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Chapter 1

Controlled Invariant Distributions and Differential Properties

Arjan van der Schaft

Abstract The theory of (controlled) invariant (co-)distributions is reviewed, emphasizing the theory of liftings of vector fields, one-forms and (co-)distributions to the tangent and cotangent bundle. In particular, it is shown how invariant distributions can be equivalently described as invariant submanifolds of the tangent and cotangent bundle. This naturally leads to the notion of an invariant Lagrangian subbundle of the Whitney sum of tangent and cotangent bundle, which amounts to a special case of the central equation of contraction analysis. The interconnection of the prolongation of a nonlinear control system (living on the tangent bundle of the state space manifold) with its Hamiltonian extension (defined on the cotangent bundle) is shown to result in a differential Hamiltonian system. The invariant submanifolds of this differential Hamiltonian system corresponding to Lagrangian subbundles are seen to result in general differential Riccati and differential Lyapunov equations. The established framework thus yields a geometric underpinning of recent advances in contraction analysis and convergent dynamics.

1.1 Introduction

In this chapter I will first recall the classical concepts of *invariant* and *controlled invariant* (co-)distributions, which are fundamental notions in nonlinear geometric control theory as developed starting from the seventies of the previous century. This will be done in the spirit of the paper [17] and of Chap. 13 in the book [18], by using the notions of liftings of vector fields, functions, and one-forms from a manifold to its tangent and cotangent bundle in a systematic way. Furthermore, I will provide two equivalent characterizations of invariance, and elucidate the necessary and sufficient condition for (local) controlled invariance as obtained in [17].

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Secondly, starting from [25], I will discuss how the notion of invariance of (co-)distributions can be generalized to invariance of *subbundles* of the Whitney sum of the tangent and cotangent bundle. By specializing to *Lagrangian* subbundles this is shown to lead to the basic equation of contraction theory, and thus under an additional zero-curvature condition to the classical Demidovich condition for *convergent dynamics*. Furthermore, I will argue that this geometric approach can be seen to be part of a larger story, where incremental analysis approaches are formulated geometrically using *differential properties*.

This chapter is dedicated to my colleague *from the very start* Henk Nijmeijer, at the occasion of the celebration of his 60th birthday. This seems to be fit, since the contents of this chapter start with the theory of our first joint publication [17] in 1982, and link, via our book *Nonlinear Dynamical Control Systems* [18], to later fundamental contributions of Henk within the area of convergent dynamics and synchronization. Furthermore, the chapter stresses the importance of a *geometric* language for nonlinear control.

The scientific work and careers of Henk and myself have been very close for a long time, with our Ph.D. defenses at the same day with the same “promotor” (Jan C. Willems) in the Aula of the University of Groningen, our joint Alma Mater. Furthermore, we were for a very long time close colleagues at the University of Twente, turning Twente into a center for nonlinear control theory. Moreover, despite the unavoidable turmoils in our careers, we have always remained good friends. Happy 60th birthday Henk!

1.2 Lifts to the Tangent and Cotangent Bundle

Throughout this chapter all objects (manifolds, functions, vector fields, one-forms, (co-)distributions, subbundles, etc.) will be assumed to be *smooth* (infinitely often differentiable).

Recall from [27] (see also [2, 4, 18]), how functions, vector fields and one-forms on a state space manifold $\mathcal{X}$ can be lifted to functions, vector fields and one-forms on the tangent and cotangent bundles $T\mathcal{X}$ and $T^*\mathcal{X}$. This will also lead to the definition of the lift of distributions and co-distributions to the tangent bundle.

First we introduce the notions of *complete* and *vertical* lifts of functions, vector fields and one-forms to the *tangent bundle*. Given a function h on $\mathcal{X}$, the *complete lift* of h to $T\mathcal{X}$, $h^c : T\mathcal{X} \rightarrow \mathbb{R}$, is defined by $h^c(x, \delta x) = \langle dh, \delta x \rangle(x)$, with $\langle \cdot, \cdot \rangle(x)$ denoting the duality pairing between elements of the cotangent space and the tangent space at $x \in \mathcal{X}$. In local coordinates $x = (x^1, \dots, x^n)$ for $\mathcal{X}$ and the induced local coordinates $(x, \delta x) = (x^1, \dots, x^n, \delta x^1, \dots, \delta x^n)$ for $T\mathcal{X}$ this reads

$$h^c(x, \delta x) = \sum_{a=1}^n \frac{\partial h}{\partial x^a}(x) \delta x^a. \quad (1.1)$$

The *vertical lift* of a function h to a function on $T\mathcal{X}$, denoted by $h^\vee : T\mathcal{X} \rightarrow \mathbb{R}$, is defined by $h^\vee(x, \delta x) = h \circ \tau_{\mathcal{X}}$, where $\tau_{\mathcal{X}} : T\mathcal{X} \rightarrow \mathcal{X}$ denotes the tangent bundle projection $\tau_{\mathcal{X}}(x, \delta x) = x$. In local induced coordinates $h^\vee(x, \delta x) = h(x)$.

Given a vector field f on $\mathcal{X}$, the *complete lift* f^c of f to $T\mathcal{X}$ is defined as the unique vector field satisfying $L_{f^c}h^c = (L_f h)^c$, for any function h on $\mathcal{X}$ (with $L_f h$ denoting the Lie-derivative of the function h along the vector field f , and similarly for $L_{f^c}h^c$). It can be proved [27] that this requirement uniquely determines f^c . Alternatively, if $\Phi_t : \mathcal{X} \rightarrow \mathcal{X}, t \in [0, \varepsilon)$, denotes the flow of f , then f^c is the vector field whose flow is given by $(\Phi_t)_* : T\mathcal{X} \rightarrow T\mathcal{X}$. In induced local coordinates $(x^1, \dots, x^n, \delta x^1, \dots, \delta x^n)$ for $T\mathcal{X}$,

$$f^c(x, \delta x) = \sum_{a=1}^n f_a(x) \frac{\partial}{\partial x^a} + \sum_{a,b=1}^n \frac{\partial f_a}{\partial x^b}(x) \delta x^b \frac{\partial}{\partial (\delta x^a)}. \quad (1.2)$$

Finally, the *vertical lift* $f^\vee$ of f to $T\mathcal{X}$ is the vector field on $T\mathcal{X}$ such that $L_{f^\vee}h^c = (L_f h)^\vee$, for any function h . In induced local coordinates for $T\mathcal{X}$

$$f^\vee(x, \delta x) = \sum_{a=1}^n f_a(x) \frac{\partial}{\partial (\delta x^a)}. \quad (1.3)$$

Thirdly, let α be a *differential one-form* on $\mathcal{X}$. The *complete lift* of α is the differential one-form on $T\mathcal{X}$ defined by setting

$$\alpha^c(q)(Z) := Z(\hat{\alpha}), \quad Z \in T_z T\mathcal{X}, \quad z \in T\mathcal{X}, \quad (1.4)$$

where $\hat{\alpha} : T\mathcal{X} \rightarrow \mathbb{R}$ is the function given as

$$\hat{\alpha}(X_p) = \alpha(p)(X_p), \quad X_p \in T_p \mathcal{X}, \quad p \in \mathcal{X}. \quad (1.5)$$

If α is given in local coordinates as $\alpha = \sum_{i=1}^n \alpha_i(x) dx^i$, then it readily follows that

$$\alpha^c(x, \delta x) = \sum_{i=1}^n \frac{\partial \alpha_i}{\partial x^j}(x) \delta x^j dx^i + \sum_{i=1}^n \alpha_i(x) d\delta x^i. \quad (1.6)$$

Finally the *vertical lift* of α is the differential one-form $\alpha^\vee$ on $T\mathcal{X}$ defined as

$$\alpha^\vee := \pi_{\mathcal{X}}^* \alpha, \quad \pi_{\mathcal{X}} : T\mathcal{X} \rightarrow \mathcal{X} \text{ projection}, \quad (1.7)$$

i.e., in local coordinates

$$\alpha^\vee(x, \delta x) = \sum_{i=1}^n \alpha_i(x) dx^i. \quad (1.8)$$

Using the local coordinate expressions the following useful identities are easily verified.

Proposition 1.1 *For any function $h: \mathcal{X} \rightarrow \mathbb{R}$, any vector fields X, X_1 and X_2 on $\mathcal{X}$, and any differential one-form α on $\mathcal{X}$, we have*

$$X^C(h^\vee) = (X(h))^\vee = X^\vee(h^C), \quad X^C(h^C) = (X(h))^C, \quad X^\vee(h^\vee) = 0, \quad (1.9)$$

$$\alpha^C(X^\vee) = (\alpha(X))^\vee = \alpha^\vee(X^C), \quad \alpha^C(X^C) = (\alpha(X))^C, \quad \alpha^\vee(X^\vee) = 0, \quad (1.10)$$

$$[X_1^C, X_2^C] = [X_1, X_2]^C, \quad [X_1^C, X_2^\vee] = [X_1, X_2]^\vee, \quad [X_1^\vee, X_2^\vee] = 0, \quad (1.11)$$

$$dh^C = (dh)^C, \quad dh^\vee = (dh)^\vee. \quad (1.12)$$

Liftings of *distributions* and *co-distributions* to the tangent bundle are now defined as follows. Recall [18] that a distribution D on $\mathcal{X}$ is a map that assigns to every $p \in \mathcal{X}$ a subspace $D(p) \subset T_p \mathcal{X}$ in such a way that around each $p \in \mathcal{X}$ there exist vector fields $X_i, i \in I$ (with I an arbitrary index set), such that for $q \in \mathcal{X}$ close to p the subspace $D(q)$ is given as $D(q) = \text{span}\{X_i(q) \mid i \in I\}$. We will also call D a *subbundle* of the tangent bundle $T\mathcal{X}$.

Similarly [18], a co-distribution P on $\mathcal{X}$ (or, a subbundle of $T^*\mathcal{X}$) is locally given as $\text{span}\{\alpha_i(q) \mid i \in I\}, q \in \mathcal{X}$, with α_i differential one-forms on $\mathcal{X}$.

Recall [18] that given a distribution D we can define the *annihilating* co-distribution $P = \text{ann } D$. Dually, given a co-distribution P we can define the *kernel* distribution $D = \ker P$. For any D and P we have $D \subset \ker(\text{ann } D)$ respectively $P \subset \text{ann}(\ker D)$, while if D and P are *constant-dimensional* then equality in both expressions holds.

Definition 1.1 Let the distribution D on $\mathcal{X}$ be locally given as $D(q) = \text{span}\{X_i(q) \mid i \in I\}, q \in \mathcal{X}$, with X_i vector fields on $\mathcal{X}$. Then the lift (also called prolongation) of D is the distribution $\dot{D}$ on $T\mathcal{X}$ defined as

$$\dot{D}(z) = \text{span}\{X_i^C(z), X_i^\vee(z) \mid i \in I\}, \quad z \in T\mathcal{X}. \quad (1.13)$$

Analogously, let the co-distribution P on $\mathcal{X}$ be given as $P(q) = \text{span}\{\alpha_i(q) \mid i \in I\}, q \in \mathcal{X}$, with α_i differential one-forms on $\mathcal{X}$, then the lift of P is the co-distribution $\dot{P}$ on $T\mathcal{X}$ defined as

$$\dot{P}(z) = \text{span}\{\alpha_i^C(z), \alpha_i^\vee(z) \mid i \in I\}, \quad z \in T\mathcal{X}. \quad (1.14)$$

In case D and P are *involutive* and constant-dimensional we obtain the following simple local representations of $\dot{D}$ and $\dot{P}$. By Frobenius' Theorem we can find local coordinates $x = (x^1, \dots, x^n)$ such that $D = \text{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}\}, k = \dim D$. Then in the induced local coordinates $(x, \delta x)$ for $T\mathcal{X}$ we have

$$\dot{D} = \text{span}\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial \delta x^1}, \dots, \frac{\partial}{\partial \delta x^k}\right\}. \quad (1.15)$$

Similarly if $P = \text{span}\{dx^{l+1}, \dots, dx^n\}$, $n - l = \dim P$, then

$$\dot{P} = \text{span}\{dx^{l+1}, \dots, dx^n, d\delta x^{l+1}, \dots, d\delta x^n\}. \quad (1.16)$$

In general we obtain:

Proposition 1.2 *Let D be a distribution, and P be a co-distribution on $\mathcal{X}$. Then*

- (a) *If $D = \ker P$, then $\dot{D} = \ker \dot{P}$.*
- (b) *If $P = \text{ann } D$, then $\dot{P} = \text{ann } \dot{D}$.*
- (c) *If D (resp. P) has constant dimension then $\dot{D}$ (resp. $\dot{P}$) has constant dimension.*
- (d) *If D (resp. P) is involutive then $\dot{D}$ (resp. $\dot{P}$) is involutive.*
- (e) *If $X^C \in \dot{D}$ for a vector field X on $\mathcal{X}$, then $X \in D$.*

Next, the lifts of functions and vector fields to the *cotangent bundle* are defined; see again [27]. As before, for the tangent bundle case, the *vertical lift* $h^\vee : T^*\mathcal{X} \rightarrow \mathbb{R}$ of a function $h : \mathcal{X} \rightarrow \mathbb{R}$ is defined by $h^\vee = h \circ \pi_{\mathcal{X}}$, where $\pi_{\mathcal{X}} : T^*\mathcal{X} \rightarrow \mathcal{X}$ denotes the cotangent bundle projection $\pi_{\mathcal{X}}(x, p) = x$. In induced local coordinates $(x, p) := (x^1, \dots, x^n, p_1, \dots, p_n)$ for $T^*\mathcal{X}$ we have $h^\vee(x, p) = h(x)$.

Since there is a natural symplectic form on the cotangent bundle $T^*\mathcal{X}$ we can define the Hamiltonian vector field on $T^*\mathcal{X}$ corresponding to $h^\vee$, denoted by $X_{h^\vee}$, and called the *vertical Hamiltonian lift*. In induced local coordinates (x, p) for $T^*\mathcal{X}$

$$X_{h^\vee} = - \sum_{a=1}^n \frac{\partial h}{\partial x^a}(x) \frac{\partial}{\partial(p_a)}. \quad (1.17)$$

Furthermore, for any vector field f on $\mathcal{X}$ define the *Hamiltonian function* $H^f : T^*\mathcal{X} \rightarrow \mathbb{R}$ as

$$H^f(x, p) = \langle p, f(x) \rangle. \quad (1.18)$$

In local induced coordinates (x, p) we have $H^f(x, p) = p^T f(x)$. The corresponding Hamiltonian vector field on $T^*\mathcal{X}$, denoted by X_{H^f} , is called the *complete Hamiltonian lift*. In induced local coordinates for $T^*\mathcal{X}$

$$X_{H^f} = \sum_{a=1}^n f_a(x) \frac{\partial}{\partial x^a} - \sum_{a,b=1}^n \frac{\partial f_b}{\partial x^a}(x) p_b \frac{\partial}{\partial(p_a)}. \quad (1.19)$$

For later use we mention that given a lift (complete or vertical) of a vector field f to the tangent bundle, as well as a lift (Hamiltonian or vertical) to the cotangent bundle, we can *combine* the two lifts into a vector field defined on the Whitney sum $T\mathcal{X} \oplus T^*\mathcal{X}$ (that is, the base manifold $\mathcal{X}$ together with the fiber space $T_x\mathcal{X} \times T_x^*\mathcal{X}$ at any point $x \in \mathcal{X}$). An example (to be used later on) is the combination of the complete lift f^C on $T\mathcal{X}$ with the Hamiltonian extension X_{H^f} on $T^*\mathcal{X}$, which defines a vector field on $T\mathcal{X} \oplus T^*\mathcal{X}$, which will be denoted as $f^C \oplus X_{H^f}$. Furthermore, since the

vertical lifts $f^\vee$ (to $T\mathcal{X}$) and $X_{h^\vee}$ (to $T^*\mathcal{X}$) do not have components on the base manifold $\mathcal{X}$ we may also define the combined vector field $f^\vee \oplus X_{h^\vee}$ on the Whitney sum $T\mathcal{X} \oplus T^*\mathcal{X}$ for any vector field f and function h .

1.3 Invariance of Distributions and Co-Distributions

First recall the standard definitions of invariance of a (co-)distribution with respect to a vector field from geometric control theory, see e.g., [18]. Consider an n -dimensional state space manifold $\mathcal{X}$ with tangent bundle $T\mathcal{X}$ and cotangent bundle $T^*\mathcal{X}$. Consider furthermore a vector field f on $\mathcal{X}$. A distribution D on $\mathcal{X}$ is called *invariant* with respect to f if $L_f X \in D$ for any vector field X in D , or equivalently, around p it holds that $L_f X_i \in D, i \in I$. Here L_f denotes the Lie-derivative with respect to f ; i.e., $L_f X = [f, X]$. Invariance of D with respect to f will be abbreviated to $L_f D \subset D$.

Similarly, a co-distribution P on $\mathcal{X}$ (or, a subbundle of $T^*\mathcal{X}$) locally given as $\text{span}\{\alpha_i(q) \mid i \in I\}, q \in \mathcal{X}$, with α_i differential one-forms on $\mathcal{X}$, is called invariant if $L_f P \subset P$, that is, $L_f \alpha \in P$ for any one-form α in P , or equivalently around p , $L_f \alpha_i \in P, i \in I$. Invariance of P with respect to f will be abbreviated to $L_f P \subset P$.

Note that $L_f D \subset D$ implies $L_f(\text{ann } D) \subset \text{ann } D$ for any distribution D , and that $L_f P \subset P$ implies $L_f(\ker P) \subset \ker P$ for any co-distribution P .

Using the geometric notions of lifting discussed in the previous section, the invariance of (co-)distributions can be described in the following equivalent ways. Let us start with the case of distributions. Note that any distribution D on $\mathcal{X}$ can be also described by the following *submanifold* of $T\mathcal{X}$:

$$\mathcal{D} := \{(x, \delta x) \in T\mathcal{X} \mid \delta x \in D(x)\}. \quad (1.20)$$

Secondly, note that a vector field f on $\mathcal{X}$ is described by the map $F : \mathcal{X} \rightarrow T\mathcal{X}$ given as $F(x) = (x, f(x)) \in T\mathcal{X}$.

Proposition 1.3 *The distribution D on $\mathcal{X}$ is invariant with respect to the vector field f on $\mathcal{X}$ if and only if one of the following two equivalent conditions is satisfied:*

- (i) $F_* D \subset \dot{D}$,
- (ii) f^c is tangent to $\mathcal{D}$.

Proof The equivalence with (i) is already proved in [18]. For the equivalence with condition (ii) consider local coordinates x and induced local coordinates $x, \delta x$ for $T\mathcal{X}$. Then the vector field f^c at a point $(x, \delta x = X(x)) \in \mathcal{D} \subset T\mathcal{X}$ is given by

$$\begin{bmatrix} f(x) \\ \frac{\partial f}{\partial x}(x)X(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ \frac{\partial X}{\partial x}(x)f(x) - L_f X(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ \frac{\partial X}{\partial x}(x)f(x) \end{bmatrix} - \begin{bmatrix} 0 \\ L_f X(x) \end{bmatrix}, \quad (1.21)$$

where the first vector in the last term denotes a tangent vector to $\mathcal{D}$. Hence if $L_f D \subset D$ then the vector field f^c is tangent to $\mathcal{D}$. Conversely, if f^c is tangent to $\mathcal{D}$ then this implies that the second vector in the last term is tangent to $\mathcal{D}$ for all $X(x) \in D(x)$, which amounts to $L_f D \subset D$. $\square$

Note that the equality expressed in (1.21) can be written in a coordinate-free way as

$$F_* X = X^c - [f, X]^v \quad (1.22)$$

for any vector field X .

Similarly, any co-distribution P on $\mathcal{X}$ can be described by the following *submanifold* of $T^*\mathcal{X}$:

$$\mathcal{P} := \{(x, p) \in T^*\mathcal{X} \mid p \in P(x)\}. \quad (1.23)$$

Then we obtain:

Proposition 1.4 *The distribution P on $\mathcal{X}$ is invariant with respect to the vector field f on $\mathcal{X}$ if and only if one of the following two equivalent conditions is satisfied:*

- (i) $F^* \dot{P} = P$,
- (ii) X_{Hf} is tangent to $\mathcal{P}$.

Proof Again, the equivalence with condition (i) is already shown in [18]. For the second condition, consider coordinates x and induced local coordinates (x, p) for $T^*\mathcal{X}$. Then the vector field X_{Hf} at a point $(x, p = \alpha(x)) \in \mathcal{P} \subset T^*\mathcal{X}$ is given by

$$\begin{bmatrix} f(x) \\ -(\frac{\partial f}{\partial x})^T(x)\alpha(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ \frac{\partial \alpha}{\partial x}(x)f(x) - L_f \alpha(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ \frac{\partial \alpha}{\partial x}(x)f(x) \end{bmatrix} - \begin{bmatrix} 0 \\ L_f \alpha(x) \end{bmatrix},$$

where the first vector in the last term denotes a tangent vector to $\mathcal{P}$. Thus if $L_f P \subset P$ holds then the vector field X_{Hf} is tangent to $\mathcal{P}$. Conversely, if X_{Hf} is tangent to $\mathcal{P}$ then this implies that the second vector in the last term is tangent to $\mathcal{P}$ for all $\alpha(x) \in P(x)$, which amounts to $L_f P \subset P$. $\square$

Note the slight asymmetry with respect to the condition $F_* D \subset \dot{D}$ in the invariant distribution case. This is due to the definitions of $F(x) = (x, f(x))$ and $\dot{P}$ implying that for any P and f we have $P \subset F^* \dot{P}$.

Conditions (ii) in the above two propositions are especially appealing because they imply that invariance of (co-)distributions can be equivalently interpreted as invariance of *submanifolds* (of the tangent, respectively cotangent, bundle). In a certain sense this unifies the treatment of invariance in geometric control theory, where usually [18] the distinction is made between invariant (co-)distributions and invariant manifolds (of the state space manifold).

1.3.1 Controlled Invariance

The above geometric framework turned out to be especially useful for a generalization of the notion of invariance for distributions and co-distributions to *controlled invariance*, cf. [17] and Chap. 13 of [18]. Consider a nonlinear control system

$$\dot{x} = f(x, u), \quad x \in \mathcal{X}, u \in \mathcal{U}, \quad (1.24)$$

where $\mathcal{U}$ is an m -dimensional control manifold (e.g., $\mathcal{U} = \mathbb{R}^m$). A distribution D on the n -dimensional state space manifold $\mathcal{X}$ is called *invariant* with respect to the system (1.24) if D is invariant with respect to the vector fields $f(\cdot, u)$ for every $u \in \mathcal{U}$. Furthermore, the distribution D is called *locally controlled invariant* if there locally exists a feedback map $u = u(x, v)$, $v \in \mathcal{U}$, $\text{rank } \frac{\partial u}{\partial v} = m$, such that D is invariant with respect to the closed-loop dynamics $\dot{x} = f(x, u(x, v))$ with new inputs $v \in \mathcal{U}$.

The basic theorem is as follows. Define the system map $F: \mathcal{X} \times \mathcal{U} \rightarrow T\mathcal{X}$ by $F(x, u) = (x, f(x, u))$, and note the following consequence of Proposition 1.4.

Proposition 1.5 *Consider the nonlinear dynamics (1.24) given by the system map $F: \mathcal{X} \times \mathcal{U} \rightarrow T\mathcal{X}$. Let D be a distribution on $\mathcal{X}$. Define D_e as the unique distribution on $\mathcal{X} \times \mathcal{U}$ such that*

$$\pi_* D_e = D, \quad \bar{\pi}_* D_e = 0, \quad (1.25)$$

with π , resp. $\bar{\pi}$, being the natural projection of $\mathcal{X} \times \mathcal{U}$ on $\mathcal{X}$, resp. $\mathcal{U}$. Then D is invariant for (1.24) if and only if

$$F_* D_e \subset \dot{D}. \quad (1.26)$$

(Local) controlled invariance is now characterized as follows.

Theorem 1.1 *Consider the nonlinear system (1.24) with system map $F: \mathcal{X} \times \mathcal{U} \rightarrow T\mathcal{X}$. Let D be an involutive distribution of constant dimension on $\mathcal{X}$. Define the vertical distribution V on $\mathcal{X} \times \mathcal{U}$ as the kernel of π_* , where $\pi: \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$ is the projection on $\mathcal{X}$. Assume that the distribution*

$$\tilde{V} := \{Z \in V \mid F_* Z \in \dot{D}\} \quad (1.27)$$

on $\mathcal{X} \times \mathcal{U}$ has constant dimension. Then D is locally controlled invariant if and only if

$$F_* D_e \subset \dot{D} + F_* V. \quad (1.28)$$

Remark 1.1 Notice that (1.28) may be equivalently replaced by the requirement

$$F_*(\pi_*^{-1}(D)) \subset \dot{D} + F_* V. \quad (1.28')$$

From a geometric point of view the above theorem yields the following characterization of local controlled invariance. Let D satisfy the assumptions of Theorem 1.1, and denote $P = \text{ann } D$. Then $E := \ker F^* \dot{P}$ is an involutive distribution on $M \times U$. Moreover if (1.28) holds, then E is constant-dimensional and satisfies

$$\pi_* E = D. \quad (1.29)$$

Furthermore, by definition of E and by Proposition 1.2

$$F_* E = F_*(\ker F^* \dot{P}) \subset \ker \dot{P} = \dot{D}. \quad (1.30)$$

Then it is easy to see that, at least locally, there exists an involutive distribution D_e on $\mathcal{X} \times \mathcal{U}$ with $D_e \subset E$, $\pi_* D_e = D$ and with $\dim D_e = \dim D$. This distribution D_e defines a feedback map $u = u(x, v)$ that renders D invariant for the closed-loop dynamics by requiring that the leaves of D_e correspond to the submanifolds where v is constant.

1.4 Invariant Lagrangian Subbundles and Contraction Analysis

Remarkably, there is close geometric connection between the previous theory of (controlled) invariance of (co-)distributions with the theory of *convergent dynamics*, as originating in classical work in differential equations, see e.g., [10, 20] for treatments and references, and revisited in the theory of *contraction analysis*, see e.g., [12–16]. The key idea, see [25], is to interpret (co-)distributions as subbundles of $T\mathcal{X}$, respectively $T^*\mathcal{X}$, and to generalize the notion of invariance of (co-)distributions to invariance of subbundles of the Whitney sum $T\mathcal{X} \oplus T^*\mathcal{X}$.

Definition 1.2 A subbundle K of $T\mathcal{X} \oplus T^*\mathcal{X}$ is a vector bundle over $\mathcal{X}$ with fiber $K(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$ at any point $x \in \mathcal{X}$. The subbundle K is called *invariant* with respect to a vector field f on $\mathcal{X}$ if

$$(L_f X, L_f \alpha) \in K \text{ for any } (X, \alpha) \in K. \quad (1.31)$$

Remark 1.2 If K has only zero components in $T_x^*\mathcal{X}$ for any point $x \in \mathcal{X}$, then K can be identified with a *distribution* on $\mathcal{X}$. Alternatively, if K has only zero components in $T_x\mathcal{X}$ for any point $x \in \mathcal{X}$, then it can be identified with a *co-distribution* on $\mathcal{X}$. In these cases invariance of K amounts to invariance of the identified distribution, respectively co-distribution.

Remark 1.3 The above definition of invariance of K is formally identical to the definition of an *infinitesimal symmetry* of a *Dirac structure*; see [3, 6, 23] for details. (A Dirac structure is a subbundle of $T\mathcal{X} \oplus T^*\mathcal{X}$ which is maximally isotropic with respect to the duality product.)

Associated to the subbundle K define the *submanifold* $\mathcal{K}$ of $T\mathcal{X} \oplus T^*\mathcal{X}$ as follows

$$\mathcal{K} := \{(x, \delta x, p) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid (\delta x, p) \in K(x)\}. \quad (1.32)$$

We have the following characterization of invariance of K . Recall from Sect. 1.2 the definition of the lift $f^c \oplus X_{Hf}$ on the Whitney sum $T\mathcal{X} \oplus T^*\mathcal{X}$.

Proposition 1.6 *The subbundle K is invariant with respect to the vector field f on $\mathcal{X}$ if and only if the submanifold $\mathcal{K}$ is invariant for the vector field $f^c \oplus X_{Hf}$ on $T\mathcal{X} \oplus T^*\mathcal{X}$.*

Proof In coordinates x and induced local coordinates for $T\mathcal{X}$ and $T^*\mathcal{X}$ the vector field $f^c \oplus X_{Hf}$ at a point $(x, X(x), \alpha(x)) \in \mathcal{K} \subset T\mathcal{X} \oplus T^*\mathcal{X}$ is given by

$$\begin{aligned} \begin{bmatrix} f(x) \\ \frac{\partial f}{\partial x}(x)X(x) \\ -(\frac{\partial f}{\partial x})^T(x)\alpha(x) \end{bmatrix} &= \begin{bmatrix} f(x) \\ \frac{\partial X}{\partial x}(x)f(x) - L_f X(x) \\ \frac{\partial \alpha}{\partial x}(x)f(x) - L_f \alpha(x) \end{bmatrix} \\ &= \begin{bmatrix} f(x) \\ \frac{\partial X}{\partial x}(x)f(x) \\ \frac{\partial \alpha}{\partial x}(x)f(x) \end{bmatrix} - \begin{bmatrix} 0 \\ L_f X(x) \\ L_f \alpha(x) \end{bmatrix}, \end{aligned}$$

where the first vector in the last term denotes a tangent vector to $\mathcal{K}$. Thus if (1.31) holds then the vector field $f^c \oplus X_{Hf}$ is tangent to $\mathcal{K}$. Conversely, if $f^c \oplus X_{Hf}$ is tangent to $\mathcal{K}$ then this implies that the second vector in the last term is tangent to $\mathcal{K}$ for all $(X(x), \alpha(x)) \in K(x)$, which amounts to (1.31), i.e., invariance of K . $\square$

In the rest of this section we will consider a special type of subbundle of the Whitney sum $T\mathcal{X} \oplus T^*\mathcal{X}$ defined as follows.

Definition 1.3 A subbundle K of $T\mathcal{X} \oplus T^*\mathcal{X}$ is called a *Lagrangian subbundle* if $K(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$ is a Lagrangian subspace (with respect to the canonical symplectic form on $T_x\mathcal{X} \times T_x^*\mathcal{X}$ [1]) for every $x \in \mathcal{X}$.

All subbundles

$$\begin{aligned} K(x) &= \{(\delta x, p) \mid V(x)p = U(x)\delta x, \\ V(x)U^T(x) &= U(x)V^T(x), \text{rank}[U(x) \ V(x)] = n\} \end{aligned} \quad (1.33)$$

with $U(x)$, $V(x)$ $n \times n$ matrices depending on x (with n the dimension of the state space manifold $\mathcal{X}$), are Lagrangian, and conversely, all Lagrangian subbundles can be represented in this way.

Additionally assume that the projection of $K(x) \subset T_x\mathcal{X} \oplus T_x^*\mathcal{X}$ on $T_x\mathcal{X}$ is equal to the whole tangent space $T_x\mathcal{X}$ for all $x \in \mathcal{X}$. Then in any set of local

coordinates $x^1, \dots, x^n$ for $\mathcal{X}$ the Lagrangian subbundle K is spanned by pairs of vector fields and one-forms

$$\left(\frac{\partial}{\partial x^i}, \pi_i \right), \quad i = 1, \dots, n,$$

where the one-forms $\pi_i(x) = \pi_{1i}(x)dx^1 + \dots + \pi_{ni}(x)dx^n, i = 1, \dots, n$ satisfy, because of the fact that K is Lagrangian, the symmetry property

$$\pi_{ji}(x) = \pi_{ij}(x), \quad i, j = 1, \dots, n.$$

Conversely, all subbundles K with $K(x) = \{(\delta x, p) \mid p = \Pi(x)\delta x\}$, where $\Pi(x)$ is a *symmetric* matrix, are Lagrangian.

Now consider any such Lagrangian subbundle K with $K(x) = \{(\delta x, p) \mid p = \Pi(x)\delta x\}$, with $\Pi(x)$ symmetric. Invariance of such a Lagrangian subbundle K with respect to $f^c \oplus X_{Hf}$ can be seen to amount to [25]

$$\left(\frac{\partial f}{\partial x} \right)^T (x) \Pi(x) + \Pi(x) \frac{\partial f}{\partial x} (x) + \frac{\partial \Pi}{\partial x} (x) f(x) = 0. \quad (1.34)$$

This is a limiting case of the central equation of contraction analysis [10, 13, 21], which amounts to

$$\left(\frac{\partial f}{\partial x} \right)^T (x) \Pi(x) + \Pi(x) \frac{\partial f}{\partial x} (x) + \frac{\partial \Pi}{\partial x} (x) f(x) = -Q(x) \quad (1.35)$$

for some positive semi-definite matrix $Q(x)$. Geometrically, this means that the submanifold $\mathcal{K}$ corresponding to the Lagrangian subbundle K is invariant for the extended vector field $f^c \oplus X_{Hf} + q$, where q is the vector field in local coordinates x given by

$$q(x) = \sum_{i,j} Q_{ij}(x) \delta x^j \frac{\partial}{\partial p_i}. \quad (1.36)$$

We will call (1.35) a *differential Lyapunov equation*, cf. [25].

Alternatively, (1.34) can be interpreted from a Riemannian geometry point of view, by identifying it as the equation for a *Killing vector field* [21]; see also [2]. Indeed, the symmetric matrix $\Pi(x)$, in case it is positive-definite, defines a *Riemannian metric* on $\mathcal{X}$, in which case (1.34) amounts to the vector field f being a Killing vector field for this Riemannian metric, characterized by the property

$$L_f \langle X, Y \rangle = \langle L_f X, Y \rangle + \langle X, L_f Y \rangle, \quad (1.37)$$

for any vector fields X, Y , with $\langle \cdot, \cdot \rangle$ denoting the Riemannian metric defined by the positive-definite matrix $\Pi(x)$. Finally, if this Riemannian metric has curvature zero, then in appropriate coordinates x the matrix $\Pi(x)$ takes the form of a *constant*

matrix P , and (1.35) reduces to the Demidovich condition [19, 20]

$$\left(\frac{\partial f}{\partial x}\right)^T(x)P + P\frac{\partial f}{\partial x}(x) = -Q(x). \quad (1.38)$$

As an open problem we mention the characterization of *locally controlled invariant* Lagrangian subbundles, in a similar geometric spirit as the characterization of locally controlled invariant distributions sketched before.

1.4.1 Prolongation of Nonlinear Control Systems to Tangent and Cotangent Bundle and Further Differential Properties

The previous geometric interpretation of the basic equation of contraction analysis is part of a larger story translating incremental analysis tools into *differential properties*. A basic notion in this larger story is the prolongation (or lift) of a *nonlinear control system* to the tangent and cotangent bundle, using the notions of the lifts of functions and vector fields to tangent and cotangent bundle as described in the previous subsection. This can be done as in [4], see also [2, 24].

Consider a nonlinear control system Σ with state space manifold $\mathcal{X}$, affine in the inputs u and with outputs y determined by the state x ,

$$\begin{aligned} \Sigma : \quad \dot{x} &= f(x) + \sum_{j=1}^m u_j g_j(x) \\ y_j &= h_j(x), \quad j = 1, \dots, r, \end{aligned} \quad (1.39)$$

where $x \in \mathcal{X}$, and $u = (u_1, \dots, u_m) \in \mathcal{U} \subset \mathbb{R}^m$. The set $\mathcal{U}$ is the input space, which is assumed to be an open subset of $\mathbb{R}^m$. Finally, $\mathcal{Y} = \mathbb{R}^r$ is the output space.

The prolongation of the nonlinear control system to the tangent bundle and the cotangent bundle is constructed as follows; cf. [4].

Given an initial state $x(0) = x_0$, take any coordinate neighborhood of $\mathcal{X}$ containing x_0 . Let $t \in [0, T] \mapsto x(t)$ be the solution of (1.39) corresponding to the admissible input function $t \in [0, T] \mapsto u(t) = (u_1(t), \dots, u_m(t))$ and the initial state $x(0) = x_0$, such that $x(t)$ remains within the selected coordinate neighborhood. Denote the resulting output by $t \in [0, T] \mapsto y(t) = (y_1(t), \dots, y_r(t))$, with $y_j(t) = H_j(x(t))$. Then the *variational system* along the input-state-output trajectory $t \in [0, T] \mapsto (x(t), u(t), y(t))$ is given by the following time-varying system

$$\begin{aligned} \delta \dot{x}(t) &= \frac{\partial f}{\partial x}(x(t))\delta x(t) \\ &+ \sum_{j=1}^m u_j(t) \frac{\partial g_j}{\partial x}(x(t))\delta x(t) \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^m \delta u_j(t) g_j(x(t)) \\
\delta y_j(t) &= \frac{\partial h_j}{\partial x}(x(t)) \delta x(t), \quad j = 1, \dots, r,
\end{aligned} \tag{1.40}$$

with state $\delta x(t) \in T_{x(t)}^* \mathcal{X}$, where $\delta u = (\delta u_1, \dots, \delta u_m)^T$, $\delta y = (\delta y_1, \dots, \delta y_r)^T$ denote the input and output vectors of the variational system. (Note that $\frac{\partial h_j}{\partial x}(x)$ denotes a row vector.)

The reason behind the terminology “variational” comes from the following fact: let $(x(t, \varepsilon), u(t, \varepsilon), y(t, \varepsilon))$, $t \in [0, T]$, be a family of input-state-output trajectories of (1.39) parameterized by $\varepsilon \in (-c, c)$, $c > 0$, with $x(t, 0) = x(t)$, $u(t, 0) = u(t)$ and $y(t, 0) = y(t)$, $t \in [0, T]$. Then the infinitesimal variations

$$\delta x(t) = \frac{\partial x}{\partial \varepsilon}(t, 0), \quad \delta u(t) = \frac{\partial u}{\partial \varepsilon}(t, 0), \quad \delta y(t) = \frac{\partial y}{\partial \varepsilon}(t, 0), \tag{1.41}$$

satisfy (1.40).

Remark 1.4 For a linear system $\dot{x} = Ax + Bu$, $y = Cx$ the variational systems along any trajectory are simply given as $\delta \dot{x} = A\delta x + B\delta u$, $\delta y = C\delta x$.

The *prolongation* (or *prolonged system*) of (1.39) comprises the original system (1.39) together with its variational systems, that is the total system

$$\begin{aligned}
\dot{x} &= f(x) + \sum_{j=1}^m u_j g_j(x) \\
\delta \dot{x}(t) &= \frac{\partial f}{\partial x}(x(t)) \delta x(t) + \\
&\quad \sum_{j=1}^m u_j(t) \frac{\partial g_j}{\partial x}(x(t)) \delta x(t) + \\
&\quad \sum_{j=1}^m \delta u_j(t) g_j(x(t)) \\
y_j &= h_j(x), \quad j = 1, \dots, r \\
\delta y_j(t) &= \frac{\partial h_j}{\partial x}(x(t)) \delta x(t), \quad j = 1, \dots, r,
\end{aligned} \tag{1.42}$$

with inputs $u_j, \delta u_j$, $j = 1, \dots, m$, outputs $y_j, \delta y_j$, $j = 1, \dots, r$, and state vector $x, \delta x$.

Using the previous subsection, the prolonged system (1.42) on the tangent space $T\mathcal{X}$ can be intrinsically defined in the following coordinate-free way. Denote the elements of $T\mathcal{X}$ by $x_l = (x, \delta x)$, where $\tau_{\mathcal{X}}(x_l) = x \in \mathcal{X}$ with $\tau_{\mathcal{X}} : T\mathcal{X} \rightarrow \mathcal{X}$ again the tangent bundle projection.

Definition 1.4 ([4]) The prolonged system $\delta \Sigma$ of a nonlinear system Σ of the form (1.39) is defined as the system

$$\begin{aligned}
\dot{x}_l &= f^c(x_l) + \sum_{j=1}^m u_j g_j^c(x_l) + \sum_{j=1}^m \delta u_j g_j^v(x_l) \\
\delta \Sigma : \quad y_j &= h_j^v(x_l), \quad j = 1, \dots, r \\
\delta y_j &= h_j^c(x_l), \quad j = 1, \dots, r,
\end{aligned} \tag{1.43}$$

with state $x_l = (x, \delta x) \in T\mathcal{X}$, inputs $u_j, \delta u_j, j = 1, \dots, m$, and outputs $y_j, \delta y_j, j = 1, \dots, r$.

Note that the prolonged system $\delta\Sigma$ has state space $T\mathcal{X}$, input space $T\mathcal{U}$ and output space $T\mathcal{Y}$. One can easily check that in any system of local coordinates x for $\mathcal{X}$, u for $\mathcal{U}$, and y for $\mathcal{Y}$, and the induced local coordinates $x, \delta x$ for $T\mathcal{X}$, $u, \delta u$ for $T\mathcal{U}$, $y, \delta y$ for $T\mathcal{Y}$, the local expression of the system (1.43) equals (1.42).

Remark 1.5 For a linear system $\dot{x} = Ax + Bu, y = Cx$ the prolonged system is simply the *product* of the system with the copy system $\delta\dot{x} = A\delta x + B\delta u, \delta y = C\delta x$.

The prolongation of the nonlinear control system Σ to the *cotangent bundle* is defined as follows. Associated to the variational system (1.40) there is the *adjoint variational system*, defined as

$$\begin{aligned} \dot{p}(t) &= -\left(\frac{\partial f}{\partial x}\right)^T(x(t))p(t) \\ &\quad - \sum_{j=1}^m u_j(t)\left(\frac{\partial g_j}{\partial x}\right)^T(x(t))p(t) \\ &\quad - \sum_{j=1}^r du_j(t)\frac{\partial^T h_j}{\partial x}(x(t)) \\ dy_j(t) &= p^T g_j(x(t)), \quad j = 1, \dots, m, \end{aligned} \quad (1.44)$$

with state variables $p \in T_{x(t)}^*\mathcal{X}$, and adjoint variational inputs and outputs $du_j, j = 1, \dots, r, dy_j, j = 1, \dots, m$. Then, the original nonlinear system Σ together with all its adjoint variational systems defines the total system

$$\begin{aligned} \dot{x} &= f(x) + \sum_{j=1}^m u_j g_j(x) \\ \dot{p}(t) &= -\left(\frac{\partial f}{\partial x}\right)^T(x(t))p(t) \\ &\quad - \sum_{j=1}^m u_j(t)\left(\frac{\partial g_j}{\partial x}\right)^T(x(t))p(t) \\ &\quad - \sum_{j=1}^r du_j(t)\frac{\partial^T h_j}{\partial x}(x(t)) \\ y_j &= h_j(x), \quad j = 1, \dots, r \\ dy_j(t) &= p^T g_j(x(t)), \quad j = 1, \dots, m, \end{aligned} \quad (1.45)$$

with inputs u_j, du_j , outputs y_j, dy_j and state x, p . This total system is called the *Hamiltonian extension*. In a coordinate-free way the Hamiltonian extension is defined as follows.

Definition 1.5 ([4]) The Hamiltonian extension $d\Sigma$ of a nonlinear system Σ of the form (1.39) is defined as the system

$$\begin{aligned} \dot{x}_e &= X_{H^f}(x_e) + \sum_{j=1}^m u_j X_{H^{g_j}}(x_e) + \sum_{j=1}^r du_j X_{h_j^y}(x_e) \\ d\Sigma : \quad y_j &= h_j^y(x_e), \quad j = 1, \dots, r \\ dy_j &= H^{g_j}(x_e), \quad j = 1, \dots, m, \end{aligned} \quad (1.46)$$

with state $x_e = (x, p) \in T^*\mathcal{X}$, inputs u_j, du_j and outputs y_j, dy_j .

Note that the Hamiltonian extension $d\Sigma$ has state space $T^*\mathcal{X}$, and combined input and output space $T^*\mathcal{U} \times T^*\mathcal{Y}$, where $(u, \delta y) \in T^*\mathcal{U}$ and $(y, \delta u) \in T^*\mathcal{Y}$.

Remark 1.6 For a linear system $\dot{x} = Ax + Bu$, $y = Cx$ the Hamiltonian extension is the product of the system with its adjoint system $\dot{p} = -A^T p - C^T du$, $dy = B^T p$.

The prolongation $\delta\Sigma$ of Σ to the tangent bundle can be *combined* with the Hamiltonian extension $d\Sigma$ of Σ to the cotangent bundle. This will define a system on the Whitney sum $T\mathcal{X} \oplus T^*\mathcal{X}$ with inputs $u, \delta u, du$, states $x, \delta x, p$ and outputs $y, \delta y, dy$. It can be immediately verified that

$$\frac{d}{dt}\langle p, \delta x \rangle = \langle dy, \delta u \rangle - \langle du, \delta y \rangle, \quad (1.47)$$

which equality is in fact underlying the definition of the adjoint variational system; see also [4].

1.4.2 Differential Lyapunov Equations

In [25] it was shown how *differential Riccati equations* can be obtained by interconnecting the prolonged system $\delta\Sigma$ with the Hamiltonian extension $d\Sigma$ through the interconnection laws

$$\delta u = -dy, \quad du = \delta y, \quad (1.48)$$

and to consider invariant Lagrangian subbundles for the resulting *differential Hamiltonian system* living on the Whitney sum $T\mathcal{X} \oplus T^*\mathcal{X}$. This yields a geometric formulation of the approach developed e.g., in [11, 12, 14–16, 26].

Differential Lyapunov equations result from the case $g_j = 0$, $j = 1, \dots, m$, for the original system Σ (i.e., no inputs). In this case the interconnection of the prolongation $\delta\Sigma$ and the Hamiltonian extension $d\Sigma$ reduces to $du = \delta y$, and can be seen to lead to the differential Hamiltonian system (see [25] for further details)

$$\begin{aligned} \dot{z} &= f^c \oplus X_{H^f}(z) - \sum_{j=1}^r h_j^c X_{h_j^y}(z), \quad z \in T\mathcal{X} \\ y_j &= h_j(x), \quad j = 1, \dots, r. \end{aligned} \quad (1.49)$$

The submanifold $\mathcal{K}$ corresponding to a Lagrangian subbundle K with $K(x) = \{(\delta x, p) \mid p = \Pi(x)\delta x\}$, where $\Pi(x)$ is a *symmetric* matrix, is an *invariant submanifold* for (1.49) if and only if $\Pi(x)$ satisfies the equation

$$\left(\frac{\partial f}{\partial x}\right)^T(x)\Pi(x) + \Pi(x)\frac{\partial f}{\partial x}(x) + \left(\frac{\partial h}{\partial x}\right)^T(x)\frac{\partial h}{\partial x}(x) + \frac{\partial \Pi}{\partial x}(x)f(x) = 0. \quad (1.50)$$

This is nothing else than the *differential Lyapunov equation* (1.35) as considered before, where the matrix $Q(x)$ in (1.35) is of a somewhat special form.

1.5 Conclusions and Outlook

In this chapter (controlled) invariance of (co-)distributions has been reviewed using the geometric notions of liftings to the (co-)tangent bundle, continuing upon the approach initiated in [17] and further developed in [18]. It has been shown how the geometric treatment of invariance of (co-)distributions can be extended to invariance of Lagrangian subbundles, making a direct link to the theory of *convergent dynamics* and *contraction analysis*.

Next to the classical topic of (controlled) invariant (co-)distributions, and the recent developments on differential Lyapunov and differential Riccati equations, the established geometric framework also opens other possibilities. An important avenue for further research is the extension of the classical passivity and small-gain framework, see e.g., [22], to the differential level. Indeed, in [7–9, 24] differential passivity was defined and explored in this spirit. Furthermore, in [15, 24] differential L_2 -gain was initiated. Despite the beauty of this approach and the insight provided, the implications for control design of these recent developments are yet to be seen, and probably require the combination with dynamical analysis tools.

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