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Full Length Article

Perspectives in modeling for control of power networks

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ABSTRACT

Increasing integration of renewable energy sources in the power grid leads to a complete re-thinking of its operation. This necessitates the consideration of new modeling and analysis approaches, which can serve as natural starting points for robust and scalable control and design strategies.

In this 'vision' paper we will highlight two topics within the broad area of power networks: the modeling and analysis of the synchronous generator, and the modeling and analysis of power networks using the swing equation as an approximate model for the generator. In both cases we will discuss a port-Hamiltonian formulation, which reflects the underlying physics of power flow and energy storage. Although the port-Hamiltonian model of the synchronous generator reveals a clear structure it still poses fundamental challenges for its non-zero steady state stability analysis. It is shown how the swing equation can be directly deduced from the power balance of the port-Hamiltonian model of the synchronous generator. Under the phasor assumption, this leads to a port-Hamiltonian model of power networks of generators, which enables a straightforward stability analysis and provides a starting point for control.

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1. Introduction

The study of power networks has recently gained much attention from different angles, including the systems and control point of view. Fundamental reasons for the growing interest are the increasing share of *renewables*, such as wind and solar energy, in the electricity production and resulting fundamental challenges in the operation of the network, and at the same time the availability of advanced information technology. This has led, or will lead in the near future, to a re-thinking of the operation of the power grid, replacing the classical top-down structure of energy distribution, from a small number of major power plants to consumers, by a much more horizontal and distributed structure. In such a distributed structure many of the consumers are partially or temporarily producers of energy as well (called *prosumers*). The matching of supply and demand in the resulting complex multi-player setting, subject to a large degree of uncertainty in the energy production of renewables, leads to an operation of the power grid that is increasingly often near its capacity constraints and necessitates the use of advanced communication and control techniques ('*smart grids*'). Another important aspect in the restructuring of

the operation of the power grid is the concept of a *micro-grid*. A micro-grid is generally understood as a small-scale power network that is able to operate independently of the main (high-voltage) grid, and therefore is less vulnerable to faults and power outages elsewhere.

In the light of these developments it seems relevant to fundamentally reconsider the basic modeling approaches to power networks, and to investigate how more accurate models reflecting the underlying physics, as well as improved analysis and control concepts, can contribute to novel robust and scalable control methodologies. Indeed, while up to now for many of the control problems in the operation of the power network fairly simple models or linearizations of more involved models could be satisfactory, this is not the case anymore for the control challenges that are posed by the new ways of operating the power grid. In addition, also the classical problem of the stability of the large-scale power network after major disruptions (faults) is still largely open.

Instead of providing a general vision on the area, we restrict ourselves in this introductory and tutorial paper to two topics within the modeling and analysis of power networks. The first topic concerns the detailed modeling for control of the *synchronous generator*, which is (still) the main working horse of the large power grid. The synchronous generator constitutes a fascinating multi-physics system which can be modeled from first principles resulting in a classical eight-dimensional model. We will describe the main characteristics and the mathematical structure of this

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model by utilizing the *port-Hamiltonian* framework. This framework is ideally suited for multi-physics systems and emphasizes aspects of power flow and energy storage, and therefore provides a natural setting for the analysis and control of the dynamics of (networks of) synchronous generators. Although the structure of the port-Hamiltonian formulation of the classical eight-dimensional model is very clear, the analysis and control of this model poses fundamental challenges. Indeed, while the Hamiltonian (total energy) is a natural Lyapunov function for stability of the zero equilibrium, the nonlinearity of the model makes it intrinsically hard to study the stability of steady state values corresponding to non-zero mechanical torque (the interesting case in applications). In this paper we will discuss some of the available analysis methods, and end with a number of numerical simulation studies that motivate further research.

The second topic of this paper concerns the modeling of power networks under the assumption of pure sinusoidal currents and voltages of nominal frequency. Under this assumption the currents and voltages can be compactly represented by their *phasors*, while the network of linear transmission lines linking the generators and the loads can be modeled by an admittance matrix. A trait-d'union between the phasor representation of the transmission line network and the eight-dimensional model of the synchronous generator can be established by a crude approximation of the eight-dimensional model by means of the *swing equation*. We present a simple reasoning leading to the swing equation based on the power-balance of the port-Hamiltonian model of the eight-dimensional model, which we believe to be fairly novel. It turns out that the resulting system of generators and loads linked by the network of transmission lines again defines a port-Hamiltonian system, which admits a straightforward steady state stability analysis. In particular, the model enjoys the property of *shifted passivity* with respect to steady state values, which is an ideal starting point for robust control.

Disclaimer: This paper is not meant to be an introductory survey to modeling and control of power networks, neither a balanced view on new research directions. Instead it focusses on some of our personal (and limited) understandings and tastes in this area, and we are sure that others will write about their own.

2. Port-Hamiltonian modeling of the synchronous generator

An essential component of the alternating-current power grid is the *synchronous generator*; see e.g. Kundur (1993), Machowski, Bialek, and Bumby (2008), and Sauer and Pai (1998). Although the contribution of renewables such as solar and wind energy is growing, the synchronous generators still are the main working horses of the large power grid. While less dominant, they are also important in micro-grids.

The standard model for the synchronous generator, as described e.g. in Kundur (1993), is based on first principles physical modeling. As such it can be naturally written into port-Hamiltonian form given as, see Fiaz, Zonetti, Ortega, Scherpen, and van der Schaft (2013) for details,

$$\begin{bmatrix} \dot{\psi}_s \\ \dot{\psi}_r \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R_s & 0_{33} & 0_{31} & 0_{31} \\ 0_{33} & -R_r & 0_{31} & 0_{31} \\ 0_{13} & 0_{13} & -d & -1 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix} + \begin{bmatrix} I_3 & 0_{31} & 0_{31} \\ 0_{33} & e_1 & 0_{31} \\ 0_{13} & 0 & 1 \\ 0_{13} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_s \\ V_f \\ \tau \end{bmatrix}$$

Fig. 1. The state and port variables of the synchronous generator.

$$\begin{bmatrix} I_s \\ I_f \\ \omega \end{bmatrix} = \begin{bmatrix} I_3 & 0_{33} & 0_{31} & 0_{31} \\ 0_{13} & e_1^T & 0 & 0 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix}, \quad (1)$$

where 0_{lk} denotes the $l \times k$ zero matrix, I_3 denotes the 3×3 identity matrix, and e_1 is the first basis vector of $\mathbb{R}^3$. Eq. (1) is in port-Hamiltonian input-state-output form (van der Schaft, 1996; van der Schaft & Jeltsema, 2014; van der Schaft & Maschke, 1995)

$$\dot{x} = (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + g(x)u,$$

$$y = g^T(x) \frac{\partial H}{\partial x}(x),$$

$$J(x) = -J^T(x), \quad R(x) = R^T(x) \geq 0, \quad (2)$$

with a *Poisson structure matrix* $J(x)$ given by the constant matrix

$$J = \begin{bmatrix} 0_{66} & 0_{62} \\ 0_{26} & 0_{22} \end{bmatrix}, \quad (3)$$

and a *resistive structure matrix* $R(x)$, which is also constant, having diagonal blocks

$$R_s = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix}, \quad R_r = \begin{bmatrix} r_f & 0 & 0 \\ 0 & r_{kd} & 0 \\ 0 & 0 & r_{kq} \end{bmatrix}, \quad d, \quad 0, \quad (4)$$

denoting, respectively, the *stator resistances*, *rotor resistances* and *mechanical friction*. Here $\frac{\partial H}{\partial x}(x)$ throughout denotes the column vector of partial derivatives of the function H .

The state variables x of the eight-dimensional model of the synchronous generator comprise of

- $\psi_s \in \mathbb{R}^3$, the *stator fluxes*,
- $\psi_r \in \mathbb{R}^3$, the *rotor fluxes*; the first one corresponding to the *field winding* and the remaining two to the *damper windings*,
- p , the *angular momentum* of the rotor,
- θ , the *angle* of the rotor.

Moreover, $V_s \in \mathbb{R}^3$, $I_s \in \mathbb{R}^3$ are the three-phase¹ *stator terminal voltages and currents*, V_f , I_f are the rotor *field winding voltage and current*, and τ , ω are the *mechanical torque and angular velocity*; see Fig. 1 for a schematic view.

The *Hamiltonian* H (total stored energy of the synchronous generator) is the sum of the magnetic energy of the generator and the kinetic energy of the rotating rotor, given as the sum of the two nonnegative terms:

$$H(\psi_s, \psi_r, p, \theta) = \frac{1}{2} \begin{bmatrix} \psi_s^T & \psi_r^T \end{bmatrix} L^{-1}(\theta) \begin{bmatrix} \psi_s \\ \psi_r \end{bmatrix} + \frac{1}{2J_r} p^2$$

= magnetic energy H_m + kinetic energy H_k , (5)

¹ Note that the vectors V_s , I_s are not assumed to be *balanced* (sum zero); see also (21) in the next section.

where J_r is the rotational inertia of the rotor (not to be confused with the Poisson structure matrix J), and $L(\theta)$ is a 6×6 inductance matrix.

In the *round rotor* case (no saliency; cf. Kundur, 1993; Machowski et al., 2008)

$$L(\theta) = \begin{bmatrix} L_{ss} & L_{sr}(\theta) \\ L_{sr}^T(\theta) & L_{rr} \end{bmatrix} \quad (6)$$

where

$$L_{ss} = \begin{bmatrix} L_{aa} & -L_{ab} & -L_{ab} \\ -L_{ab} & L_{aa} & -L_{ab} \\ -L_{ab} & -L_{ab} & L_{aa} \end{bmatrix}, \quad L_{rr} = \begin{bmatrix} L_{ffd} & L_{akd} & 0 \\ L_{akd} & L_{kkd} & 0 \\ 0 & 0 & L_{kkq} \end{bmatrix} \quad (7)$$

while

$$L_{sr}(\theta) = \begin{bmatrix} \cos \theta & \cos \theta & -\sin \theta \\ \cos(\theta - \frac{2\pi}{3}) & \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) \\ \cos(\theta + \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \end{bmatrix} \times \begin{bmatrix} L_{afd} & 0 & 0 \\ 0 & L_{akd} & 0 \\ 0 & 0 & L_{akq} \end{bmatrix} \quad (8)$$

Remark 2.1. This can be extended to the saliency case, and to *saturation*. Saliency corresponds to a more complicated expression for $L(\theta)$ (see Kundur, 1993), while saturation can be captured by replacing the purely quadratic dependence of the magnetic energy on the flux variables ψ_s , ψ_r by a more involved one.

Remark 2.2. Many synchronous generators, in particular turbo-generators, have a solid-steel rotor body which acts as a screen in the q -axis. This effect is often taken into account by an additional damper winding along the q -axis, leading to a nine-dimensional model (Machowski et al., 2008) and corresponding port-Hamiltonian formulation (Stegink, De Persis, & van der Schaft, in preparation).

A crucial feature of the magnetic energy term H_m in the Hamiltonian H is its dependency on the mechanical rotor angle θ ; see the formula (8) for $L_{sr}(\theta)$. This dependence is responsible for the interaction between the mechanical domain of the generator (the mechanical motion of the rotor) and the electro-magnetic domain (the dynamics of the magnetic fields in the rotor and stator), and thus for the functioning of the synchronous generator as an energy-conversion device, transforming mechanical power into electrical power.

The synchronous generator is a fascinating case of a multi-physics system, connected to its environment by three types of ports; see Fig. 1. First, there is the scalar *mechanical port* with power variables τ , ω , to be interconnected to a *prime mover*, such as a turbine. This port is also used for control purposes, e.g., via so-called *droop control*. Second, there are three *stator terminal ports*, with vectors of power variables V_s , I_s , which denote the vectors of three-phase voltages and currents to be delivered to the electrical network. Third, there is the port with scalar power variables V_f , I_f , which is responsible for the magnetization of the rotor, and which is controlled by the *excitation system* (Kundur, 1993; Machowski et al., 2008; Pai, 1989).

Note that as always in a port-Hamiltonian formulation the partial derivatives of the Hamiltonian H have a direct physical interpretation (Fiaz et al., 2013; van der Schaft, 1996; van der Schaft & Jeltsema, 2014; van der Schaft & Maschke, 1995). In fact,

$$I_s = \frac{\partial H}{\partial \psi_s}, \quad I_r = \frac{\partial H}{\partial \psi_r} \quad (9)$$

are the vectors of *stator currents* and *rotor currents* respectively, and

$$\omega = \frac{\partial H}{\partial p} \quad (10)$$

is the *angular velocity* of the rotor. Furthermore,

$$\tau_m = \frac{\partial H}{\partial \theta} = \frac{1}{2} \psi^T \frac{\partial}{\partial \theta} L^{-1}(\theta) \psi \quad (11)$$

is the *air-gap torque*, i.e., the mechanical torque acting on the rotor due to rotating magnetic field.

It follows from (1) that the torque balance in the mechanical domain is given by

$$\dot{p} = -d\omega - \tau_m + \tau, \quad (12)$$

where τ is the external torque exerted on the rotor (e.g., by a turbine), $d\omega$ is the torque due to friction in the rotational motion of the rotor (for simplicity assumed to be linear in the angular velocity ω), and τ_m is the air-gap torque.

The system, being port-Hamiltonian, satisfies the following power-balance with respect to its three ports

$$\frac{d}{dt} H = I_s^T V_s + I_f V_f + \omega \tau - I_s^T R_s I_s - I_r^T R_r I_r - d\omega^2 \quad (13)$$

Here $\omega \tau$ defines the *mechanical power* delivered to the synchronous generator (by a prime mover such as a turbine), $-I_s^T V_s$ denotes the *electrical power* delivered to the network, $I_f V_f$ is the power delivered by the excitation system, and finally $I_s^T R_s I_s$, $I_r^T R_r I_r$, $d\omega^2$ denote the power losses due to resistance (in the stator, respectively rotor) and mechanical friction. In particular, the synchronous generator (1) is *passive* (van der Schaft, 1996) with respect to its three ports, with storage function H .

In normal circumstances, the magnitude of the terms $\omega \tau$ and $-I_s^T V_s$ will be large as compared to the terms $I_f V_f$, $I_s^T R_s I_s$, $I_r^T R_r I_r$, $d\omega^2$. In fact, in steady-state operation one would expect that $\omega \tau \approx -I_s^T V_s$ (the ideal scenario being that all mechanical power is converted into electrical power).

Although typically the power at the excitation port (V_f , I_f) will be small, its presence is crucial for the magnetization of the rotor and thus for the functioning of the synchronous generator; see also Section 4.1.1. The excitation port is also important for control purposes, in particular for regulating the voltage amplitude at the stator terminals.

We finally note that, as is commonly the case for physical models, the model (1) is *bi-directional*. In particular, it also models a *synchronous motor*. In this case, $I_s^T V_s$ would be the electrical power delivered to the motor, and $-\omega \tau$ would be the mechanical power delivered by the motor. Thus the above port-Hamiltonian model (1) is really the model of a *synchronous machine*, covering both synchronous generators and motors.

3. Analysis of the synchronous generator by the Blondel–Park transformation

The dynamics of the eight-dimensional model (1) is difficult to analyze; especially for non-zero mechanical torque τ (from an application perspective the only interesting situation). A main reason for this is the same fact that is responsible for the functioning of the synchronous generator: the magnetic energy $H_m = [\psi_s^T \quad \psi_r^T]^T L^{-1}(\theta) [\psi_s \quad \psi_r]^T$ is depending on the mechanical angle θ .

There is a way, however, to eliminate the dependence of the magnetic energy on θ , by *transforming* the stator flux variables ψ_s depending on θ . This is known as the *Blondel–Park* or *dq0* transformation, defined as the coordinate transformation

(Fiaz et al., 2013; Kundur, 1993; Machowski et al., 2008)

$$\begin{aligned} \psi_{dq0} &= \begin{bmatrix} \psi_d \\ \psi_q \\ \psi_0 \end{bmatrix} := \mathbb{T}_{dq0}(\theta) \psi_s \\ &:= \sqrt{\frac{2}{3}} \begin{bmatrix} \cos \theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ \sin \theta & \sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \psi_s \end{aligned} \quad (14)$$

Then the vector V_s transforms in the same way to $V_{dq0} := \mathbb{T}_{dq0}(\theta)V_s$, while the vector of currents I_s transforms to $I_{dq0} := \mathbb{T}_{dq0}^T(\theta)I_s$, implying that $I_s^T V_s = I_{dq0}^T V_{dq0}$; and thus power is preserved.

Since the Blondel–Park transformation satisfies² the orthonormality property

$$\mathbb{T}_{dq0}^{-1}(\theta) = \mathbb{T}_{dq0}^T(\theta), \quad (15)$$

I_s and V_s actually transform in the same way; that is, $I_{dq0} = \mathbb{T}_{dq0}(\theta)I_s$, $V_{dq0} := \mathbb{T}_{dq0}(\theta)V_s$.

Note that I_0 and V_0 measure the ‘unbalance’ in the three-phase current and voltage; in ideal operation I_0 and V_0 are zero.

The Hamiltonian H transforms to a Hamiltonian $\mathcal{H}$ expressed in the new coordinates $x_{dq0} := (\psi_{dq0}, \psi_r, p, \theta)$, whose magnetic part is given by

$$\mathcal{H}_m = \frac{1}{2} \begin{bmatrix} \psi_{dq0}^T & \psi_r^T \end{bmatrix} \mathcal{L}^{-1} \begin{bmatrix} \psi_{dq0} \\ \psi_r \end{bmatrix}, \quad (16)$$

where

$$\begin{aligned} \mathcal{L} &= \begin{bmatrix} \mathbb{T}_{dq0}(\theta) & 0 \\ 0 & I_3 \end{bmatrix} L(\theta) \begin{bmatrix} \mathbb{T}_{dq0}^T(\theta) & 0 \\ 0 & I_3 \end{bmatrix} \\ &= \begin{bmatrix} L_d & 0 & 0 & \sqrt{\frac{3}{2}}L_{afd} & \sqrt{\frac{3}{2}}L_{akd} & 0 \\ 0 & L_q & 0 & 0 & 0 & -\sqrt{\frac{3}{2}}L_{akq} \\ 0 & 0 & L_0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}L_{afd} & 0 & 0 & L_{ffd} & L_{akd} & 0 \\ \sqrt{\frac{3}{2}}L_{akd} & 0 & 0 & L_{akd} & L_{kkd} & 0 \\ 0 & -\sqrt{\frac{3}{2}}L_{akq} & 0 & 0 & 0 & L_{kkq} \end{bmatrix}, \end{aligned} \quad (17)$$

and we conclude that $\mathcal{L}$ is independent of θ . Thus the Blondel–Park transformation results in an expression of the magnetic energy $\mathcal{H}_m$, given by (16), which is independent of the mechanical angle θ .

On the other hand, the elimination of θ from the magnetic energy expression $\mathcal{H}_m$ has other consequences. In fact, in the new coordinates $x_{dq0} = (\psi_{dq0}, \psi_r, p, \theta)$ the eight-dimensional system takes the form (Fiaz et al., 2013)

$$\begin{bmatrix} \dot{\psi}_{dq0} \\ \dot{\psi}_r \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R_s & 0 & \begin{bmatrix} -\psi_q \\ \psi_d \end{bmatrix} & 0 \\ 0 & -R_r & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & 0 \\ \begin{bmatrix} \psi_q & -\psi_d & 0 \end{bmatrix} & 0 & -d & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi_{dq0}} \\ \frac{\partial \mathcal{H}}{\partial \psi_r} \\ \frac{\partial \mathcal{H}}{\partial p} \\ 0 \end{bmatrix}$$

² Actually, the choice of the constants in (14) is dictated by this orthonormality property.

$$\begin{aligned} &+ \begin{bmatrix} I_3 & 0_{31} & 0_{31} \\ 0_{33} & e_1 & 0_{31} \\ 0_{13} & 0 & 1 \\ 0_{13} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{dq0} \\ V_f \\ \tau \end{bmatrix} \\ \begin{bmatrix} I_{dq0} \\ I_f \\ \omega \end{bmatrix} &= \begin{bmatrix} I_3 & 0_{33} & 0_{31} & 0_{31} \\ 0_{13} & e_1^T & 0 & 0 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi_{dq0}} \\ \frac{\partial \mathcal{H}}{\partial \psi_r} \\ \frac{\partial \mathcal{H}}{\partial p} \\ 0 \end{bmatrix} \end{aligned} \quad (18)$$

Here the resistive matrix R_s is the same as before (due to the orthonormality of $\mathbb{T}_{dq0}$), but the constant Poisson structure matrix J given by (3) has been transformed into the *state dependent* Poisson structure

$$\mathcal{J}(x_{dq0}) := \begin{bmatrix} & 0_{33} & 0_{33} & \begin{bmatrix} -\psi_q \\ \psi_d \\ 0 \end{bmatrix} & 0_{31} \\ & 0_{33} & 0_{33} & \begin{bmatrix} 0 \\ 0_{31} \end{bmatrix} & 0_{31} \\ \begin{bmatrix} \psi_q & -\psi_d & 0 \end{bmatrix} & 0_{13} & 0 & 0 & -1 \\ 0_{13} & 0_{13} & 1 & 0 & 0 \end{bmatrix} \quad (19)$$

Hence, in the new coordinates the expression $\frac{\partial \mathcal{H}}{\partial \theta}$ equals zero (no dependence on θ), but on the other hand, the air-gap torque τ_m now shows up as the product of the seventh row of the transformed Poisson structure matrix $\mathcal{J}(x_{dq0})$ with $\frac{\partial \mathcal{H}}{\partial x_{dq0}}$. Indeed, the air-gap torque is equivalently given as

$$\tau_m = \psi_d I_q - \psi_q I_d \quad (20)$$

Thus from a mathematical point of view we have reduced the complexity of the expression of the magnetic energy, however sacrificing the simplicity of the Poisson structure matrix.

An important advantage of the transformed model (18) is the fact that we can split off the dynamics of θ , and consider *steady states* of the reduced seven-dimensional model, instead of *periodic motions* of the original eight-dimensional model. Furthermore, it is clear that the dynamics of ψ_0 is given by

$$\dot{\psi}_0 = -r_s \frac{\partial \mathcal{H}}{\partial \psi_0} + V_0, \quad (21)$$

which in view of the form of $\mathcal{L}$ in (17) is independent of the rest of the dynamics. Moreover, for a constant V_0 , the dynamics (21) is asymptotically stable if and only if $r_s > 0$.

Remark 3.1. Notice that $L_0 \approx 0$ as stated on Kundur (1993, p. 65). In fact, if the so-called *leakage flux not crossing the air-gap* would be neglected then $L_0 = 0$. In this case (21) reduces to the algebraic equation $-r_s I_0 + V_0 = 0$.

Hence for analysis purposes we can reduce the model (18) to the six-dimensional port-Hamiltonian system

$$\begin{aligned} \begin{bmatrix} \dot{\psi}_d \\ \dot{\psi}_q \\ \dot{\psi}_r \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} -r_s & 0 \\ 0 & r_s \\ 0_{32} & -R_r \\ \begin{bmatrix} \psi_q & -\psi_d \end{bmatrix} & 0_{13} \end{bmatrix} \begin{bmatrix} -\psi_q \\ \psi_d \\ \psi_d \\ -d \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi_d} \\ \frac{\partial \mathcal{H}}{\partial \psi_q} \\ \frac{\partial \mathcal{H}}{\partial \psi_r} \\ \frac{\partial \mathcal{H}}{\partial p} \end{bmatrix} \\ &+ \begin{bmatrix} I_2 & 0_{21} & 0_{21} \\ 0_{32} & e_1 & 0_{31} \\ 0_{12} & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{dq} \\ V_f \\ \tau \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} I_{dq} \\ I_f \\ \omega \end{bmatrix} = \begin{bmatrix} I_2 & 0_{23} & 0_{21} \\ 0_{12} & e_1^T & 0 \\ 0_{12} & 0_{13} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\mathcal{H}}}{\partial \psi_d} \\ \frac{\partial \hat{\mathcal{H}}}{\partial \psi_q} \\ \frac{\partial \hat{\mathcal{H}}}{\partial \psi_r} \\ \frac{\partial \hat{\mathcal{H}}}{\partial p} \end{bmatrix}, \quad (22)$$

where $V_{dq}, I_{dq} \in \mathbb{R}^2$ are the first two components of V_{dq0} , respectively I_{dq0} . Furthermore, the Hamiltonian $\hat{\mathcal{H}}$ of the six-dimensional port-Hamiltonian system is equal to the Hamiltonian $\mathcal{H}$ of the original eight-dimensional model minus the term $\frac{1}{2L_0} \psi_d^2$.

Note that also the six-dimensional model is *passive* with respect to the inputs V_f, V_{dq}, τ and outputs I_f, I_{dq}, ω , with storage function given by $\hat{\mathcal{H}}$. In fact, the following power balance is derived from (13)

$$\frac{d}{dt} \hat{\mathcal{H}} = I_{dq}^T V_{dq} + I_f V_f + \omega \tau - r_s I_{dq}^T I_{dq} - I_f^T R_f I_f - d \omega^2 \quad (23)$$

Hence, for zero inputs $V_f = 0, V_{dq} = 0, \tau = 0$ the zero equilibrium $\psi_d = \psi_q = 0, \psi_r = 0, p = 0$ is *globally asymptotically stable*.

This can be extended to *shifted passivity* with respect to any constant $\bar{V}_f, \bar{V}_{dq}$ and $\bar{\tau} = 0$ as follows. The standard method (Maschke, Ortega, & van der Schaft, 2000; van der Schaft, 1996; van der Schaft & Jeltsema, 2014) of studying shifted passivity for a general port-Hamiltonian system (2) is based on adding to its Hamiltonian H a function determined by the input matrix $g(x)$ and the complete structure matrix $J(x) - R(x)$, provided additional *integrability conditions* are satisfied (Fiaz et al., 2013; Maschke et al., 2000).

In case of constant matrices $J - R$ and g (not depending on x) these integrability conditions are automatically satisfied, and the method is equivalent to considering the *shifted Hamiltonian* (sometimes called Bregman or availability function)

$$\tilde{H}(x) := H(x) - \frac{\partial^T H}{\partial x}(\bar{x})(x - \bar{x}) - H(\bar{x}), \quad (24)$$

with respect to a steady state $\bar{x}$ corresponding to $\bar{u}$, that is

$$(J - R) \frac{\partial H}{\partial x}(\bar{x}) + g \bar{u} = 0$$

Then

$$\begin{aligned} \frac{d}{dt} \tilde{H} &= -\frac{\partial^T \tilde{H}}{\partial x}(x) R \frac{\partial \tilde{H}}{\partial x}(x) + (y - \bar{y})^T (u - \bar{u}) \\ &\leq (y - \bar{y})^T (u - \bar{u}), \end{aligned} \quad (25)$$

where $\bar{y} = g^T \frac{\partial H}{\partial x}(\bar{x})$ is the corresponding steady state output value.

In case the original Hamiltonian H is *convex*, the shifted Hamiltonian $\tilde{H}$ will have a minimum at $\bar{x}$, thus proving passivity with respect to the shifted inputs $u - \bar{u}$ and outputs $y - \bar{y}$, with storage function $\tilde{H}$. This is called *shifted passivity*.

Although the Poisson structure matrix in (22) is state-dependent, the integrability conditions mentioned above turn out to be satisfied for any constant $\bar{V}_{dq}, \bar{V}_f$ and $\bar{\tau} = 0$ (Fiaz et al., 2013). In fact, continuing on Fiaz et al. (2013), we can define the *augmented Hamiltonian*

$$\hat{\mathcal{H}}_{\text{aug}}(\psi_{dq}, \psi_r, p) := \hat{\mathcal{H}}(\psi_{dq}, \psi_r, p) - \frac{1}{r_f} \bar{V}_f \psi_f - \frac{1}{r_s} \bar{V}_{dq}^T \psi_{dq} \quad (26)$$

(with ψ_f the first component of the rotor flux vector ψ_r), which has a minimum at the steady state values $(\bar{\psi}_{dq}, \bar{\psi}_r, \bar{p})$, and is in fact, up to a constant, equal to the shifted Hamiltonian derived from $\hat{\mathcal{H}}$ as in (24). With the aid of this augmented Hamiltonian $\hat{\mathcal{H}}_{\text{aug}}$ the six-dimensional model (22) can then be rewritten as the

following port-Hamiltonian system

$$\begin{bmatrix} \dot{\psi}_d \\ \dot{\psi}_q \\ \dot{\psi}_r \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -[r_s & 0] & 0_{23} & [-\psi_q] \\ 0 & r_s & -R_r & \psi_d \\ \psi_q & -\psi_d & 0_{13} & -d \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial \psi_d} \\ \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial \psi_q} \\ \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial \psi_r} \\ \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial p} \end{bmatrix} + \begin{bmatrix} I_2 & 0_{21} & 0_{21} \\ 0_{32} & e_1 & 0_{31} \\ 0_{12} & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{dq} - \bar{V}_{dq} \\ V_f - \bar{V}_f \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} I_{dq} - \bar{I}_{dq} \\ I_f - \bar{I}_f \\ \omega \end{bmatrix} = \begin{bmatrix} I_2 & 0_{23} & 0_{21} \\ 0_{12} & e_1^T & 0 \\ 0_{12} & 0_{13} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial \psi_d} \\ \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial \psi_q} \\ \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial \psi_r} \\ \frac{\partial \hat{\mathcal{H}}_{\text{aug}}}{\partial p} \end{bmatrix}, \quad (27)$$

where $\bar{I}_{dq}, \bar{I}_f$ are the steady-state values. This immediately implies that the system is passive with respect to the shifted inputs $V_{dq} - \bar{V}_{dq}, V_f - \bar{V}_f$ and shifted outputs $I_{dq} - \bar{I}_{dq}, I_f - \bar{I}_f$, and the original input τ and output ω at the mechanical port, that is

$$\frac{d}{dt} \hat{\mathcal{H}}_{\text{aug}} \leq (I_{dq} - \bar{I}_{dq})^T (V_{dq} - \bar{V}_{dq}) + (I_f - \bar{I}_f) (V_f - \bar{V}_f) + \omega \tau$$

This also implies that for $V_{dq} = \bar{V}_{dq}, V_f = \bar{V}_f$ and $\tau = 0$ the steady state is *globally asymptotically stable*.

Unfortunately, as shown in Fiaz et al. (2013), this analysis cannot be extended to non-zero constant $\bar{\tau}$, since in this case the integrability conditions are *not* satisfied for $r_s \neq 0$. Hence, the case of non-zero $\bar{\tau}$ (the case of interest for applications) remains open. In the next Section 3.1 we will study the steady-state stability of the six-dimensional model for non-zero constant $\bar{\tau}, \bar{V}_f, \bar{V}_{dq}$ by an alternative, partial, approach based on Caliskan and Tabuada (2014). Furthermore, in Section 4 we will discuss the stability of the synchronous generator coupled to a linear resistive load for non-zero constant $\bar{\tau}, \bar{V}_f$.

3.1. The shifted Hamiltonian as storage function for non-zero constant torque

As concluded above, the integrability conditions for shifted passivity of the six-dimensional port-Hamiltonian system (22) for non-zero constant $\bar{\tau}$ are *not* satisfied. In this subsection we will discuss another approach, utilized in Caliskan and Tabuada (2014) for a simplified model of the synchronous generator, which starts from the definition of the shifted Hamiltonian as in (24), and then derives *additional* conditions under which this shifted Hamiltonian serves as a storage function guaranteeing passivity with respect to the shifted inputs and outputs.

Denote for compactness of notation

$$s = (\psi_d, \psi_q, \psi_r, p), \quad u = (V_{dq}, V_f, \tau), \quad y = (I_{dq}, I_f, \omega)$$

Let $\bar{u}$ be a constant input, and consider any corresponding steady-state value $\bar{s}$ of the system (22), resulting in a steady state output $\bar{y}$. Exploiting, as in Caliskan and Tabuada (2014), the linear dependence of the Poisson structure matrix in (22) on ψ_d, ψ_q , necessary and sufficient conditions on the system parameters for shifted passivity of (22) with respect to the shifted Hamiltonian³ can be obtained as follows.

³ Note that these conditions may not be necessary for shifted passivity with respect to another storage function.

As in (24), define the shifted Hamiltonian $\tilde{\mathcal{H}}$ corresponding to the Hamiltonian $\hat{\mathcal{H}}$ of the six-dimensional model (22) by

$$\tilde{\mathcal{H}}(s) := \hat{\mathcal{H}}(s) - (\nabla \hat{\mathcal{H}}(\bar{s}))^T (s - \bar{s}) - \hat{\mathcal{H}}(\bar{s}), \quad (28)$$

where we are using the shorthand notation ∇H for $\frac{\partial H}{\partial x}$.

Denote furthermore the 6×6 Poisson structure matrix of (22) by $\hat{\mathcal{J}}(s)$, the resistive structure matrix by $\hat{\mathcal{R}}$, and the input matrix by $\hat{g}$. Then

$$\begin{aligned} \tilde{\mathcal{H}} &= (\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T ([\hat{\mathcal{J}}(s) - \hat{\mathcal{R}}] \nabla \hat{\mathcal{H}}(s) + \hat{g}\bar{u} + \hat{g}(u - \bar{u})) \\ &= (\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T ([\hat{\mathcal{J}}(s) - \hat{\mathcal{R}}] \nabla \hat{\mathcal{H}}(s) - [\hat{\mathcal{J}}(\bar{s}) \\ &\quad - \hat{\mathcal{R}}] \nabla \hat{\mathcal{H}}(\bar{s}) + \hat{g}(u - \bar{u})) \\ &= (\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T (\hat{\mathcal{J}}(s - \bar{s}) \nabla \hat{\mathcal{H}}(\bar{s}) - \hat{\mathcal{R}}(\nabla \hat{\mathcal{H}}(s) \\ &\quad - \nabla \hat{\mathcal{H}}(\bar{s})) + \hat{g}(u - \bar{u})) \end{aligned} \quad (29)$$

where we have used, guided by Caliskan and Tabuada (2014),

$$\begin{aligned} &(\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T (\hat{\mathcal{J}}(s) \nabla \hat{\mathcal{H}}(s) - \hat{\mathcal{J}}(\bar{s}) \nabla \hat{\mathcal{H}}(\bar{s})) \\ &= (\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T (\hat{\mathcal{J}}(s) \nabla \hat{\mathcal{H}}(s) - \hat{\mathcal{J}}(\bar{s}) \nabla \hat{\mathcal{H}}(\bar{s})) \\ &\quad + \hat{\mathcal{J}}(s) \nabla \hat{\mathcal{H}}(\bar{s}) - \hat{\mathcal{J}}(\bar{s}) \nabla \hat{\mathcal{H}}(\bar{s}) \\ &= (\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T (\hat{\mathcal{J}}(s) \nabla \hat{\mathcal{H}}(\bar{s}) - \hat{\mathcal{J}}(\bar{s}) \nabla \hat{\mathcal{H}}(\bar{s})) \\ &= (\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T \hat{\mathcal{J}}(s - \bar{s}) \nabla \hat{\mathcal{H}}(\bar{s}), \end{aligned}$$

exploiting skew-symmetry of $\tilde{\mathcal{J}}$.

Define $\tilde{I}_d := I_d - \bar{I}_d$ and use likewise notation for the remaining variables with respect to their steady state values. Then the term $(\nabla \hat{\mathcal{H}}(s) - \nabla \hat{\mathcal{H}}(\bar{s}))^T \hat{\mathcal{J}}(s - \bar{s}) \nabla \hat{\mathcal{H}}(\bar{s})$ in the last line of (29) can be written in the form

$$\begin{aligned} &\begin{bmatrix} \tilde{I}_d \\ \tilde{I}_q \\ \tilde{I}_f \\ \tilde{I}_{kd} \\ \tilde{I}_{kq} \\ \tilde{\omega} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\tilde{\psi}_q \\ 0 & 0 & 0 & 0 & 0 & \tilde{\psi}_d \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{\psi}_q & -\tilde{\psi}_d & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{I}_d \\ \tilde{I}_q \\ \tilde{I}_f \\ \tilde{I}_{kd} \\ \tilde{I}_{kq} \\ \tilde{\omega} \end{bmatrix} \\ &= -\tilde{I}_d \tilde{\psi}_q \tilde{\omega} + \tilde{I}_q \tilde{\psi}_d \tilde{\omega} + \tilde{\omega} (\tilde{\psi}_q \tilde{I}_d - \tilde{\psi}_d \tilde{I}_q) \end{aligned}$$

Since $\tilde{\psi} = \mathcal{L}\tilde{I}$, this term can be further rewritten as

$$\frac{1}{2} \begin{bmatrix} \tilde{I}_d \\ \tilde{I}_q \\ \tilde{I}_f \\ \tilde{I}_{kd} \\ \tilde{I}_{kq} \\ \tilde{\omega} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & k\tilde{\omega}L_{akq} & -L_d\tilde{I}_q \\ 0 & 0 & k\tilde{\omega}L_{afd} & k\tilde{\omega}L_{akd} & 0 & L_q\tilde{I}_d \\ 0 & k\tilde{\omega}L_{afd} & 0 & 0 & 0 & -k\tilde{I}_qL_{afd} \\ 0 & k\tilde{\omega}L_{akd} & 0 & 0 & 0 & -k\tilde{I}_qL_{akd} \\ k\tilde{\omega}L_{akq} & 0 & 0 & 0 & 0 & -k\tilde{I}_dL_{akq} \\ -L_d\tilde{I}_q & L_q\tilde{I}_d & -k\tilde{I}_qL_{afd} & -k\tilde{I}_qL_{akd} & -k\tilde{I}_dL_{akq} & 0 \end{bmatrix} \begin{bmatrix} \tilde{I}_d \\ \tilde{I}_q \\ \tilde{I}_f \\ \tilde{I}_{kd} \\ \tilde{I}_{kq} \\ \tilde{\omega} \end{bmatrix}$$

where $k = \sqrt{\frac{3}{2}}$. By substitution into (29) we obtain the following result.

Proposition 3.2. Consider a constant $\bar{u}$, and a corresponding steady state value $\bar{s}$ of the system (22), resulting in a steady state output $\bar{y}$. Then the system (22) is passive with respect to the shifted inputs and outputs $u - \bar{u}, y - \bar{y}$, using the shifted Hamiltonian $\tilde{\mathcal{H}}$ as storage

Fig. 2. The synchronous generator SG coupled to a linear resistive load r_l .

function, if

$$\begin{bmatrix} -2r_s & 0 & 0 & 0 & k\tilde{\omega}L_{akq} & -L_d\tilde{I}_q \\ 0 & -2r_s & k\tilde{\omega}L_{afd} & k\tilde{\omega}L_{akd} & 0 & L_q\tilde{I}_d \\ 0 & k\tilde{\omega}L_{afd} & -2r_f & 0 & 0 & -k\tilde{I}_qL_{afd} \\ 0 & k\tilde{\omega}L_{akd} & 0 & -2r_{kd} & 0 & -k\tilde{I}_qL_{akd} \\ k\tilde{\omega}L_{akq} & 0 & 0 & 0 & -2r_{kq} & -k\tilde{I}_dL_{akq} \\ -L_d\tilde{I}_q & L_q\tilde{I}_d & -k\tilde{I}_qL_{afd} & -k\tilde{I}_qL_{akd} & -k\tilde{I}_dL_{akq} & -2d \end{bmatrix} < 0 \quad (30)$$

Corollary 3.3. Let $u = \bar{u}$ with steady state value $\bar{s}$. If (30) holds then $\bar{s}$ is a globally asymptotically stable steady state of (22).

Basically, the condition (30) is satisfied if the damping constant d and the resistances r_s, r_f, r_{kd}, r_{kq} are large enough and the steady state $\bar{s}$ is located in a sufficiently small neighborhood around the origin.

4. Stability analysis of the synchronous generator coupled to a linear load

By far the simplest scenario of a power network containing synchronous generators is the case of a single synchronous generator connected via its stator terminals to a three-phase linear resistive load, with identical resistances (r_l, r_l, r_l), cf. Fig. 2. We will see that already in this simplest case a full dynamical analysis becomes rather complicated for non-zero torque τ . Substitution of the characteristics of the three-phase linear load in (22), using the orthonormality of $\mathbb{T}_{dq0}$ in (15), leads to the following equations (almost identical to (22))

$$\begin{aligned} \begin{bmatrix} \dot{\psi}_d \\ \dot{\psi}_q \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} -\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} & 0_{23} \\ 0_{32} & -R_r \\ \begin{bmatrix} \psi_q & -\psi_d \end{bmatrix} & 0_{13} & -d \end{bmatrix} \begin{bmatrix} -\psi_q \\ \psi_d \\ \psi_r \end{bmatrix} + \begin{bmatrix} 0_{21} & 0_{21} \\ e_1 & 0_{31} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_f \\ \tau \end{bmatrix} \\ &\quad + \begin{bmatrix} \frac{\partial \tilde{\mathcal{H}}}{\partial \psi_d} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial \psi_q} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial \psi_r} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial p} \end{bmatrix} \\ \begin{bmatrix} I_f \\ \omega \end{bmatrix} &= \begin{bmatrix} 0_{12} & e_1^T & 0 \\ 0_{12} & 0_{13} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{\mathcal{H}}}{\partial \psi_d} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial \psi_q} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial \psi_r} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial p} \end{bmatrix}, \end{aligned} \quad (31)$$

with $r := r_s + r_l$. Furthermore, consider constant values $\bar{\tau}$ of the mechanical torque and $\bar{V}_f$ of the excitation voltage. The resulting

steady state values of (31), denoted by $\bar{\cdot}$, are given as

$$\begin{aligned} 0 &= -r \begin{bmatrix} \bar{I}_d \\ \bar{I}_q \end{bmatrix} + \begin{bmatrix} -\bar{\psi}_q \\ \bar{\psi}_d \end{bmatrix} \bar{\omega} \\ 0 &= -R_r \bar{I}_r + \begin{bmatrix} \bar{V}_f \\ 0 \\ 0 \end{bmatrix} \\ 0 &= -\bar{\tau}_m - d\bar{\omega} + \bar{\tau}, \end{aligned} \quad (32)$$

while in view of (21) $\bar{I}_0 = 0$. As proved in Fiaz et al. (2013), given a steady state value $\bar{\omega}$ of the frequency ω the steady state values $\bar{I}_d, \bar{I}_q, \bar{I}_r, \bar{\psi}_{d0}, \bar{\psi}_r$ of the rest of the state variables are uniquely determined. The complication is therefore in the determination of the steady state values of $\bar{\omega}$, which are solutions of a polynomial equation of degree 3 as detailed in the next section.⁴

Note that a direct consequence of this analysis is that steady states of (31) correspond to sinusoidal solutions of the original system (1), with frequency equal to a steady-state value $\bar{\omega}$.

4.1. Equilibrium frequencies

In this subsection we look more carefully at the steady state frequencies $\bar{\omega}$. They satisfy the equation (Fiaz et al., 2013)

$$\bar{\tau} = \omega \left(d + \frac{k^2 L_{afd}^2 r \bar{V}_f^2}{r_f^2 (L_d^2 \omega^2 + r^2)} \right),$$

and thus are solutions of the polynomial equation

$$p(\omega) = 3\bar{V}_f^2 L_{afd}^2 r \omega - 2r_f^2 (\bar{\tau} - d\omega) (L_d^2 \omega^2 + r^2) = 0 \quad (33)$$

Recall the following standard result concerning polynomials of degree 3; see e.g. Irving (2004).

Proposition 4.1. Given a polynomial $p(x) = ax^3 + bx^2 + cx + d$ with discriminant

$$\Delta = b^2 c^2 - 4ac^3 - 4b^3 d - 27a^2 d^2 + 18abcd$$

Then the following holds true:

- if $\Delta > 0$ then $p(x) = 0$ has 3 distinct real roots.
- if $\Delta < 0$ then $p(x) = 0$ has 1 real root and 2 complex conjugate roots.
- if $\Delta = 0$ then at least two 2 roots coincide and all roots are real.

By considering the discriminant Δ of (33) it thus follows that (33) has only one real zero $\bar{\omega}$ if and only if

$$\begin{aligned} &24drL_{afd}^2 r_f^4 \bar{V}_f^2 (3d^2 r^2 - 5\bar{\tau}^2 L_d^2) \\ &+ 9L_{afd}^4 r_f^2 \bar{V}_f^4 (12d^2 r^2 - \bar{\tau}^2 L_d^2) \\ &+ 54drL_{afd}^6 \bar{V}_f^6 + 16r_f^6 (d^2 r^2 + \bar{\tau}^2 L_d^2)^2 > 0 \end{aligned}$$

4.1.1. Equilibrium currents

From Proposition 4.2 of Fiaz et al. (2013), given a steady state frequency $\bar{\omega}$, there is a unique flux vector $\bar{\psi}_d, \bar{\psi}_q, \bar{\psi}_r$, and therefore a unique steady state current vector $\bar{I}_d, \bar{I}_q, \bar{I}_r$ associated to it; see (32). More explicitly, the steady state currents are given by

$$\begin{aligned} \bar{I}_d &= -\sqrt{\frac{3}{2}} \frac{L_{afd} L_d \bar{\omega}^2 \bar{V}_f}{(r^2 + L_d^2 \bar{\omega}^2) r_f}, \bar{I}_q = \sqrt{\frac{3}{2}} \frac{L_{afd} r \bar{\omega} \bar{V}_f}{(r^2 + L_d^2 \bar{\omega}^2) r_f} \\ \bar{I}_f &= \frac{\bar{V}_f}{r_f}, \bar{I}_{kd} = \bar{I}_{kq} = 0 \end{aligned} \quad (34)$$

Among others, this shows the (physically obvious) fact that a nonzero excitation voltage $\bar{V}_f$ is required for a nonzero steady-state current.

4.1.2. Desired equilibria

Now consider the problem of designing the constant mechanical torque $\bar{\tau}$ and excitation voltage $\bar{V}_f$ in such a way that the steady state frequency $\bar{\omega}$ is equal to a desired value ω_{des} , and that in addition the voltage magnitude $\sqrt{\bar{V}_d^2 + \bar{V}_q^2}$ is equal to a desired value V_{des} . It can be noted, cf. Fig. 2, that $V_l = r_l I_{dq}$, where V_l is the voltage across the load. Hence, it should hold that

$$r_l \sqrt{\bar{I}_d^2 + \bar{I}_q^2} = V_{des}. \quad (35)$$

By solving equations (33)–(35) we obtain the following equations for the required mechanical torque and excitation voltage:

$$\begin{aligned} \bar{\tau} &= d\omega_{des} + \frac{V_{des}^2 (r_l + r_s)}{r_l^2 \omega_{des}} \\ \bar{V}_f &= \pm \sqrt{\frac{2}{3} \frac{V_{des} r_f \sqrt{L_d^2 \omega_{des}^2 + (r_l + r_s)^2}}{L_{afd} r_l \omega_{des}}} \end{aligned} \quad (36)$$

Suppose that the voltage amplitude over the load is constant and is equal to V_{des} . Then, because of equation (35), a lower load resistance r_l implies that the current flowing through the load has a larger magnitude and therefore the power consumption of the load increases. Observe from equation (36) that a larger power consuming load (a lower r_l) implies a higher required mechanical torque and excitation voltage magnitude, as we would expect.

Remark 4.2. One should bear in mind that if the constant inputs $\bar{\tau}, \bar{V}_f$ are chosen as in (36), then this does not necessarily mean that there is only one steady state frequency $\bar{\omega}$, nor that the steady state corresponding to $\bar{\omega}$ is stable. These aspects are investigated more thoroughly in the following subsections.

4.2. Steady state stability analysis based on linearization

In this section we linearize the dynamics of the synchronous generator coupled to a linear resistive load around a given steady state, denoted by $\bar{\cdot}$. Rewrite (31) as

$$\begin{bmatrix} \dot{\psi}_d \\ \dot{\psi}_q \\ \dot{\psi}_f \\ \dot{\psi}_{kd} \\ \dot{\psi}_{kq} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -rI_d - \psi_q \omega \\ -rI_q + \psi_d \omega \\ -r_f I_f + V_f \\ -r_{kd} I_{kd} \\ -r_{kq} I_{kq} \\ \psi_q I_d - \psi_d I_q - d\omega + \tau \end{bmatrix},$$

whereas before $r = r_s + r_l$, and

$$\begin{bmatrix} \psi_d \\ \psi_q \\ \psi_f \\ \psi_{kd} \\ \psi_{kq} \\ p \end{bmatrix} = \begin{bmatrix} L_d I_d + kL_{afd} I_f + kL_{akd} I_{kd} \\ L_q I_q - kL_{akq} I_{kq} \\ kL_{afd} I_d + L_{ffd} I_f + L_{akd} I_{kd} \\ kL_{akd} I_d + L_{akd} I_f + L_{kdd} I_{kd} \\ -kL_{akq} I_q + L_{kkq} I_{kq} \\ \frac{1}{M} \omega \end{bmatrix}$$

and $k = \sqrt{\frac{3}{2}}$. Furthermore, rewrite the system in terms of the currents, instead of the fluxes. Then its linearization around $(\bar{I}, \bar{\omega})$ is given by (37), where

$$E = \begin{bmatrix} \mathcal{L}_{red} & 0 \\ 0 & M \end{bmatrix}, \quad L_d = L_q$$

⁴ This holds for the round rotor case considered here. Without the round rotor assumption one needs to consider a polynomial equation of degree 5, cf. Fiaz et al. (2013).

Fig. 3. The mechanical torque $\bar{\tau}$ (blue) and excitation voltage $\bar{V}_f$ (orange) as functions of the load resistance r_l . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Here $\mathcal{L}_{\text{red}}$ is the 5×5 inductance matrix obtained from the 6×6 inductance matrix $\mathcal{L}$ defined in (17) by deleting the third row and third column (corresponding to the 0-coordinate).

A sufficient condition for asymptotic stability of the linearization is given below.

Proposition 4.3. *The linearized system (37) is asymptotically stable if $A + A^T < 0$.*

Proof. Since $E > 0$ it follows from Theorem 3.1 of Duan and Patton (1998) that $E^{-1}A$ is Hurwitz. $\square$

Remark 4.4. Remarkably, there is a close connection between the matrix A in (37) and the matrix displayed in (30). In fact, the matrix $A + A^T$ is equal to the matrix in (30) if $r_s = r$ and $L_d = L_q$.

4.3. Numerical analysis of the stability of the linearized system

Proposition 4.3 only states a sufficient condition for the stability of the linearized system. In this subsection we will perform a numerical analysis of the stability of the linearization of the synchronous generator coupled with a resistive load. We first consider the case of constant inputs $\bar{\tau}, \bar{V}_f$ given by (36). Afterwards, we consider the case of constant inputs plus proportional control.

We throughout take $\omega_{\text{des}} = 1$ and $V_{\text{des}} = 1$. Furthermore, we take $r_s = 0.003$, $d = 0.01$. The other parameters correspond to a 555 MVA, 24 kV, 0.9 p.f. 60 Hz, 3600 RPM turbine-generator, see Kundur (1993, p. 102).

4.3.1. Constant control

The constant input values (36) as a function of the resistive load r_l are depicted in Fig. 3. The stability of the linearized system is determined by the eigenvalues of $E^{-1}A$, which are plotted as functions of the resistive load r_l in Fig. 4.

As noted before, there are three steady state frequencies $\bar{\omega}$, which are plotted as functions of r_l in Fig. 5. Remarkably, the point where there are three real steady state frequencies with two of them coinciding ($r_l \approx 1.8$) happens at the same point as where the linearized system becomes (un)stable, see Fig. 4. This suggests that the steady state associated with the *middle* steady state frequency is unstable if there are three distinct real steady state frequencies. In fact, as suggested in the following subsection, the two steady

Fig. 4. The real part of the eigenvalues of the matrix $E^{-1}A$. The blue lines correspond to stable eigenvalues (these eigenvalues should be multiplied by -1), whereas the red line corresponds to unstable eigenvalues. This plot shows that the linearized system is asymptotically stable if $r_l \geq 1.8$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 5. The steady state frequencies as functions of the load resistance r_l . The blue lines are the real parts of the steady state frequencies whereas the red line corresponds to the imaginary part. Remarkably, the point where the linearized system becomes (un)stable (at $r_l \approx 1.8$, see Fig. 4), is the point where two of the three steady state frequencies coincide. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

states associated to the minimum and maximum real steady state frequencies are locally asymptotically stable.

$$E \begin{bmatrix} \dot{\bar{I}}_d \\ \dot{\bar{I}}_q \\ \dot{\bar{I}}_f \\ \dot{\bar{I}}_{kd} \\ \dot{\bar{I}}_{kq} \\ \dot{\bar{\omega}} \end{bmatrix} = \underbrace{\begin{bmatrix} -r & -L_q \bar{\omega} & 0 & 0 & kL_{akq} \bar{\omega} & -L_d \bar{I}_q \\ L_d \bar{\omega} & -r & kL_{afd} \bar{\omega} & kL_{akd} \bar{\omega} & 0 & L_d \bar{I}_d + kL_{afd} \bar{I}_f \\ 0 & 0 & -r_f & 0 & 0 & 0 \\ 0 & 0 & 0 & -r_{kd} & 0 & 0 \\ 0 & 0 & 0 & 0 & -r_{kq} & 0 \\ 0 & -kL_{afd} \bar{I}_f & -kL_{afd} \bar{I}_q & -kL_{akd} \bar{I}_q & -kL_{akq} \bar{I}_d & -d \end{bmatrix}}_A \begin{bmatrix} \bar{I}_d \\ \bar{I}_q \\ \bar{I}_f \\ \bar{I}_{kd} \\ \bar{I}_{kq} \\ \bar{\omega} \end{bmatrix} \quad (37)$$

4.3.2. Proportional control

In case a frequency proportional controller (plus a constant part as before) is used then there is additional freedom in choosing

Fig. 6. The equilibrium frequencies as function of r_l in case $d_{cl} = 10$. The black line shows the real part of the (unstable) eigenvalues. The red line shows the (non-negative) imaginary part of the equilibrium frequencies. In this case the linearized system is asymptotically stable if $r_l \geq 0.1$. This plot confirms that the two outer equilibrium frequencies are stable and the middle one is unstable. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

d . This can be done by the introduction of a controller of the form $\tau = -\hat{d}(\omega - \omega_{des}) + \bar{\tau}$, where $\bar{\tau}$ is as in (36). In this way the damping coefficient d_{cl} of the closed-loop system becomes equal to $d_{cl} = d + \hat{d}$; and thus can be chosen as desired. It can be shown that if d_{cl} is larger, then the stability margin in terms of r_l is larger, see Fig. 6. By manipulating d_{cl} , numerical results even suggest that for any given $r_l > 0$ there exists a sufficiently large d_{cl} such that the linearized system is asymptotically stable. This claim is strengthened by the following result which holds for all positive parameter values.

Proposition 4.5. *Given constant $r_l > 0$ and let the polynomial p be given by (33), where $r = r_s + r_l$. If the constant input values (36) (depending on d) are substituted in p , then the discriminant Δ of the resulting polynomial p satisfies*

$$\lim_{d \rightarrow \infty} \Delta = -\infty.$$

Proposition 4.5 states that if d_{cl} is sufficiently large, then $\Delta < 0$ and thus there exists only one real equilibrium. As numerical results show, if there is only one real equilibrium, then the linearized system is asymptotically stable. The result of Proposition 4.5 is in line with Proposition 4.3, in the sense that the conditions of Proposition 4.3 also imply that r_l is sufficiently large.

5. From the eight-dimensional model to the swing equation

In this section we will discuss an approach to drastically simplify the eight-dimensional model of the synchronous generator into another standard model, called the *swing equation*. This simplification is based on neglecting the dynamics of the magnetic flux variables and replacing it by a static relationship between mechanical power and electrical power supplied at the stator terminals. This leads to a one-dimensional model, still capturing the mechanical dynamics of the system. The resulting swing equation is again in port-Hamiltonian form; however, with respect to a completely different set of variables than the original eight-dimensional system.

The simple approach taken in this section seems to be relatively new. In the literature, see e.g. Kundur (1993), Machowski et al. (2008), and Galaz, Ortega, and Bazanella (2002), the simplification towards the swing equation is typically undertaken through a number of intermediate steps; each of which has its own merits. In

Stegink et al. (in preparation) it will be shown how also these intermediate steps can be understood from a port-Hamiltonian point of view, emphasizing the expressions of the Hamiltonian function (stored energy) for each of the steps. See also the recent papers (Natarajan & Weiss, 2014a; 2014b) for an alternative approach.

The simplification of the eight-dimensional model to the swing equation as followed in this section is directly based on the energy balance (13) of the original eight-dimensional model (1). As before, split the total Hamiltonian as $H = H_m + H_k$, where the magnetic energy $H_m(\psi_s, \psi_r, \theta)$ and the kinetic energy $H_k(p)$ are given respectively as

$$H_m := \frac{1}{2} \begin{bmatrix} \psi_s^T & \psi_r^T \end{bmatrix} L^{-1}(\theta) \begin{bmatrix} \psi_s \\ \psi_r \end{bmatrix}, \quad H_k := \frac{1}{2J_r} p^2 \quad (38)$$

Using this splitting we can decompose the original power balance (13) into a power balance for the magnetic energy

$$\frac{d}{dt} H_m = -I_s^T R_s I_s - I_r^T R_r I_r + I_s^T V_s + I_f V_f + \omega \tau_m, \quad (39)$$

together with a power balance for the kinetic energy

$$\frac{d}{dt} H_k = -d\omega^2 + \omega\tau - \omega\tau_m, \quad (40)$$

by simply *adding* the term $\omega\tau_m$ to the power balance for the magnetic energy, and *subtracting* the same term from the power balance for the kinetic energy. Equivalently, we are thus looking at the 8-dimensional model as the *interconnection* of a seven-dimensional magnetic subsystem (involving the variables ψ_s, ψ_r, θ appearing in the expression for the magnetic energy) together with a one-dimensional mechanical subsystem (just involving the angular momentum p). This interconnection takes place via the port variables ω, τ_m , with τ_m the air-gap torque (cf. (13)), and with $\omega\tau_m$ the power flowing between the mechanical subsystem and the magnetic subsystem.

The swing equation is now obtained by neglecting the seven-dimensional dynamics of the magnetic subsystem, and replacing it by a *static* relationship around a reference value $\bar{\omega}$ (e.g., 50Hz) of the frequency (angular velocity) ω . This is accomplished by making the *key assumption* that the increase in magnetic energy is, apart from losses and the power delivered by the excitation system, proportional to the *deviation* of the frequency ω with respect to its nominal value $\bar{\omega}$, that is,

$$\frac{d}{dt} H_m = -I_s^T R_s I_s - I_r^T R_r I_r + A(\omega - \bar{\omega}) + I_f V_f, \quad (41)$$

for some positive constant A . Physically, this corresponds to the fact that once a deviation of the frequency from a given value $\bar{\omega}$ occurs, then this will lead to the creation of a magnetic field associated with the damper windings and a resulting restoring torque (Lenz' force). Therefore, A is also called the *damping constant* corresponding to the damper windings (although it does not correspond to dissipation of energy).

In view of the balance equation (13) the assumption (41) is equivalent to the assumption that the port variables ω, τ on the mechanical side of the seven-dimensional system are related with the port variables on the electrical network side by the expression

$$\omega\tau_m = -I_s^T V_s + A(\omega - \bar{\omega}) \quad (42)$$

This has a clear interpretation: around steady state $\bar{\omega}$ the mechanical power $\omega\tau_m$ delivered by the rotor is directly transformed into electrical power $-I_s^T V_s$ at the stator terminals transmitted to the network, *plus* the term $A(\omega - \bar{\omega})$ due to the restoring torque of the damper windings.

With regard to the one-dimensional system in the angular momentum p this leads to the following expression, directly relating

the change in kinetic energy of the rotor to the power delivered to the electrical network:

$$\begin{aligned} \frac{d}{dt}H_k &= -d\omega^2 + \omega\tau - \omega\tau_m \\ &= -d\omega^2 - A(\omega - \tilde{\omega}) + I_s^T V_s + \omega\tau \\ &\approx -A(\omega - \tilde{\omega}) + I_s^T V_s + \omega\tau \\ &= -A(\omega - \tilde{\omega}) + P_{\text{electric}} + P_{\text{mech}}, \end{aligned} \quad (43)$$

where we additionally assume that the mechanical losses $d\omega^2$ due to friction can be neglected.

Furthermore, assuming that ω is close to $\tilde{\omega}$ we can make the *approximation*

$$\frac{d}{dt}H_k = \frac{d}{dt} \frac{1}{2J_r} p^2 = \frac{d}{dt} \frac{1}{2} J_r \omega^2 = (J_r \omega) \dot{\omega} \approx M \dot{\omega}, \quad (44)$$

with $M := J_r \tilde{\omega}$. Eqs. (43) and (44) lead to the following equation for the deviation $\tilde{\omega} = \omega - \tilde{\omega}$ of the frequency with respect to the nominal value $\tilde{\omega}$

$$M \dot{\tilde{\omega}} = -A \tilde{\omega} + P_{\text{electric}} + P_{\text{mech}}, \quad (45)$$

called the *swing equation*.

The assumption (42) leading to the swing equation can be only justified on a relatively short time-interval after the deviation $\tilde{\omega}$ from the nominal value $\tilde{\omega}$ occurs. Indeed, after some time no further increase of the magnetic energy in the damper windings will take place. This observation regarding the underlying assumptions and validity of the model is well-known in the power engineering literature; see e.g. (Kundur, 1993; Machowski et al., 2008; Pai, 1989). In fact, the terminology ‘swing equation’ is based on this observation: the swing equation is thought to be reasonably accurate for the ‘first swing’ of the synchronous generator after a deviation of the nominal value $\tilde{\omega}$ occurs. An *asymptotic* analysis of the dynamics of the synchronous generator based on the swing equation model, as is sometimes done in the (control) literature, is therefore debatable. We also refer to the recent paper (Caliskan & Tabuada, 2015) for an evaluation of the swing equation model in comparison with the eight-dimensional model (1).

Thus from a *modeling* point of view we may conclude that the swing equation has only limited validity. On the other hand, from a *control* point of view an extra term of the same form as the damping $A\tilde{\omega}$ appears in case a standard *droop control* scheme is used. In droop control the deviation of the mechanical torque τ with respect to the nominal value $\tilde{\tau}$ is set to be proportional to the deviation $\tilde{\omega}$. As a consequence, the swing equation may be a reasonable model for a *droop-controlled* synchronous generator.

The swing equation (8) is *formally* equivalent to the equation of motion for an actuated rotating mass subject to linear friction. However, from a physical point of view it should be emphasized that $\tilde{\omega}$ denotes *deviation* from the nominal value $\tilde{\omega}$ (instead of frequency itself), and that $A\tilde{\omega}$ is a restoring force; *not* dissipation.

Furthermore notice that the physical dimensions (units) used in the swing equation are *different* from the physical dimensions for an actuated rotating mass. In fact, the physical dimension of the input in the swing equation is *power* instead of *torque*. This means that in the following input-state-output port-Hamiltonian system based on (45), given as

$$\begin{aligned} \dot{\tilde{p}} &= -A \tilde{\omega} + u, \quad u = P_{\text{electric}} + P_{\text{mech}}, \\ y &= \frac{dH_s}{d\tilde{p}}, \quad H_s(\tilde{p}) = \frac{1}{2M} \tilde{p}^2, \end{aligned} \quad (46)$$

where $\tilde{p} := M\tilde{\omega}$, the output y is necessarily *dimensionless*, ensuring that the product $uy = P_{\text{electric}}y + P_{\text{mech}}y$ still has dimension of power.

6. Networks

In principle, it is rather straightforward to formulate time-domain models of power networks. One considers the graph with the basic components such as synchronous generators and loads as (grounded) nodes, and with edges corresponding to the transmission lines; similar to electrical circuit modeling. Various time-domain models may be used to describe the transmission lines, ranging from the telegrapher’s equations to simple RLC-circuits (such as ladder networks). In particular, the systematic approach as proposed in Fiaz et al. (2013) for time-domain modeling of power networks (approximating the transmission lines by standard Π models) has the advantage that the number of algebraic constraints in the obtained model is minimized.

On the other hand, time-domain models of large power networks necessarily are high-dimensional, while the dynamics of the transmission lines is often not the main source of complicated dynamical behavior. As a consequence, most of the models for power networks are based on modeling the transmission line network as a (complex-valued) *admittance* matrix; based on the assumptions of linearity and pure sinusoidal voltage and current signals with nominal fixed frequency. Of course, such models are not suited for the analysis of *transient* behavior.

6.1. Phasor model of the transmission line network

The quasi-steady state modeling of a transmission line network by an admittance matrix is done as follows. Consider the set of sinusoids $\mathbb{E}\cos(\tilde{\omega}t + \delta)$ with amplitude $\mathbb{E}$ and phase shift δ for a given (nominal) frequency $\tilde{\omega}$ (e.g., 50Hz). Its *phasor representation* is given by the complex number $\mathbb{E}e^{j\delta}$, with $\mathbb{E} \geq 0$ and $\delta \in [0, 2\pi)$ real variables, and $j := \sqrt{-1}$. Conversely, the sinusoid corresponding to the phasor $\mathbb{E}e^{j\phi}$ is given as $\mathbb{E}\cos(\tilde{\omega}t + \delta) = \text{Re } \mathbb{E}e^{j\delta}e^{j\tilde{\omega}t}$.

The usefulness of the phasor representation of sinusoids stems from two facts: (1) the map from sinusoid $\mathbb{E}\cos(\tilde{\omega}t + \delta)$ to phasor $\mathbb{E}e^{j\delta}$ is *linear*, (2) the phasor of the *derivative* of a sinusoid is the phasor of the original sinusoid *multiplied* by $j\tilde{\omega}$, and similarly the phasor of the integral of a sinusoid is equal to the phasor of the original sinusoid divided by $j\tilde{\omega}$.

Now consider a network composed of *linear transmission lines*, where the voltages and currents are all assumed to be pure sinusoids with equal frequency $\tilde{\omega}$. Then the phasor $\mathcal{I}$ of the current I through a transmission line is related to the phasor $\mathcal{V}$ of the voltage across the line as

$$\mathcal{I} = \tau \mathcal{V} \quad (47)$$

Here the complex number τ captures the characteristics of the transmission line: it is equal to the conductance $G > 0$ in case of a purely resistive line, equal to $j\tilde{\omega}C$, $C > 0$, in case of a purely capacitive line, equal to $\frac{1}{j\tilde{\omega}L}$, $L > 0$, in case of a purely inductive line, and combinations of these for the general case.

The transmission line network defines a graph, with m edges corresponding to the transmission lines, and n nodes corresponding to the buses (where components such as generators and loads are attached). Endowing the graph with an arbitrary orientation (specifying in which direction the currents flowing through the edges are considered to be positive, as well as specifying the signs of the voltage drops), we describe the resulting oriented graph by its $n \times m$ *incidence matrix*⁵ D . Denoting the *voltage potentials* at the nodes by $E_1, \dots, E_n$, then the m -dimensional vector V of *voltage*

⁵ The incidence matrix D of a directed graph with n nodes and m edges is an $n \times m$ matrix, with k th column the vector with -1 at the position of the tail vertex of the k th edge, 1 at the position of the head vertex of this edge, and zeros elsewhere. Obviously $\mathbb{1}^T D = 0$, where $\mathbb{1}$ is the vector of all ones.

drops across the transmission lines is given by the m -dimensional vector $V = D^T E$.

Now assume that the voltage potentials are sinusoids with phasors $\mathcal{E}_k = \mathbb{E}_k e^{j\delta_k}$, $k = 1, \dots, n$. Then the phasors of the currents through the edges are equal to the phasors of the voltage drops across these edges, multiplied by the complex numbers $\gamma_1, \dots, \gamma_m$ capturing the characteristics of the transmission lines. It follows that the vector $\mathcal{I}$ of phasors of the currents injected at the nodes of the network is related to the phasors of the voltage potentials at the nodes by the compact formula, see e.g. van der Schaft (2010), Dörfler and Bullo (2013), and Simpson-Porco, Dörfler, and Bullo (2013a)

$$\mathcal{I} = DTD^T \mathcal{E}, \quad T = \text{diag}(\tau_1, \dots, \tau_m) \quad (48)$$

The $n \times n$ complex matrix $Y := DTD^T$ is called the *admittance* matrix of the transmission line network. From the definition it is obvious that it is symmetric ($Y = Y^T$) and has zero column and row sums (since the column sums of D are zero). Furthermore, splitting between the real and imaginary part

$$Y = D \text{Re}(T) D^T + j D \text{Im}(T) D^T =: \text{Re}(Y) + j \text{Im}(Y), \quad (49)$$

it follows that both $\text{Re}(Y)$ and $\text{Im}(Y)$ are symmetric matrices with row and column sums zero. In fact, see e.g. van der Schaft (2010) and Simpson-Porco et al. (2013a), $\text{Re}(Y)$ is a Laplacian matrix in case of a purely resistive transmission line network, while $\text{Im}(Y)$ is a Laplacian matrix in case of a purely capacitive network and minus a Laplacian matrix in case of a purely inductive network.

In case the transmission lines are modeled by RLC-circuits the expressions for $\text{Re}(Y)$ and $\text{Im}(Y)$ are easily found by using the aforementioned properties of phasors of derivatives, reducing the solution of a set of linear differential equations for the RLC-circuit to the solution of complex algebraic equations.

Eq. (48) is the starting point for *power flow analysis*. For example, split up the nodes into a subset corresponding to energy production (generators g) and energy consumption (corresponding to loads l). Then (48) is correspondingly split into

$$\begin{bmatrix} \mathcal{I}_g \\ \mathcal{I}_l \end{bmatrix} = \begin{bmatrix} Y_{gg} & Y_{gl} \\ Y_{lg} & Y_{ll} \end{bmatrix} \begin{bmatrix} \mathcal{E}_g \\ \mathcal{E}_l \end{bmatrix} \quad (50)$$

The analysis of this equation is depending on the model for the loads. Two simple possibilities are: (1) linear resistive loads, (2) constant power loads. In the first case

$$\mathcal{I}_l = -G\mathcal{E}_l, \quad (51)$$

for some Laplacian matrix G modeling the conductances of the loads. In this case, equations (50) reduce to the power flow generation equations

$$\mathcal{I}_g = (Y_{gg} - Y_{gl}(Y_{ll} + G)^{-1}Y_{lg})\mathcal{E}_g =: Y_g \mathcal{E}_g, \quad (52)$$

where the admittance matrix Y_g is a Schur complement, which inherits the Laplacian properties from Y and G ; see van der Schaft (2010) and Dörfler and Bullo (2013) for the basic ideas.

In the second case, the *complex power* $S := I^* V_l$, with $*$ denoting transpose and complex conjugate, is a prescribed constant vector $\tilde{S}_l$, that is

$$\mathcal{I}_l^* V_l = \tilde{S}_l \quad (53)$$

The vector $\text{Re}(\tilde{S}_l)$ is called the vector of *active power* consumed at the loads, while $\text{Im}(\tilde{S}_l)$ is called the vector of *reactive power* consumed at the loads. See e.g. Willems (2010) for an insightful treatment of these notions; in particular, the active power can be equated (up to a constant) with the *average power* consumed at the loads. In the case of constant power consumption, the resulting relation between $\mathcal{I}_g$ and $\mathcal{E}_g$ is quadratic, and much more difficult to analyze.

6.2. Networks of generators

A simple model of a network of synchronous generators⁶ is obtained by combining the above quasi-static model of a transmission line network with the swing equation model of a synchronous generator, under the following assumptions:

(1) The amplitudes $\mathbb{E}_k$ of the voltage potentials E_k at the nodes $k = 1, \dots, n$ are all *constant*.

(2) All transmission lines are purely *inductive*, i.e., $Y = -jB$, with B an $n \times n$ real matrix, called the *susceptance* matrix of the purely inductive transmission line network.⁷

Under these assumptions the active power transfer P_{ij} from node i with voltage phasor angle δ_i to node j with voltage phasor angle δ_j is computed from (48) as

$$P_{ij} = \Gamma_{ij} \sin(\delta_i - \delta_j), \quad (54)$$

where $\Gamma_{ij} := B_{ij} \mathbb{E}_i \mathbb{E}_j$, with B_{ij} the susceptance of the transmission line between nodes i and j .

This can be combined with the dynamics of the swing equation (45) for each generator as follows. For ease of notation denote the frequency deviation $\omega - \bar{\omega}$ in the sequel simply by ω . With this abuse of notation, the power balance at the i th generator node within the network can be written as

$$M_i \dot{\omega}_i = -A_i \omega_i - \sum_{j \in \mathcal{N}_i} P_{ij} + u_i, \quad (55)$$

with u_i denoting the generated/consumed power at the i th node, and $\mathcal{N}_i$ denoting the set of neighbors of node i in the transmission line network graph. Taking all these equations together in vector notation we arrive at the following n -dimensional system

$$M\dot{\omega} = -A\omega - D\Gamma \sin(D^T \delta) + u, \quad (56)$$

where ω is the n -dimensional vector of frequency deviations, u is the n -dimensional vector of produced/consumed power at all nodes, A is the diagonal matrix of damping constants, and M is the diagonal matrix with diagonal elements $M_1, \dots, M_n$ modeling the inertia of the generators. Furthermore, $\text{Sin } z$, $z \in \mathbb{R}^m$, denotes the element-wise sinus function with i th element given as $\text{Sin } z_i$, $i = 1, \dots, m$, and Γ is the $m \times m$ diagonal matrix with k th diagonal element given by Γ_{ij} , if the k th transmission line is between node i and j .

Furthermore, it is assumed that the voltage phasor angles δ_i corresponding to the i th generator satisfy

$$\dot{\delta}_i = \omega_i, \quad i = 1, \dots, n, \quad (57)$$

with ω_i the frequency deviation of the i th generator.

The coupled differential equations (56) and (57) can be compactly written as follows. Denote the *phase difference* across the k th transmission line from node i to node j by

$$q_k := \delta_j - \delta_i, \quad k = 1, \dots, m \quad (58)$$

In vector notation $q = D^T \delta$, with D the $n \times m$ incidence matrix of the network. Therefore $\dot{q} = D^T \omega$, and equations (56) and (57) result in the $(n+m)$ -dimensional system

$$\begin{aligned} \dot{q} &= D^T \omega \\ M\dot{\omega} &= -A\omega - D\Gamma \sin q + u, \end{aligned} \quad (59)$$

This model is often taken as a fruitful starting point for control purposes, especially for control of *micro-grids*;

⁶ Instead of a network of single synchronous generators and synchronous machines this may also represent a network of aggregated *generator areas*. Furthermore, it may also represent a *micro-grid*, where (some of) the generators actually correspond to inverters (coupled to other power sources) which are controlled in such a way as to emulate synchronous generators.

⁷ Usually the inductance of the transmission line starting from a generator node is increased by adding an inductor representing the inductance of the generator.

see e.g. Dörfler, Simpson-Porco, and Bullo (2016), Simpson-Porco et al. (2013a), Simpson-Porco, Dörfler, and Bullo (2013b), Trip, Bürger, and De Persis (2016), De Persis and Monshizadeh (2015), Schiffer (2015), and Schiffer, Seel, Raisch, and Sezi (2015).

It is readily seen that the model (59) can be written in port-Hamiltonian form as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad p = M\omega$$

$$y = \frac{\partial H}{\partial p}(q, p) = \omega, \quad (60)$$

with input u the vector of generated/consumed power at the nodes, output y the vector of frequency deviations at the nodes, and with Hamiltonian

$$H(q, p) := \frac{1}{2} p^T M^{-1} p - \mathbb{1}^T \Gamma \text{Cos } q, \quad (61)$$

where Cos denotes the component-wise cosinus function.

The Poisson structure matrix of this port-Hamiltonian system is given by the constant matrix (determined by the topology of the network)

$$J = \begin{bmatrix} 0 & D^T \\ -D & 0 \end{bmatrix}, \quad (62)$$

whereas the resistive structure matrix is $R = \text{diag}(0, A)$.

The first term in the Hamiltonian (61) is the *kinetic energy*, shifted with respect to the nominal angular momentum $\bar{p} := M\bar{\omega}$ of the generators. The second term can be interpreted as a *potential energy* corresponding to the phasor approximation of the inductive transmission line network.

6.2.1. Dynamical analysis

The dynamical analysis of the port-Hamiltonian system (60) for a *constant* vector of inputs $u = \bar{u}$ turns out to be rather straightforward, and can be performed along the lines of van der Schaft and Maschke (2013). Consider the set of *steady state* values $(\bar{q}, \bar{p} = M\bar{\omega})$, given as the solutions of

$$0 = \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(\bar{q}, \bar{p}) \\ \frac{\partial H}{\partial p}(\bar{q}, \bar{p}) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{u} \end{bmatrix} \quad (63)$$

Assuming without loss of generality that the graph is connected, and thus $\ker D^T = \text{span } \mathbb{1}$, it follows from the upper half of the equations that the steady state value $\bar{\omega}$ is determined as $\bar{\omega} = \frac{\partial H}{\partial p}(\bar{q}, \bar{p}) = \mathbb{1}\omega_*$ for a certain common scalar frequency deviation ω_* . Furthermore, by pre-multiplication of the lower half of the equations by $\mathbb{1}^T$ we obtain the equation

$$\mathbb{1}^T A \mathbb{1} \omega_* = \mathbb{1}^T \bar{u}, \quad (64)$$

uniquely determining ω_* . (Note that $\mathbb{1}^T A \mathbb{1} = \sum_i A_i$, i.e., the sum of the damping constants of all the generators.)

By substitution of (64) into the second set of equations of (63) we obtain the following equation in the steady state value $\bar{q}$:

$$0 = -D\Gamma \text{Sin } \bar{q} - A \mathbb{1} \omega_* + \bar{u} \quad (65)$$

Eq. (65) does not necessarily have a solution $\bar{q}$ (since the mapping $\text{Sin} : \mathbb{R}^M \rightarrow \mathbb{R}^M$ is bounded), and if there exists a solution then this solution is *unique* if and only if $\ker D = 0$ (no cycles in the transmission line network graph). At the same time, if $\ker D \neq 0$, also *conserved quantities* are present. Indeed, all functions $C(q) = k^T q$ with $k \in \ker D$ are conserved quantities⁸ for (60); that is, $\frac{d}{dt} C(q) = k^T D^T \omega = 0$ along solutions of (60).

This can be summarized in the following proposition:

Proposition 6.1. *The set of steady states of (60) is given as*

$$\mathcal{E} := \{(\bar{q}, \bar{p}) \mid \bar{p} = M \mathbb{1} \omega_*, D\Gamma \text{Sin } \bar{q} = -A \mathbb{1} \omega_* + \bar{u}\} \quad (66)$$

If $\mathcal{E}$ is non-empty, then it has dimension equal to $\ell := \dim \ker D$, while the dynamics (60) has ℓ independent conserved quantities $C(q) = k^T q$ with $k \in \ker D$. Furthermore, $\bar{p}$ is unique, and determined by $\bar{u}$.

In the sequel let us assume that $\mathcal{E}$ is non-empty. By the existence of conserved quantities (or equivalently, from the first line of (59)) it follows that the solution $(q(\cdot), p(\cdot))$ of (60) starting from an arbitrary initial condition (q_0, p_0) at time 0 will satisfy

$$q(t) - q_0 \in \text{im } D^T, \quad t \geq 0 \quad (67)$$

In the so-called ‘DC-approximation’, where we replace $\text{Sin } q$ by q the set of steady states $\mathcal{E}_{DC}$ is given as the affine space

$$\mathcal{E}_{DC} = \{(\bar{q}, \bar{p}) \mid \bar{p} = M \mathbb{1} \omega_*, D\Gamma \bar{q} = -A \mathbb{1} \omega_* + \bar{u}\}$$

It follows that for any q_0 there exists a *unique* $v_0 \in \text{im } D^T$ such that $q_0 + v_0 = \bar{q}_\infty$, with $(\bar{q}_\infty, \bar{p}) \in \mathcal{E}_{DC}$ (recall that $\bar{p}$ is uniquely determined). Hence, $\bar{q}_\infty$ is the *unique* steady state value corresponding to initial value q_0 .

One may expect that this can be generalized⁹ to the original equations involving the sin mapping, in the sense that for any initial condition (q_0, p_0) there exists a unique (modulo π -shifts) steady state value $(\bar{q}_\infty, \bar{p}) \in \mathcal{E}$.

Since the Poisson structure matrix J and the resistive structure matrix R in (60) are independent of the state (q, p) , we may use the general construction of the *shifted Hamiltonian* as in (24) for the stability analysis of the steady states in $\mathcal{E}$. With respect to the Hamiltonian H given by (61) the shifted Hamiltonian with respect to a steady state $(\bar{q}, \bar{p})$ is computed as (modulo a constant)

$$\begin{aligned} \tilde{H}(q, p) &:= \frac{1}{2} (p - \bar{p})^T M^{-1} (p - \bar{p}) - \mathbb{1}^T \Gamma \text{Cos } q - (q - \bar{q})^T \Gamma \text{Sin } \bar{q} \end{aligned} \quad (68)$$

By construction of the shifted Hamiltonian the dynamics (60) can then be rewritten as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial q}(q, p) \\ \frac{\partial \tilde{H}}{\partial p}(q, p) \end{bmatrix} + \begin{bmatrix} 0 \\ u - \bar{u} \end{bmatrix}, \quad (69)$$

where the constant input value $\bar{u}$ has been incorporated in the definition of $\tilde{H}$. It follows that

$$\begin{aligned} \frac{d}{dt} \tilde{H}(q, p) &= -(p - \bar{p})^T M^{-1} A M^{-1} (p - \bar{p}) + (y - \bar{y})^T (u - \bar{u}) \\ &\leq (y - \bar{y})^T (u - \bar{u}) \end{aligned} \quad (70)$$

In fact, the shifted Hamiltonian $\tilde{H}$ defines a storage function for all p and all q with $q_k \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if $\bar{q}_k \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $k = 1, \dots, m$. In fact, by convexity of minus the cosinus function on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ it follows that

$$\tilde{H}(\bar{q}, \bar{p}) \leq \tilde{H}(q, p) \quad (71)$$

Hence shifted passivity with respect to the shifted inputs and outputs $u - \bar{u}, y - \bar{y}$ results, with the steady state output value $\bar{y} = \frac{\partial \tilde{H}}{\partial p}(\bar{q}, \bar{p}) = M \mathbb{1} \omega_*$ determined by $\bar{u}$.

Furthermore, for $u = \bar{u}$ it follows that the largest invariant set within the set $\{(q, p) \mid \frac{d}{dt} \tilde{H}(q, p) = 0\}$ is equal to $\mathcal{E}$, and thus by LaSalle’s Invariance principle the dynamics will converge to $\mathcal{E}$, where it is to be expected (see the above discussion) that it will converge to a steady state value which is solely determined by the initial condition.

⁸ In fact, being conserved quantities for all possible Hamiltonians they are *Casimirs*; solely determined by the Poisson structure matrix.

⁹ See van der Schaft, Rao, and Jayawardhana (2013) for a very similar statement in the context of equilibria of mass-action kinetics chemical reaction networks, and van der Schaft and Maschke (2013) for a general theory in the linear case.

6.3. Networks of generators and frequency dependent loads

A direct extension of the previous analysis is to networks consisting of generators (denoted by g), and frequency dependent and constant loads¹⁰ (denoted by l). This leads to a bipartite directed graph with nodes representing either a generator or a load. The incidence matrix of the graph is split correspondingly into

$$D = \begin{bmatrix} D_g \\ D_l \end{bmatrix} \quad (72)$$

Modeling as before the generators by swing equations the total system takes the form

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & D_g^T \\ -D_g & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v_g + \begin{bmatrix} D_l^T \\ 0 \end{bmatrix} \omega_l \\ \omega_g &= \frac{\partial H}{\partial p} \\ v_l &= D_l \frac{\partial H}{\partial q}, \end{aligned} \quad (73)$$

together with damping relations, between the delivered power v_g , v_l and the frequencies ω_g , ω_l at the generator and load nodes, given by

$$\begin{aligned} v_g &= -A_g \omega_g + u_g && \text{generators} \\ v_l &= -A_l \omega_l + u_l && \text{loads} \end{aligned} \quad (74)$$

Here A_g , A_l are the diagonal damping matrices of the generators, respectively, loads. Furthermore, H is the same Hamiltonian as defined before in (61), with p corresponding to the angular momenta $p = M\omega_g$ of the generators (there is no inertia associated to the loads).

Solving for $\omega_l = -A_l^{-1}v_l + A_l^{-1}u_l$ this leads to

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -D_l^T A_l^{-1} D_l & D_g^T \\ -D_g & -A_g \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 & D_l^T A_l^{-1} \\ I & 0 \end{bmatrix} \begin{bmatrix} u_g \\ u_l \end{bmatrix} \quad (75)$$

Note that this again is a port-Hamiltonian model, with Poisson structure matrix J and resistive structure matrix R respectively given by

$$J = \begin{bmatrix} 0 & D_g^T \\ -D_g & 0 \end{bmatrix}, \quad R = \begin{bmatrix} D_l^T A_l^{-1} D_l & 0 \\ 0 & A_g \end{bmatrix} \quad (76)$$

The dynamical analysis can be performed in the same way as for the network of generators. Indeed, a steady state analysis for constant generated power $\bar{u}_g, \bar{u}_l$ leads to the requirement

$$\begin{bmatrix} D_g^T & D_l^T \end{bmatrix} \begin{bmatrix} \bar{\omega}_g \\ \bar{\omega}_l \end{bmatrix} = 0, \quad (77)$$

which again, under the standing assumption that the transmission line network is connected ($\ker D^T = \text{span } \mathbb{1}$), implies

$$\bar{\omega}_g = \mathbb{1} \omega_*, \quad \bar{\omega}_l = \mathbb{1} \omega_*, \quad (78)$$

where the steady state frequency deviation ω_* is determined by

$$(\mathbb{1}^T A_g \mathbb{1} + \mathbb{1}^T A_l \mathbb{1}) \omega_* = \mathbb{1}^T \bar{u}_g + \mathbb{1}^T \bar{u}_l \quad (79)$$

Furthermore, the steady state value $\bar{q}$ satisfies

$$D_g \sin \bar{q} = -A_g \mathbb{1} \omega_* + \bar{u}_g \quad (80)$$

By using as Lyapunov function the same shifted Hamiltonian (68) as in the previous subsection shifted passivity and asymptotic stability of the steady states results.

¹⁰ Also many classes of inverters can be modeled this way. See De Persis and Monshizadeh (2015) for a related set-up.

7. Conclusions

Power networks constitute an important research area with many challenges to the systems and control community. In this paper we have concentrated on two topics within the broad area of *modeling for control* of power networks: the dynamics of the synchronous generator and a simple model of power networks based on the quasi-steady state approximation of the transmission line network, the assumption of constant voltage potentials, and the swing equation model of the synchronous generator. Both cases lead to an insightful port-Hamiltonian formulation, with the dynamical analysis of the eight-dimensional synchronous generator model still posing fundamental questions. We believe both cases illustrate the relevance of a reconsideration of the modeling approaches to power networks for control purposes, and motivate a further research focus on this. Especially the property of shifted passivity with respect to steady state values has turned out to be a very useful starting point for control (De Persis & Monshizadeh, 2015; Stegink, De Persis, & van der Schaft, 2015a; 2015b; Trip et al., 2016).

Next to the modeling, analysis and control of the *physical* power network a major challenge is the design and control of the *communication and control* layer ('*smart grids*'). The overall aim is to produce and consume energy in such a way that there is balance between supply and demand, while the physical network is operating within a desired range (frequency regulation, no voltage collapse, sharing of power, regulation of 'reactive power', satisfaction of flow and storage constraints). Furthermore, there is a need to achieve this in an optimal manner, while at the same time guaranteeing robustness with respect to e.g. variations in uncontrollable energy supply/demand and faults.

This is also linked to *economic* considerations and dynamic pricing of energy supply and demand, for which there is a large literature. From a port-Hamiltonian perspective it was recently shown in Stegink et al., (2015a, 2015b) how the primal-dual gradient algorithm for the maximization of the social welfare can be intrinsically formulated as another port-Hamiltonian system, which can be coupled to the physical network so as to obtain an overall port-Hamiltonian system composed of a *physical* and a *cyber-economic* part. The combination between the physical network and the dynamical control architecture, together with a strong optimization and economic perspective, poses many interesting research questions.

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