and relatively high applied stress $\sigma_o$ such that $E/\sigma_o = 5$. (b) Calculations of the roughness factor $G$ vs roughness correlation length $\xi$ for various roughness exponents $H$, $G_o = (1/8)[E/(1 - v^2)\sigma_o]^2$, $E/\sigma_o = 5$, and $\lambda = 100$ nm.
with \( T_{YL} = (1 + aY^2q_L^2\xi^2) \) and \( T_L = (1 + aq_L^2\xi^2) \). Equation (4) will be used for the calculation of \( P(Y) \) or \( P(\lambda) \) in combination with Eq. (2). For \( H = 0 \) and \( H = 1 \) we obtain from Eq. (4) the limiting forms, if we consider the identity \( \ln(T) = \lim_{n \to 0} (1/n)(T^n - 1) \)

\[
G(Y)_{H-0} = \frac{1}{16} \left[ \frac{E}{(1 - v^2)\sigma_o} \right]^2 \frac{w^2}{a^2\xi^2} \times \left[ \frac{1}{1 - \frac{1}{H}(T_{YL}^{-H} - T_L^{-H})} + \frac{1}{H} \frac{T_{YL}^{-H} - T_L^{-H}}{T_{YL}^{-H} T_L^{-H}} \right].
\]

(6)

Furthermore our calculations are performed for macroscopic contact area of size \( L = 100 \mu m, a_o = 0.3 \text{ nm}, \) Poisson ratio \( v = 0.3, \) and rms roughness amplitude \( w = 10 \text{ nm} \) such that \( w \ll \xi \) (assuming nanometer scale roughness). Our investigations span a wide range of magnification values \( Y \) such that the corresponding length scales \( \lambda = L/Y \) range from macroscopic dimensions \( \lambda \gg \xi \) down to length scales smaller that the corresponding roughness correlation length \( \xi. \)

Figure 1 shows calculations of the ratio \( P(\lambda) = A(\lambda)/A(L) \) versus the lateral length scale \( \lambda \) for various roughness correlation lengths \( \xi \gg w. \) The function \( P(\lambda) \) is obtained from \( P(Y) \) if we substitute \( Y = L/\lambda. \) Clearly \( P(\lambda) \) and thus the real contact area \( A(\lambda) \) increases with lateral length scale \( \lambda \) and approaches values close to the macroscopic area \( A(L) \) for \( \lambda \gg \xi. \) As the correlation length \( \xi \) decreases, i.e., the surface becomes rougher because the rms amplitude \( w \) is assumed fixed, the increment of \( P(\lambda) \) and thus of the contact area becomes sharper for \( \lambda > \xi. \) The latter becomes more pronounced for roughness ratios \( w/\xi > 0.1 \) characterizing a rather rough surface at long wavelengths \((\gg \xi).\)

On the other hand, if we plot \( P(\lambda) \) as a function of the roughness correlation length \( \xi \) [Fig. 2(a)], we observe that as \( \xi \) increases and thus the surface smoothen at long wavelengths, the real contact area initially decreases and after passing through a minimum (for \( \xi < \lambda \)) it further increases. However, as Fig. 2(a) indicates, such an increment takes place at a faster rate for correlation lengths \( \xi > \lambda, \) and larger roughness exponents \( H, \) i.e., for smoother surfaces at short roughness wavelengths \((< \xi). \) The minimum in Fig. 2(a) is due to the maximum of the factor \( G \) [Eq. (4)] which will lead to the inverse behavior for \( P(\lambda) \) due to the Gaussian factor
in the integrand of Eq. (2). Physically, as the correlation
length $\xi$ increases the ratio $\lambda/\xi$ (for fixed contact length $\lambda$)
decreases which favors lowering of the contact area, while an
increment of the correlation length (for fixed roughness am-
plitude $w$) leads to surface smoothening which favors larger
contact area. The competition between these processes lead
to the minimum observed in Fig. 2(a).

Next, we investigate the dependence of $P(\lambda)$ on the
applied load $\sigma_o$ at some fixed lateral length scale $\lambda$ comparable
with the lateral roughness correlation length $\xi$. As Fig. 3
indicates for small loads $\sigma_o/E<0.2$, the dependence of
$P(\lambda)$ on the ratio $\sigma_o/E$ is linear [also see the inset of Fig.
3(a)], which is in agreement with former findings for weak
loads.\textsuperscript{4,5} However, Fig. 3 indicates that the roughness effect
either as a function of the roughness exponent $H$ or the
correlation length $\xi$ is more pronounced in the nonlinear or rela-
tively high load regime such that $0.2<\sigma_o/E<1$. Indeed, in
this case rougher surfaces lead to lower values of the ratio
$P(\lambda)$ and thus of the real contact area when $\lambda$ is sufficiently
smaller than $\xi$ [see also Fig. 2(a)]. The opposite behavior
takes place for small correlation lengths $\xi<\lambda/4$ [Fig. 2(a)].
Nevertheless, as the load approaches high values close to the
elastic modulus, $\sigma_o\sim E$, the effect of roughness becomes
insignificant. This is also depicted in Fig. 4 which shows that
the ratio $P(\lambda)$ increases faster with increasing applied load
$\sigma_o$ or equivalently the real contact area approaches faster
values close to the nominal macroscopic value $A(L)$.

Now we will compare our calculations with the approxima-
tions used in earlier works\textsuperscript{4,5} where the surface roughness
spectrum is considered only by its asymptotic limits [following
Eq. (3)]

$$C(q) = \begin{cases} 
\frac{w^2}{2} \frac{\xi^2}{2\pi} & \text{if } q \ll \sqrt{\frac{a}{\xi}} \\
\frac{w^2}{2} \frac{(2\pi)^{1+H} \xi^2}{2H} q^{2(1+H)} & \text{if } q \gg \sqrt{\frac{a}{\xi}} 
\end{cases}$$  \hspace{1cm} (7)

with the wave vector $\vec{q} = 1/\sqrt{a}$ defined by the intersection of
the saturation value of $C(q)$ and its power law regime. In
such a case we obtain for the factor $G(Y)$ from Eq. (7)

$$G(Y) \approx \frac{1}{8} \left[ \frac{E}{(1-v^2)\sigma_o} \right]^2 \begin{cases} 
\frac{w^2}{2a^{1+H} \xi^2H} \left[ \frac{1}{1-H} \left( \frac{2\pi Y}{L} \right)^{2(1-H)} - \left( \frac{2\pi}{\xi \sqrt{a}} \right)^{2(1-H)} \right] & \text{if } Y > Y_o \\
\frac{w^2}{4} \left( \frac{1}{\sqrt{a}} \right)^4 - \left( \frac{2\pi}{L} \right)^2 & \text{if } Y > Y_o \\
\frac{w^2}{2a^{1+H} \xi^2H} \left[ \frac{1}{1-H} \left( \frac{2\pi Y}{L} \right)^{2(1-H)} - \left( \frac{2\pi}{L} \right)^{2(1-H)} \right] & \text{if } Y < Y_o
\end{cases}$$  \hspace{1cm} (8)
with \( Y_o = L/2\pi\xi \sqrt{a} \). Calculation of \( P(Y) \) or \( P(\lambda) \) versus lateral length scale \( \lambda \) (Fig. 5) indicate that such an approximation gives significantly smaller values for \( P(\lambda) \) and thus for the real contact area \( A(\lambda) \) especially for large roughness exponents \( H > 0.5 \) (see also Fig. 6). The estimated deviations for \( P(\lambda) \) occur at length scales \( \lambda \) comparable to or smaller than the roughness correlation length \( \xi \) (Fig. 5).

Finally, we point out that for weak loads such that \( \sigma_o \ll E \) and \( G(Y) \gg 1 \), it has been shown to lowest order in the expansion of \( \sin x = x \) in Eq. (3) that \( P(Y) \approx [\pi G(Y)]^{-1/2} \). However, for higher order terms, since \( \sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} / (2n+1)! \), we obtain from Eqs. (3) and (4) the more precise expression and still in analytic form

\[
A(\lambda) = A(L) \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} 2^{2n+1} \frac{(-1)^n a^{2n+1}}{n!(2n+1)} \left( \frac{1}{E} \right)^{2n+1} \left( \frac{\xi}{w} \right)^{2n+1} \left( \frac{1}{1-H} \frac{T_{\lambda}^{-1-H}}{T_{L}^{-1-H}} - \frac{T_{L}^{-1-H}}{T_{\lambda}^{-1-H}} \right) ^{n+(1/2)},
\]

which shows that \( A(\lambda) \) depends inversely proportional to the long wavelength roughness ratio \( w/\xi \) in powers of \( 2n+1 \), and proportionally to the stress ratio \( \sigma_o / E \) in powers of \( 2n+1 \).

V. CONCLUSIONS

In summary, we have calculated the real contact area between elastic bodies with self-affine rough surfaces, which are described in terms of analytic correlation models in Fourier space. It is found that the roughness has a strong influence on the real contact area \( A \) at lateral length scales \( \lambda \) which are comparable with the in-plane roughness correlation length \( \xi \), and for significant applied loads \( \sigma_o \) beyond the linear regime (or \( A \propto \sigma_o \)). The effect of the roughness exponent \( H \) can be rather complex, depending on the relative magnitude of the roughness correlation length \( \xi \) with respect to the lateral length scale \( \lambda \) where the contact area is calculated. We also show that descriptions of the influence of the roughness only based on power law approximations of the self-affine roughness spectrum can be rather inadequate, especially for large roughness exponents \( H > 0.5 \).

Finally, we should point out that the calculation of the roughness influence is performed in terms of a specific roughness model\(^{11} \) which gives an analytic form of the factor \( G(Y) \) by incorporating the effect of intermediate lateral roughness wavelengths \( q \sim 2 \pi / \xi \). Clearly for other correlation models\(^ {10,12} \) there can be deviations, since they differ mainly around the lateral roughness wavelengths \( q \sim 2 \pi / \xi \), however, not as strong as those that are obtained by use of extrapolation schemes as those of Eq. (8).

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11. G. Palasantzas, Phys. Rev. B 48, 14472 (1993); 49, 5785 (1994); Besides the simplicity of \( C(q) \), its Fourier transform yields the analytically solvable correlation function \( C(r) = [w^2 / \pi] (1+H) (r^{2a} / \xi^2)^H K_0 (r^{2a} / \xi^2) \) with \( K_0 \) the second kind of Bessel function of order \( H \).