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Stabilization of collocated systems by nonlinear boundary control

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ABSTRACT

We design control laws for a class of dissipative infinite-dimensional systems using nonlinear boundary control action. The applications are to saturated control of SCOLE-type models.

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1. Introduction and motivation

A popular way of stabilizing beam-like models is to apply collocated control and measurement action. In many cases, by a suitable choice of an extended state space, even collocated systems with boundary control action can be formulated as systems with bounded control. For examples of this formulation see Slemrod [1] and the examples in Oostveen [2, Chapter 9]. This results in a linear system

$$\dot{z}(t) = Az(t) + Bu(t), \quad y(t) = B^*z(t), \quad z(0) = z_0, \quad t \geq 0$$

on the state space Z , where A the infinitesimal generator of a contraction C_0 -semigroup, $B \in \mathcal{L}(U, Z)$, and U is another Hilbert space. To illustrate the formulation of boundary control as bounded control in an appropriate space we recall the example used in [1]. It was motivated by the SCOLE-related model¹ from Bailey and Hubbard [3] of one of the arms of a satellite, consisting of a central hub with four flexible beams attached to it.

Example 1.1. This model describes the transverse vibrations of a beam of length L , which is clamped at one end and to which a point mass m is attached at the tip. Here $w(x, t)$ denotes

the displacement of the beam and $u(t)$ is a scalar control. A piezoelectric film is bonded to the beam of length L , which applies a bending moment to the beam when a voltage is applied to it. This voltage is the control input of the system and the angular velocity at the tip is the measurement. The mathematical formulation is given by

$$\frac{\partial^2 w}{\partial t^2}(x, t) + \frac{\partial^4 w}{\partial x^4}(x, t) = 0 \quad \text{for } 0 < x < L, \quad (1)$$

with boundary conditions

$$w(0, t) = \frac{\partial w}{\partial x}(0, t) = 0, \quad (2)$$

$$m \frac{\partial^2 w}{\partial t^2}(L, t) = \frac{\partial^3 w}{\partial x^3}(L, t), \quad (3)$$

$$J \frac{\partial^3 w}{\partial t^2 \partial x}(L, t) = -\frac{\partial^2 w}{\partial x^2}(L, t) + u(t), \quad (4)$$

and measurement

$$y(t) = \frac{\partial^2 w}{\partial t \partial x}(L, t), \quad (5)$$

where m is the mass and J is its moment of inertia. In the rest of the paper we assume that both are equal to one. Let $\mathcal{L}_2(0, L)$ denote the linear space of square integrable functions on the interval $[0, L]$ and let $H^2(0, L)$ and $H^4(0, L)$ denote the standard Sobolev spaces,

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¹ NASA Spacecraft Laboratory Control Experiment.

i.e.,

$$H^2(0, L) := \left\{ h \in L_2(0, L) \mid h, \frac{dh}{dx} \text{ are abs. continuous,} \right. \\ \left. \frac{d^2h}{dx^2} \in L_2(0, L) \right\}$$

$$H^4(0, L) := \left\{ h \in L_2(0, L) \mid h, \frac{dh}{dx}, \frac{d^2h}{dx^2}, \frac{d^3h}{dx^3} \right. \\ \left. \text{are abs. continuous, } \frac{d^4h}{dx^4} \in L_2(0, L) \right\}.$$

Define the state space by

$$Z = \left\{ h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} \in H^2(0, L) \right. \\ \left. \times L_2(0, L) \times \mathbb{R} \times \mathbb{R} \mid h_1(0) = \frac{dh_1}{dx}(0) = 0 \right\}$$

with the inner product

$$\langle h, \tilde{h} \rangle_Z = \int_0^L \frac{d^2h_1}{dx^2}(x) \frac{d^2\tilde{h}_1}{dx^2}(x) dx + \int_0^L h_2\tilde{h}_2 dx + h_3\tilde{h}_3 + h_4\tilde{h}_4.$$

The input space and output space are $U = \mathbb{R}$. We define the operators $A : \mathcal{D}(A) \subset Z \rightarrow Z$ and $B \in \mathcal{L}(U, Z)$ by

$$\mathcal{D}(A) = \left\{ h \in H^4(0, L) \times H^2(0, L) \times \mathbb{R} \times \mathbb{R} \mid h_1(0) = \frac{dh_1}{dx}(0) = 0, \right. \\ \left. h_2(0) = \frac{dh_2}{dx}(0) = 0, h_2(L) = h_3, \frac{dh_2}{dx}(L) = h_4 \right\},$$

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ -\frac{d^4}{dx^4} & 0 & 0 & 0 \\ \frac{d^3}{dx^3}|_{x=L} & 0 & 0 & 0 \\ -\frac{d^2}{dx^2}|_{x=L} & 0 & 0 & 0 \end{pmatrix}, \quad Bu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u.$$

When

$$z_1(t) = w(t), \quad z_2(t) = \dot{w}(t), \quad z_3(t) = \frac{\partial w}{\partial t}(L, t),$$

$$z_4(t) = \frac{\partial^2 w}{\partial t \partial x}(L, t).$$

Eqs. (1)–(5) are equivalent to

$$\dot{z}(t) = Az(t) + Bu(t), \quad z_0(0) = z_0, \\ y(t) = B^*z(t).$$

In Slemrod [1] it is shown that B is bounded, A is skew-adjoint with compact resolvent, and $\Sigma(A, B, B^*, 0)$ is approximately observable. In particular, A generates a unitary group on Z . ■

It is well-known that under mild conditions the output feedback action $u(t) = -y(t)$ will produce an asymptotically stable closed-loop system. We shall use the following formulation from Oostveen [2, Chapter 2].

Theorem 1.2. *Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U, Z)$ and A the infinitesimal generator of a contraction C_0 -semigroup. Assume that A has compact resolvent, and the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable. Then*

- for all $\kappa > 0$, the operator $A - \kappa BB^*$ generates a strongly stable semigroup, $T_{-\kappa BB^*}(t)$;
- the closed-loop system $\Sigma(A - \kappa BB^*, B, B^*, 0)$ is input stable, i.e.,

$$\int_0^\infty \|T_{-\kappa BB^*}(s)Bu(s)\|^2 ds \leq \frac{1}{2} \|u\|_{L_2(0, \infty)}^2, \quad u \in L_2(0, \infty);$$

- for all $u \in L_2(0, \infty)$ we have

$$\int_0^t T_{-\kappa BB^*}(t-s)u(s)ds \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

So the feedback $u(t) = -y(t)$ in the above example would produce a strongly stable system. However, in practice, the control action that one can apply is often limited by physical constraints. An example is when the magnitude of the control is constrained to be bounded. Slemrod [1] considered the case of saturated boundary control: $u(t) = -y(t)\chi(y(t))$, where

$$\chi(y) = \begin{cases} 1, & \|y\| < 1 \\ \frac{1}{\|y\|}, & \|y\| \geq 1. \end{cases} \quad (6)$$

This results in the following closed-loop semilinear system with a uniform Lipschitz nonlinearity:

$$\dot{z}(t) = Az(t) - BB^*z(t)\chi(B^*z(t)), \quad z(0) = z_0. \quad (7)$$

His main theoretical result [1, Theorem 7.1] was the following.

Theorem 1.3. *Let Z be a Hilbert space, $B \in \mathcal{L}(\mathbb{R}, Z)$ and A the infinitesimal generator of a contraction semigroup. Assume that A has compact resolvent. Then the semilinear differential equation (7) has a global mild solution $z(t; z_0)$ and if $\Sigma(A, B, B^*, 0)$ is approximately observable, i.e.,*

$$B^*T(t)z_0 = 0 \quad \text{for all } t \geq 0 \implies z = \{0\},$$

then $z(t; z_0) \rightarrow 0$ as $t \rightarrow \infty$.

Slemrod used a Lyapunov approach and several abstract results from the theory of nonlinear contraction semigroups. Using a system theoretic approach, we obtain a more general result with much shorter proofs. In our main result, Theorem 2.2, we allow for a significantly larger class of nonlinearities and a general Hilbert space for the input space.

2. Stabilization of collocated systems

Throughout the paper we consider the collocated state linear system $\Sigma(A, B, B^*, 0)$ on a real Hilbert space. In the formulation of Theorems 1.2 and 1.3 we see that the first assumes approximately controllability whereas the second assumes approximately observability.

We show that under certain assumptions on A either condition will ensure that $A - \kappa BB^*$ generates a strongly stable semigroup.

Theorem 2.1. *Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U, Z)$ and A the infinitesimal generator of a contraction C_0 -semigroup. Assume that A has compact resolvent, and that the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable or approximately observable. Then $A - \kappa BB^*$ and $A^* - \kappa BB^*$ generate strongly stable semigroups for all $\kappa > 0$.*

Proof. If the system is approximately controllable, then the assertion follows from Theorem 1.2. So it remains to show that the assertion also holds when the system $\Sigma(A, B, B^*, -)$ is approximately observable. By [4, Section 4.1] we have that the dual system $\Sigma(A^*, B, B^*, -)$ is approximately controllable, and since A^*

has a compact resolvent, we see by [Theorem 1.2](#) that the semigroup generated by $A^* - \kappa BB^*$ is strongly stable. This implies that the semigroup generated by $A - \kappa BB^*$ is weakly stable, and since $A - \kappa BB^*$ has compact resolvent, we conclude that $A - \kappa BB^*$ generates a strongly stable semigroup, see e.g. [\[5, Proposition 3.21\]](#). ■

Consider the problem of stabilizing the following collocated system by a suitable choice of control:

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), & z(0) &= z_0, \\ y(t) &= B^*z(t). \end{aligned} \quad (8)$$

The following theorem shows that by using [Theorem 1.2](#), asymptotic stability of this system holds for a wide class of nonlinear output feedbacks and general Hilbert spaces U . For the proof we refer to [Section 3](#).

Theorem 2.2. *Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U, Z)$ and A the infinitesimal generator of a contraction C_0 -semigroup. Assume that A has compact resolvent, and that the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable or approximately observable. Furthermore, let f be a function from U to U which satisfies:*

- f is locally Lipschitz continuous and $f(0) = 0$,
- there exist positive constants δ, α, γ such that
 - $\langle y, f(y) \rangle \geq \alpha \|y\|^2$ when $\|y\| < \delta$, and
 - $\langle y, f(y) \rangle \geq \gamma$ when $\|y\| \geq \delta$,

then the origin of the system

$$\dot{z}(t) = Az(t) - Bf(B^*z(t)), \quad z(0) = z_0 \quad (9)$$

is globally asymptotically stable.

Remark 2.3. The above theorem also holds when the assumption on controllability/observability is replaced by the condition that $A - BB^*$ generates a strongly stable semigroup.

[Theorem 2.2](#) leads to the following improvement on the Slemrod result.

Corollary 2.4. *Let Z, U be Hilbert spaces, $B \in \mathcal{L}(U, Z)$ and A the infinitesimal generator of a contraction C_0 -semigroup. If A has compact resolvent, and the state linear system $\Sigma(A, B, B^*, 0)$ is approximately controllable or approximately observable, then with $u(t) = -\kappa y(t)\chi(y(t))$, $\kappa > 0$, the origin of the closed-loop system*

$$\dot{z}(t) = Az(t) - \kappa BB^*z(t)\chi(B^*z(t)), \quad z(0) = z_0 \quad (10)$$

is globally asymptotically stable.

In [\[1\]](#), Slemrod remarked that his theory did not apply to the NASA spacecraft Laboratory Control Experiment (SCOLE) described in Taylor and Balakrishnan [\[6\]](#) because his [Theorem 1.3](#) requires the assumption that $U = \mathbb{R}$ and in the SCOLE model $U = \mathbb{R}^{10}$. Our [Corollary 2.4](#) does apply to this situation. While the SCOLE models are from the last century, there has been renewed interest in the stabilization of SCOLE-like models. For example, Zhao and Weiss [\[7\]](#) use a SCOLE-like model for the problem of vibration reduction in wind turbines using a tuned mass damper. They assume no restrictions on the magnitude of the control action. Our [Theorem 2.2](#) opens the possibility of designing more realistic controllers for the SCOLE-type models.

Example 2.5. As an example we take the two-dimensional input non-linearity

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(u_1, u_2) = (\phi_1(u_1), \phi_2(u_2)),$$

where $\phi_i(v) = \alpha_i \max(a_i, \min(v, b_i))$ with $a_i < 0 < b_i$ and $\alpha_i > 0$.

Note that

$$v_i \phi_i(v_i) \geq \begin{cases} \alpha_i v_i^2 & |v_i| < \delta_i = \min(-a_i, b_i) \\ \gamma_i = \alpha_i \min(a_i^2, b_i^2) & |v_i| \geq \delta_i. \end{cases}$$

To show that this nonlinearity satisfies the conditions in [Theorem 2.2](#) we introduce the rectangle in $R \subset \mathbb{R}^2$ with edges passing through the points $(a_1, 0)$, $(0, a_2)$, $(b_1, 0)$ and $(0, b_2)$. Let $\delta > 0$ be the radius of a circle with center $(0, 0)$ which is included in the rectangle, and let ε be the radius of a circle with center $(0, 0)$ that includes the rectangle. Then for $u = (u_1, u_2)$ with $\|u\| \leq \delta$, we have that $u^T f(u) \geq \min\{\alpha_1, \alpha_2\} \|u\|^2$. For $\|u\| > \varepsilon$ we have that one of the inputs is saturated. Assuming that it is u_1 that gives $u^T f(u) \geq \min\{-a_1, b_1\} |u_1|$. So in this area we have $u^T f(u) > \tilde{\gamma}$. On the compact set $\{u \in \mathbb{R}^2 \mid \delta \leq \|u\| \leq \varepsilon\}$ the continuous function $u^T f(u)$ will attain its minimum. Since $u^T f(u)$ is strictly positive on this set, this minimum is larger than zero. Combining this with the above, we see that we can find a $\gamma > 0$ such that $u^T f(u) > \gamma$ for $\|u\| \geq \delta$. Hence for this nonlinearity the conditions in [Theorem 2.2](#) are satisfied. These observations generalize to higher dimensional input nonlinearities.

3. Proofs

Proof of Theorem 2.2. Since f is (locally) Lipschitz continuous applying [\[8, Theorem 6.1.2\]](#), we conclude that for $z_0 \in Z$ there exists $t_{\max} > 0$ such that (9) has a unique mild solution on $[0, t_{\max})$ and $z(\cdot; z_0) \in C(0, \tau; Z)$ for $0 \leq \tau < t_{\max}$. Furthermore, for $z_0 \in D(A)$, $z(\cdot; z_0)$ is a classical solution, i.e., $z(\cdot; z_0) \in C^1([0, \tau]; Z)$, $z(t; z_0) \in D(A)$ for all $t \in [0, \tau]$ and $z(t; z_0)$ satisfies (9) for all $t \in [0, \tau]$ (see [\[8, Theorem 6.1.5\]](#)). Define the Fréchet differentiable functional $V(z) := \|z\|^2$. If $z_0 \in D(A)$, then $V(t) := V(z(t; z_0))$ is differentiable for $t \in [0, t_{\max})$ and using (9) we find

$$\begin{aligned} \dot{V}(t) &= \langle Az(t), z(t) \rangle + \langle z(t), Az(t) \rangle \\ &\quad - \langle z(t), Bf(B^*z(t)) \rangle - \langle Bf(B^*z(t)), z(t) \rangle \\ &\leq -\langle B^*z(t), f(B^*z(t)) \rangle - \langle f(B^*z(t)), B^*z(t) \rangle \\ &\leq \begin{cases} -2\alpha \|B^*z(t)\|^2 & \|B^*z(t)\| < \delta \\ -2\gamma & \|B^*z(t)\| \geq \delta \end{cases} \\ &\leq 0, \end{aligned}$$

where we have used the assumptions that U is a real Hilbert space, A generates a contraction semigroup and the assumptions on f . So integrating gives

$$\begin{aligned} V(t) &= \|z(t; z_0)\|^2 \\ &\leq \|z_0\|^2 - 2 \int_0^t \langle f(B^*z(s; z_0)), B^*z(s; z_0) \rangle ds \leq \|z_0\|^2. \end{aligned} \quad (11)$$

By the continuity with respect to initial conditions, this extends to all $z_0 \in Z$. Hence by [\[8, Theorem 6.1.4\]](#) we have $t_{\max} = \infty$ and there exists a bounded global solution. Moreover, for all $t \geq 0$, from [\[9, Theorem 3.9\]](#) we have

$$\begin{aligned} \dot{V}_+(z(t; z_0)) &= \lim_{h \downarrow 0} \frac{V(z(t+h; z_0)) - V(z(t; z_0))}{h} \\ &\leq -2 \langle B^*z(t; z_0), f(B^*z(t; z_0)) \rangle \leq 0. \end{aligned}$$

Hence V is a Lyapunov functional for (9) and the origin is Lyapunov stable.

Let $\Omega_1 := \{t : \|B^*z(t; z_0)\| \geq \delta\}$ and $\Omega_2 := \{t : \|B^*z(t; z_0)\| < \delta\}$. So from the conditions on f we obtain

$$\int_{\Omega_1} \langle B^*z(s; z_0), f(B^*z(s; z_0)) \rangle ds \geq \gamma \mu(\Omega_1),$$

and taking the limit as $t \rightarrow \infty$ in (11) implies that Ω_1 has finite measure. Moreover, from (11) and the conditions on f we obtain

$$\begin{aligned} \infty &> \int_{\Omega_2} \langle B^*z(s; z_0), f(B^*z(s; z_0)) \rangle ds \\ &\geq \alpha \int_{\Omega_2} \|B^*z(s; z_0)\|^2 ds. \end{aligned} \quad (12)$$

Thus

$$\int_0^\infty \|B^*z(s; z_0)\|^2 ds = \left(\int_{\Omega_1} + \int_{\Omega_2} \right) \|B^*z(s; z_0)\|^2 ds < \infty. \quad (13)$$

Reformulate the closed-loop system (9) as

$$\dot{z}(t) = (A - BB^*)z(t) + B[B^*z(t) - f(B^*z)], \quad z(0) = z_0,$$

so that the closed-loop solution is also given by

$$\begin{aligned} z(t; z_0) &= T_{-BB^*}(t)z_0 + \int_0^t T_{-BB^*}(t-s)B[B^*z(s; z_0) \\ &\quad - f(B^*z(s; z_0))]ds. \end{aligned} \quad (14)$$

By Theorem 1.2.a. or Theorem 2.1 the semigroup $T_{-BB^*}(t)$ is strongly stable.

Since $\mu(\Omega_1)$ is finite and $f(B^*z(t; z_0))$ is a continuous function of t , we have

$$\int_{\Omega_1} \|f(B^*z(s; z_0))\|^2 ds < \infty.$$

Now $\|B^*z(s; z_0)\| < \delta$ for $s \in \Omega_2$ and since f is locally Lipschitz continuous we have that there exists a $L_\delta > 0$ such that

$$\int_{\Omega_2} \|f(B^*z(s; z_0))\|^2 ds \leq L_\delta \int_{\Omega_2} \|B^*z(s; z_0)\|^2 ds < \infty,$$

where we have also used (12).

Thus $f(B^*z(\cdot; z_0)) \in L_2(0, \infty)$ and so with (13) we can apply Theorem 1.2 to the closed loop system (14) to complete the proof. ■

Proof of Corollary 2.4. We have $f(y) = \kappa y \chi(y)$ and so

$$\langle y, f(y) \rangle = \begin{cases} \kappa \|y\|^2 & \|y\| < 1 \\ \kappa \|y\| \geq \kappa & \|y\| \geq 1. \end{cases}$$

First we show that $y \chi(y)$ is uniformly Lipschitz continuous, by proving the following inequality:

$$\|y_1 \chi(y_1) - y_2 \chi(y_2)\| \leq 2\|y_1 - y_2\|. \quad (15)$$

If $\|y_1\|, \|y_2\| < 1$ then (15) is trivially satisfied. So we consider the other two cases. Note that without loss of generality, we may assume that $\|y_1\| \leq \|y_2\|$.

Case 1: $\|y_1\| < 1, \|y_2\| \geq 1$.

First we find that

$$\|y_2\| \leq \|y_1 - y_2\| + \|y_1\| < \|y_1 - y_2\| + 1.$$

With this we find that

$$\begin{aligned} &\|y_1 \chi(y_1) - y_2 \chi(y_2)\| \\ &\leq \|y_1 - y_2\| + \left\| y_2 - \frac{y_2}{\|y_2\|} \right\| \\ &= \|y_1 - y_2\| + \|y_2\| \left[1 - \frac{1}{\|y_2\|} \right] \\ &= \|y_1 - y_2\| + \|y_2\| - 1 \leq 2\|y_1 - y_2\|. \end{aligned}$$

Case 2: $\|y_1\| \geq 1, \|y_2\| \geq 1$.

As in case 1. we find that $\|y_2\| \leq \|y_1 - y_2\| + \|y_1\|$ or equivalently $\|y_2\| - \|y_1\| \leq \|y_1 - y_2\|$. Furthermore,

$$\begin{aligned} &\|y_1 \chi(y_1) - y_2 \chi(y_2)\| \\ &= \frac{1}{\|y_1\|} \left\| y_1 - \frac{\|y_1\|}{\|y_2\|} y_2 \right\| \\ &\leq \frac{1}{\|y_1\|} \left(\|y_1 - y_2\| + \|y_2\| \left[1 - \frac{\|y_1\|}{\|y_2\|} \right] \right) \\ &= \frac{1}{\|y_1\|} (\|y_1 - y_2\| + \|y_2\| - \|y_1\|) \\ &\leq 2\|y_1 - y_2\|, \end{aligned}$$

where in the last step we used the assumption $\|y_1\| \geq 1$. So we have shown that $y \mapsto y \chi(y)$ and hence f are uniformly Lipschitz continuous and Theorem 2.2 completes the proof. ■

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