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Published in:
Automatica

DOI:
[10.1016/j.automatica.2017.06.001](https://doi.org/10.1016/j.automatica.2017.06.001)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2017

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Wei, J., Everts, A. R. F., Camlibel, M. K., & van der Schaft, A. J. (2017). Consensus dynamics with arbitrary sign-preserving nonlinearities. *Automatica*, 83, 226-233. <https://doi.org/10.1016/j.automatica.2017.06.001>

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Consensus dynamics with arbitrary sign-preserving nonlinearities[☆]

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ARTICLE INFO

Article history:

Received 4 September 2015

Received in revised form 20 April 2017

Accepted 12 May 2017

Available online 3 July 2017

Keywords:

Multi-agent systems

Consensus

Nonsmooth analysis

Port-Hamiltonian systems

ABSTRACT

This paper studies consensus problems for multi-agent systems defined on directed graphs where the consensus dynamics involves general nonlinear and discontinuous functions. Sufficient conditions, only involving basic properties of the nonlinear functions and the topology of the underlying graph, are derived for the agents to converge to consensus.

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1. Introduction

Nonlinear agreement protocols have recently attracted the attention of many researchers. They may arise due to the nature of the controller, see e.g. Jafarian and De Persis (2015) and Saber and Murray (2003), or may describe the physical coupling existing in the network, see e.g. Bürger, Zelazo, and Allgöwer (2014) and Monshizadeh and De Persis (2015). In this paper, we consider a general nonlinear consensus protocol. The topology among the agents is assumed to be a directed graph containing a directed spanning tree, which for the linear consensus protocol is known to be a sufficient and necessary condition for reaching state consensus.

Previous work related to this paper can be divided into two categories, depending on whether the dynamical systems are continuous or not. For the case of continuous dynamical systems, closely related to this paper are Lin, Francis, and Maggiore (2007) and Papachristodoulou, Jadbabaie, and Münz (2010). In Papachristodoulou et al. (2010), a general first-order consensus protocol with a continuous nonlinear function is considered for the case that there is delay in the communication. In Lin et al. (2007), the authors considered a nonlinear consensus protocol with Lipschitz continuous functions, under a switching topology. For the case of discontinuous dynamical system, Cortés (2006) is one of the major motivations of this paper. Nonlinearities of the form of sign functions were considered in Cortés (2006), where the notion

of Filippov solutions is employed. However, in order to guarantee asymptotic consensus of the second network protocol in Section 4 of Cortés (2006), further conditions turn out to be necessary. This is formulated as the main result in Section 3.2 of this paper. In De Persis and Frasca (2013), the authors considered a similar control protocol as in Cortés (2006) in a hybrid dynamical systems framework with a self-triggered communication policy, which avoids the notion of Filippov solutions. In addition, in De Persis and Frasca (2013) practical consensus is considered, that is, consensus within a predefined margin. The results presented in Cortés (2006) and De Persis and Frasca (2013) are restricted to undirected graphs. In Dimarogonas and Johansson (2010), the authors considered quantized communication protocols within the framework of hybrid dynamical systems, without using the notion of Filippov solutions.

The contribution of this paper is to provide a uniform framework to analyze the convergence towards consensus of a first-order consensus protocol for a very general class of discontinuous nonlinear functions, under the weakest fixed topology assumption, i.e., a directed graph containing a directed spanning tree. The analysis is conducted with the notion of Filippov solutions, and generalizes and corrects the second network consensus protocol in Cortés (2006).

The structure of the paper is as follows. In Section 2, we introduce some terminology and notation in the context of graph theory and stability analysis of discontinuous dynamical systems. The main results are presented in Theorems 7 and 19 in Section 3. The general problem is introduced in Section 3.1, whereafter in Sections 3.2 and 3.3 two important subcases are considered, which are then combined in Section 3.4. In Section 4 we study error dynamics corresponding to the systems considered in Section 3.2,

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Dimos V. Dimarogonas under the direction of Editor Christos G. Cassandras.

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and provide sufficient conditions for the equivalence between the convergence of the error to zero and the convergence of the original states to consensus. Conclusions follow in Section 5.

2. Preliminaries and notations

In this section we briefly review some notions from graph theory, and give some definitions and notation regarding Filippov solutions.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a weighted digraph with node set $\mathcal{V} = \{v_1, \dots, v_n\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements a_{ij} . An edge of $\mathcal{G}$ is denoted by $e_{ij} := (v_i, v_j)$ and we write $\mathcal{I} = \{1, 2, \dots, n\}$. The adjacency elements a_{ij} are associated with the edges of the graph such that $a_{ij} > 0$ if and only if $e_{ji} \in \mathcal{E}$, while $a_{ii} = 0$ for all $i \in \mathcal{I}$. For undirected graphs $A = A^\top$.

The set of neighbors of node v_i is denoted by $N_i := \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$. For each node v_i , its in-degree and out-degree are defined as

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^n a_{ij}, \quad \deg_{\text{out}}(v_i) = \sum_{j=1}^n a_{ji}.$$

The degree matrix of the digraph $\mathcal{G}$ is a diagonal matrix Δ where $\Delta_{ii} = \deg_{\text{in}}(v_i)$. The graph Laplacian is defined as $L = \Delta - A$ and satisfies $L\mathbf{1} = 0$, where $\mathbf{1}$ is the n -vector containing only ones. We say that a node v_i is *balanced* if its in-degree and out-degree are equal. The graph $\mathcal{G}$ is called *balanced* if all of its nodes are balanced or, equivalently, if $\mathbf{1}^\top L = 0$.

A directed path from node v_i to node v_j is a chain of edges from $\mathcal{E}$ such that the first edge starts from v_i , the last edge ends at v_j and every edge in between starts where the previous edge ends. If for every two nodes v_i and v_j there is a directed path from v_i to v_j , then the graph $\mathcal{G}$ is called *strongly connected*. A subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', A')$ of $\mathcal{G}$ is called a *directed spanning tree* for $\mathcal{G}$ if $\mathcal{V}' = \mathcal{V}$, $\mathcal{E}' \subseteq \mathcal{E}$, and for every node $v_i \in \mathcal{V}'$ there is exactly one node v_j such that $e_{ji} \in \mathcal{E}'$, except for one node, which is called the root of the spanning tree. Furthermore, we call a node $v \in \mathcal{V}$ a *root* of $\mathcal{G}$ if there is a directed spanning tree for $\mathcal{G}$ with v as a root. In other words, if v is a root of $\mathcal{G}$, then there is a directed path from v to every other node in the graph. A digraph $\mathcal{G}$ is called *weakly connected* if $\mathcal{G}^0$ is connected, where $\mathcal{G}^0$ is the undirected graph obtained from $\mathcal{G}$ by ignoring the orientation of the edges.

The multi-dimensional saturation function *sat* and sign function *sign* are defined as follows. For any $x \in \mathbb{R}^n$,

$$\text{sat}(x; u^-, u^+)_i = \begin{cases} u_i^- & \text{if } x_i < u_i^-, \\ x_i & \text{if } x_i \in [u_i^-, u_i^+], \\ u_i^+ & \text{if } x_i > u_i^+ \end{cases}, \quad i \in \mathcal{I}, \quad (1)$$

$$\text{sign}(x)_i = \begin{cases} -1 & \text{if } x_i < 0, \\ 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_i > 0, \end{cases} \quad i \in \mathcal{I}, \quad (2)$$

where u^- and u^+ are n -vectors containing the lower and upper bounds respectively.

With $\mathbb{R}_-$, $\mathbb{R}_+$ and $\mathbb{R}_{\geq 0}$ we denote the sets of negative, positive and nonnegative real numbers respectively. The vectors $e_1, e_2, \dots, e_n$ denote the canonical basis of $\mathbb{R}^n$. The i th row and j th column of a matrix M are denoted by M_i and M_j respectively. For the empty set, we adopt the convention that $\max \emptyset = -\infty$.

In the rest of this section we give some definitions and notations regarding Filippov solutions (see, e.g., Cortés (2008)). Let F be a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $2^{\mathbb{R}^n}$ denote the collection of all subsets of $\mathbb{R}^n$. The map F is *essentially bounded* if there is a bound B such that $\|F(x)\|_2 < B$ for almost every $x \in \mathbb{R}^n$. The map F is *locally essentially bounded* if the restriction of F to every compact subset of

$\mathbb{R}^n$ is essentially bounded. The Filippov set-valued map of F , denoted $\mathcal{F}[F] : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, is given as

$$\mathcal{F}[F](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} \overline{\text{co}}\{F(B(x, \delta) \setminus S)\}, \quad (3)$$

where $B(x, \delta)$ is the open ball centered at x with radius $\delta > 0$, $S \subset \mathbb{R}^n$, μ denotes the Lebesgue measure and $\overline{\text{co}}$ denotes the convex closure. The zero measure set S is arbitrarily chosen. Hence, the set $\mathcal{F}[F](x)$ is independent of the value of $F(x)$. If F is continuous at x , then $\mathcal{F}[F](x) = \{F(x)\}$. A Filippov solution of the differential equation $\dot{x}(t) = F(x(t))$ on $[0, t_1] \subset \mathbb{R}$ is an absolutely continuous function $x : [0, t_1] \rightarrow \mathbb{R}^n$ that satisfies the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[F](x(t)) \quad (4)$$

for almost all $t \in [0, t_1]$. Let f be a map from $\mathbb{R}^n$ to $\mathbb{R}$. We use the same definition of regular function as in Clarke (1990) and recall that convex functions are regular. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz, then its *generalized gradient* $\partial f : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is defined by

$$\partial f(x) := \text{co}\{\lim_{i \rightarrow \infty} \nabla f(x_i) \mid x_i \rightarrow x, x_i \notin S \cup \Omega_f\}, \quad (5)$$

where ∇ denotes the gradient operator, $\Omega_f \subset \mathbb{R}^n$ denotes the set of points where f is not differentiable, and $S \subset \mathbb{R}^n$ is an arbitrary set of measure zero. ($\partial f(x)$ is independent of the choice of S Clarke (1990).) Given a set-valued map $\mathcal{F} : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, the *set-valued Lie derivative* $\tilde{\mathcal{L}}_{\mathcal{F}} f : \mathbb{R}^n \rightarrow 2^{\mathbb{R}}$ of a locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to $\mathcal{F}$ at x is defined as

$$\tilde{\mathcal{L}}_{\mathcal{F}} f(x) := \{a \mid \exists v \in \mathcal{F}(x) \text{ s.t. } \zeta^\top v = a \forall \zeta \in \partial f(x)\}.$$

A Filippov solution $t \mapsto x(t)$ is *maximal* if it cannot be extended forward in time. Since the Filippov solutions of a discontinuous system (4) are not necessarily unique, we need to specify two types of invariant sets. A set $\mathcal{R} \subset \mathbb{R}^n$ is called *weakly invariant* for (4) if, for each $x_0 \in \mathcal{R}$, at least one maximal solution of (4) with initial condition x_0 is contained in $\mathcal{R}$. Similarly, $\mathcal{R} \subset \mathbb{R}^n$ is called *strongly invariant* for (4) if, for each $x_0 \in \mathcal{R}$, every maximal solution of (4) with initial condition x_0 is contained in $\mathcal{R}$. For more details, see Cortés (2008).

3. Main results

3.1. Problem formulation

In this work we consider a network of n agents, who communicate according to a communication topology given by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. In this network, agent i receives information from agent j if and only if there is an edge from node v_j to node v_i in the graph $\mathcal{G}$. We denote the state of agent i at time t as $x_i(t) \in \mathbb{R}$, and consider the following dynamics for agent i

$$\dot{x}_i = f_i \left(\sum_{j=1}^n a_{ij} g_{ij}(x_j - x_i) \right) =: h_i(x), \quad (6)$$

where f_i and g_{ij} are functions, a_{ij} are the elements of the adjacency matrix A .

Each function f_i describes how agent i handles incoming information, while g_{ij} are concerned with the flow of information along the edges of the graph $\mathcal{G}$. All these functions are nonlinear and may have discontinuities, but we will use the concept of sign-preserving functions.

Definition 1. We say that a function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is *sign preserving* if $\varphi(0) = 0$ and for each $y \in \mathbb{R} \setminus \{0\}$ we have both $y\varphi(y) > 0$ and $\min y\mathcal{F}[\varphi](y) > 0$.

Fig. 1. Two digraphs for Examples 3, 11 and 12.

Notice that $y\mathcal{F}[\varphi](y)$ attains a minimum since it is closed. Examples of sign-preserving functions are e.g., *sign* and *sat*. If a function φ has only finitely many discontinuities, e.g. when it is piecewise continuous, then the condition $y\mathcal{F}[\varphi](y) > 0$ only needs to be checked for its discontinuity points. The condition $\min y\mathcal{F}[\varphi](y) > 0$ for all $y \in \mathbb{R} \setminus \{0\}$, will be illustrated in Example 4.

Throughout this paper, we assume the following.

Assumption 2. The functions f_i and g_{ij} are sign preserving, Lebesgue measurable, and locally essentially bounded.

To handle possible discontinuities in the right-hand side of (6), we consider Filippov solutions of the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[h](x(t)). \quad (7)$$

The existence of a Filippov solution for each initial condition is guaranteed by the Lebesgue measurability and the local essential boundedness of the functions f_i and g_{ij} in Assumption 2. In this paper we assume the completeness of Filippov solutions of (7) for any initial condition. Notice that when the functions h_i are globally bounded, the completeness of Filippov solution of (7) is guaranteed by Theorem 1 in Chapter 2 §7 in Filippov (1988). Moreover, by Proposition 3 of Theorem 1 in Paden and Sastry (1987), we have

$$\mathcal{F}[h](x(t)) \subset \bigtimes_{i=1}^n \mathcal{F}[h_i](x(t)).^1 \quad (8)$$

The agents of the network are said to achieve *consensus* if they all converge to the same value, that is, $\lim_{t \rightarrow \infty} x(t) = \eta \mathbf{1}$ for some constant $\eta \in \mathbb{R}$, where $x(t) = [x_1(t), \dots, x_n(t)]^\top$ is a solution of (6) with $x(0) = x_0$. It is well known that if all functions f_i and g_{ij} are the identity function, in which case (6) boils down to the linear consensus protocol, then the agents will achieve consensus iff the graph $\mathcal{G}$ contains a directed spanning tree (Agaev & Chebotarev, 2005; Ren, Beard, & McLain, 2005). In this work, we investigate the consensus problem for general functions f_i and g_{ij} satisfying Assumption 2. First, in Section 3.2, we consider the special case that the functions g_{ij} are equal to the identity function, that is $\dot{x}_i = f_i(\sum_{j=1}^n a_{ij}(x_j - x_i))$. Thereafter, in Section 3.3, we consider the case where the functions f_i are the identity function, that is $\dot{x}_i = \sum_{j=1}^n a_{ij}g_{ij}(x_j - x_i)$. Finally, in Section 3.4, we will combine these results.

The following examples motivate the sign-preserving condition by showing what happens if the functions f_i and g_{ij} do not satisfy this property.

Example 3. Consider the following system defined on the graph given in Fig. 1a

$$\begin{aligned} \dot{x}_1 &= f_1(0) \\ \dot{x}_2 &= f_2(x_1 - x_2) \end{aligned} \quad (9)$$

with $f_i(y) = \text{sat}(y; 0, 1)$ for $i = 1, 2$. Notice that f_i satisfies $yf_i(y) = 0$ for all $y < 0$, and hence f_i is not sign preserving. In this case the

Fig. 2. Two graphs with three nodes used in Examples 5, 10, 14 and 17.

existence of a directed spanning tree is not a sufficient condition for convergence to consensus. Indeed, if the initial condition satisfies $x_2(0) > x_1(0)$, then $x_1(t) = x_1(0)$ and $x_2(t) = x_2(0)$ for all $t \geq 0$. Hence, the agents do not reach consensus.

Example 4. Consider the system (9) defined on the digraph in Fig. 1a with f_i given by

$$f_i(y) = \begin{cases} y + 1 & \text{if } y < -1 \\ y & \text{if } y \in [-1, 1], \\ y - 1 & \text{if } y > 1. \end{cases} \quad i = 1, 2 \quad (10)$$

Then the function f_i satisfies $f_i(0) = 0$ and $yf_i(y) > 0$ for all $y \neq 0$. However, since $\mathcal{F}[f_i](1) = [0, 1]$ and $\mathcal{F}[f_i](-1) = [-1, 0]$, we have that $\min y\mathcal{F}[f_i](y) > 0$ is not satisfied for $y = \pm 1$. Hence, f_i is not sign preserving. Consider the point $x^* = [0, 1]^\top$, we have

$$\mathcal{F}[h](x^*) = \overline{\text{co}}\{[0, -1]^\top, [0, 0]^\top\},$$

which contains the point $[0, 0]^\top$. Consequently, x^* is an equilibrium point of the differential inclusion $\dot{x}(t) \in \mathcal{F}[h](x(t))$. For example, the trajectory

$$x_1(t) = 0, \quad x_2(t) = 1 + e^{-t}$$

is a solution of (9) which converges to x^* . Therefore, the agents do not reach consensus.

3.2. Node nonlinearity

We first consider the system (6) where the functions g_{ij} are all the identity function, and focus our attention on the functions f_i , which describe how agent i handles the incoming information flow. In this case, the total dynamics of the agents can be written as

$$\dot{x} = f(-Lx), \quad (11)$$

where L is the graph Laplacian induced by the information flow digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, and $f(y) = [f_1(y_1), f_2(y_2), \dots, f_n(y_n)]^\top$, $\forall y \in \mathbb{R}^n$. In this case we consider Filippov solutions of the differential inclusion

$$\dot{x}(t) \in \mathcal{F}[h](x(t)), \quad (12)$$

where $h(x) = f(-Lx)$. Note that, since L is a singular matrix, we have $\mathcal{F}[h](x(t)) \neq \mathcal{F}[f](-Lx(t))$ in general.

The aim of this section is to investigate under which conditions the Filippov solutions of the system (12) achieve consensus. Because of possible discontinuity of the right-hand side of (11), there can be Filippov solutions of (12) that are unbounded. The following example illustrates this unwanted behavior.

Example 5. Consider a dynamical system (11) defined on an undirected graph as given in Fig. 2a, where the functions f_i are all signum function. Suppose $x(t_0) \in \text{span}\{\mathbf{1}\}$ at time t_0 , then

$$\mathcal{F}[h](x(t_0)) = \overline{\text{co}}\{v_1, v_2, v_3, -v_1, -v_2, -v_3\}, \quad (13)$$

¹ $\bigtimes$ denotes the Cartesian product.

where $v_1 = [1, 1, -1]^\top$, $v_2 = [1, -1, 1]^\top$, and $v_3 = [-1, 1, 1]^\top$. Since $\sum_{i=1}^3 \frac{1}{3} v_i = \frac{1}{3} \mathbf{1}$, we have that $\{\eta \mathbf{1} \mid \eta \in [-\frac{1}{3}, \frac{1}{3}]\} \subset \mathcal{F}[h](x(t_0))$. Hence, any function $x(t) = \eta(t) \mathbf{1}$ with $\eta(t)$ differentiable almost everywhere and satisfying $\dot{\eta}(t) \in [-\frac{1}{3}, \frac{1}{3}]$ is a Filippov solution, e.g., $x(t) = \frac{1}{3} t \mathbf{1}$ and $x(t) = \frac{1}{3} \sin(t) \mathbf{1}$ which exhibit sliding consensus.

The undesirable behavior $x(t) = \eta(t) \mathbf{1}$ with $\eta(t)$ a nonconstant function in the previous example will be called *sliding consensus*. Sliding consensus arises whenever $\eta \mathbf{1}$ is contained in $\mathcal{F}[h](\alpha \mathbf{1})$ for some scalars $\eta \neq 0$ and sufficiently many α . Note that this example shows that for the validity of Theorem 11(ii) in Cortés (2006) we need extra conditions. A counter example to Theorem 11(i) in Cortés (2006) can be constructed similarly to Example 5. In fact, it will turn out that the sliding consensus can be excluded by replacing the signum function for at least one node by a function that is continuous at the origin, for example the logarithmic quantizer. This motivates the introduction of the following subsets of the index set $\mathcal{I}$ corresponding to the digraph $\mathcal{G}$:

$$\mathcal{I}_r = \{i \in \mathcal{I} \mid v_i \text{ is a root of } \mathcal{G}\},$$

$$\mathcal{I}_c = \{i \in \mathcal{I} \mid f_i \text{ is continuous at the origin}\}.$$

Before we present the main result of this section, we first state a preparatory lemma.

Lemma 6 (Prop. 2.2.6, 2.3.6 in Clarke (1990)). *The following functions are regular and Lipschitz continuous,*

$$V(x) := \max_{i \in \mathcal{I}} x_i, \quad W(x) := -\min_{i \in \mathcal{I}} x_i. \quad (14)$$

Theorem 7. *Consider system (12) defined on a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$. If one of the following conditions holds,*

- (i) $\mathcal{I}_c \cap \mathcal{I}_r$ is not empty,
- (ii) $|\mathcal{I}_r| = 1$,
- (iii) $|\mathcal{I}_r| = 2$, $f_i(0^-)$ and $f_i(0^+)$ exist, and $f_i(0^-) = -f_i(0^+)$ for $i \in \mathcal{I}_r$,

then all the trajectories of system (12) achieve consensus asymptotically, for any initial condition. Furthermore, they will remain in the set $[\min_i x_i(0), \max_i x_i(0)]^n$ for all $t \geq 0$.

Proof. Notice that in all three cases the index set $\mathcal{I}_r$ is nonempty, which implies that the graph $\mathcal{G}$ contains a directed spanning tree. Condition (i) implies that the digraph $\mathcal{G}$ has a root v_i for which f_i is continuous at the origin.

Consider candidate Lyapunov functions V and W as given in (14). Let $x(t)$ be a trajectory of (12) and define

$$\alpha(t) = \{k \in \mathcal{I} \mid x_k(t) = V(x(t))\}.$$

The generalized gradient of V is given as Clarke (1990, Example 2.2.8)

$$\partial V(x(t)) = \text{co}\{e_k \in \mathbb{R}^n \mid k \in \alpha(t)\}. \quad (15)$$

Let Ψ be defined as

$$\Psi = \left\{ t \geq 0 \mid \text{both } \dot{x}(t) \text{ and } \frac{d}{dt} V(x(t)) \text{ exist} \right\}. \quad (16)$$

Since x is absolutely continuous and V is locally Lipschitz, we have that $\Psi = \mathbb{R}_{\geq 0} \setminus \tilde{\Psi}$ for a set $\tilde{\Psi}$ of measure zero. By Lemma 1 in Bacciotti and Ceragioli (1999), we have

$$\frac{d}{dt} V(x(t)) \in \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \quad (17)$$

for all $t \in \Psi$ and hence that the set $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \neq \emptyset$ for all $t \in \Psi$. For $t \in \tilde{\Psi}$, we have $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is empty, hence $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) < 0$. For $t \in \Psi$, let $a \in \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$. By definition, there exists a $v^a \in \mathcal{F}[h](x(t))$ such that $a = v^a \cdot \zeta$ for all $\zeta \in \partial V(x(t))$. Consequently, by choosing $\zeta = e_k$ for $k \in \alpha(t)$, we observe that v^a satisfies

$$v_k^a = a \quad \forall k \in \alpha(t). \quad (18)$$

Next, we want to show that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \leq 0$ for all $t \in \Psi$ by considering two possible cases: $\mathcal{I}_r \not\subseteq \alpha(t)$ or $\mathcal{I}_r \subseteq \alpha(t)$.

If $\mathcal{I}_r \not\subseteq \alpha(t)$, then there exists an $i \in \mathcal{I}_r$ such that $x_i(t) < V(x(t))$. Furthermore, since v_i is a root, we can choose an index $j \in \alpha(t)$ such that the shortest path from v_i to v_j has the least number of edges. By our choice of j , there is at least one edge $e_{kj} \in \mathcal{E}$ such that $x_k(t) < x_j(t)$, which implies that we have $-L_j x(t) < 0$. Moreover, the existence of an edge e_{kj} implies that $\text{rank } L_j = 1$, which together with property 4 of Theorem 1 in Paden and Sastry (1987) gives us

$$\mathcal{F}[h_i](x(t)) = \mathcal{F}[f_j](-L_j x(t)). \quad (19)$$

By the sign-preserving property of f_j and $-L_j x(t) < 0$, we have that $\mathcal{F}[h_i](x(t)) \subset \mathbb{R}_-$. By (8), we find that $v_j < 0$ for any $v \in \mathcal{F}[h](x(t))$. Using observation (18) for $k = j$, we see that every $a \in \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ satisfies $a < 0$. By the fact that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is a closed set, we have $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) < 0$.

If $\mathcal{I}_r \subseteq \alpha(t)$, we will consider the conditions (i), (ii) and (iii) separately and prove that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) = \{0\}$. First, we note that if a node v_k is a root, then $e_{jk} \in \mathcal{E}$ implies that v_j is a root as well, and hence we have

$$(-Lx)_k = \sum_{j \in \mathcal{I}} a_{kj}(x_j - x_k) = \sum_{j \in \mathcal{I}_r} a_{kj}(x_j - x_k). \quad (20)$$

- (i) In this case $\mathcal{I}_c \cap \mathcal{I}_r \subseteq \alpha(t)$. For any $i \in \mathcal{I}_c \cap \mathcal{I}_r$, we have that f_i is continuous at 0 and satisfies $f_i(0) = 0$. This implies that any $v \in \mathcal{F}[h](x(t))$ satisfies $v_i = 0$. Using observation (18), we can conclude that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) = \{0\}$.
- (ii) Let $\mathcal{I}_r = \{i\}$. Since there is only one root in this case, namely v_i , we have $L_i = 0$ and hence $f_i((Lx(t)))_i = f_i(0) = 0$ for all t . Consequently, each $v \in \mathcal{F}[h](x(t))$ satisfies $v_i = 0$. Using observation (18) again, we see that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) = \{0\}$.
- (iii) Let $\mathcal{I}_r = \{i, j\}$. By (20), the dynamics of x_i and x_j are given as $\dot{x}_i = f_i(a_{ij}(x_j - x_i))$ and $\dot{x}_j = f_j(a_{ji}(x_i - x_j))$ respectively. Since $\mathcal{I}_r \subseteq \alpha(t)$, we have $x_i(t) = x_j(t)$ and hence any $v \in \mathcal{F}[h](x(t))$ satisfies

$$\begin{aligned} [v_i, v_j] &\subseteq \overline{\text{co}}\{[f_i(0^-), f_j(0^+)], [f_i(0^+), f_j(0^-)]\} \\ &= \overline{\text{co}}\{[f_i(0^-), f_j(0^+)], -[f_i(0^-), f_j(0^+)]\}, \end{aligned}$$

where the last equality is implied by condition (iii). This implies that the convex set given in the above equation is a line segment that only crosses $\text{span}\{[1, 1]^\top\}$ in the origin. This implies that any $v \in \mathcal{F}[h](x(t))$ with $v_i = v_j$ must satisfy $v_i = v_j = 0$. Using $\partial V(x(t)) = \text{co}\{e_i, e_j \in \mathbb{R}^n\}$ and (18), we see that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) = \{0\}$.

Define $\beta(t) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\}$. By using similar computations and observations, we find that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) < 0$ if $\mathcal{I}_r \not\subseteq \beta(t)$, and $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) \leq 0$ if $\mathcal{I}_r \subseteq \beta(t)$.

We conclude that $V(x(t))$ and $W(x(t))$ are not increasing along the trajectories $x(t)$ of the system (12). Hence, the trajectories are bounded and remain in the set $[\min_i x_i(0), \max_i x_i(0)]^n$ for all $t \geq 0$. Therefore, for any $N \in \mathbb{R}_+$, the set $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq N\}$ is strongly invariant for (12). By Theorem 2 in Cortés (2008), we have that all solutions of (12) starting at S_N converge to the largest

weakly invariant set M contained in

$$S_N \cap \overline{\{x \in \mathbb{R}^n \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x)\}} \cap \{x \in \mathbb{R}^n \mid 0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x)\}. \quad (21)$$

From the argument above we see that $0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t))$ is only possible if $\mathcal{I}_r \subseteq \alpha(t)$, and $0 \in \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x(t))$ can only happen if $\mathcal{I}_r \subseteq \beta(t)$. This implies that for every root v_i , the state x_i converges simultaneously to the maximum and to the minimum, i.e., the trajectories $x(t)$ of the system achieve consensus for any initial condition. $\square$

Remark 8. Here we interpret condition (i), (ii) and (iii) in [Theorem 7](#). It can be seen from the proof of [Theorem 7](#) that the sliding consensus can be introduced only by the behaviors of the agents in $\mathcal{I}_r$, while the agents in $\mathcal{I} \setminus \mathcal{I}_r$ just track the trajectories of the root agents. The condition (i) is a sufficient condition for the general case, namely $\mathcal{I}_c \cap \mathcal{I}_r \neq \emptyset$. In this case, there is at least one root for which the input converges to zero as all the agents converge to consensus, which prevents sliding behavior. The conditions (ii) and (iii) are provided for two special cases, i.e., $|\mathcal{I}_r| = 1$ and $|\mathcal{I}_r| = 2$. In these cases, we allow $\mathcal{I}_r \cap \mathcal{I}_c = \emptyset$. When there is only one root in the graph, the states of the other agents will converge to the state of the root which is constant along the trajectory, excluding sliding behavior. If there are two roots, v_i and v_j , and both f_i and f_j are not continuous at the origin, then condition (iii) gives another way to eliminate sliding consensus: the limits of f_i and f_j for $t \rightarrow 0^+$ should be opposite to the limits for $t \rightarrow 0^-$. For example, $f_i = f_j = \text{sign}$ is one of this type of protocols.

Remark 9. The set $\mathcal{I}_c$ can be enlarged such that it contains all the functions which are *essentially continuous* at the origin, i.e., $\text{esslim}_{x_i \rightarrow 0^-} f_i(x_i) = \text{esslim}_{x_i \rightarrow 0^+} f_i(x_i) = 0$ (for definitions see e.g. [Arutyunov \(2000\)](#) and [Walsh \(2006\)](#)). This can be done since in the definitions of both essential limits and Filippov set-valued map, any zero measure set can be excluded. For condition (iii) in [Theorem 7](#), the same extension is possible; considering essential limits instead of limits.

The conditions (i), (ii) and (iii) in [Theorem 7](#) all exclude the possibility of sliding consensus, and guarantee asymptotic consensus. The role of each condition will be illustrated in the following examples.

Example 10. Consider system (11) defined on the undirected graph in [Fig. 2a](#) with all edge weights one. Suppose that f_1 is continuous at the origin, so that condition (i) in [Theorem 7](#) is satisfied. Then the sliding consensus is not a Filippov solution. Indeed, if at time t_0 we have $x(t_0) \in \text{span}\{\mathbf{1}\}$, then the first component of the Filippov set-valued map $\mathcal{F}[h](x(t_0))$ is equal to $\{0\}$. This implies that $x(t) = x(t_0)$, for all $t \geq t_0$.

Example 11. Consider system (9) defined on the digraph in [Fig. 1a](#). It satisfies condition (ii) of [Theorem 7](#). Since $f_1(0) = 0$, the state of the root v_1 is constant. Consensus is achieved by the fact that f_2 is sign preserving.

Example 12. Consider system (11) defined on the digraph given in [Fig. 1b](#) with $a_{12} = a_{21} = 1$.

First, we consider a case in which f_1 and f_2 satisfy condition (iii) of [Theorem 7](#) and take $f_1 = f_2 = \text{sign}(\cdot)$. If the trajectory achieves consensus at time t , the Filippov set-valued map $\mathcal{F}[h](x(t))$ equals $\overline{\text{co}}\{[1, -1]^T, [-1, 1]^T\}$, which intersects $\text{span}\{\mathbf{1}\}$ only at $[0, 0]^T$. Hence $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x) = \tilde{\mathcal{L}}_{\mathcal{F}[h]}W(x) = 0$, which implies that the trajectory remains constant, i.e., there is no sliding consensus.

Second, we consider a case in which $f_i(0^-) \neq -f_i(0^+)$ for $i = 1, 2$, which means that the condition (iii) of [Theorem 7](#) is not

satisfied. In this case, sliding consensus can be a Filippov solution. For instance, take

$$f_i(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0 \end{cases} \quad i = 1, 2.$$

Suppose that at t_0 the state x achieves consensus. Then the Filippov set-valued map at $x(t_0)$ is $\overline{\text{co}}\{[-1, 2]^T, [2, -1]^T\}$ which intersects $\text{span}\{\mathbf{1}\}$ at $[\frac{1}{2}, \frac{1}{2}]^T$. Then $x(t) = \frac{1}{2}\mathbf{1}t + x(t_0)$ is a Filippov solution for $t \geq t_0$ that exhibits sliding consensus.

3.3. Edge nonlinearity

In this section we consider the case where the functions f_i are all the identity function, that is,

$$\dot{x}_i = \sum_{j=1}^N a_{ij}g_{ij}(x_j - x_i) =: h_i(x), \quad i \in \mathcal{I}. \quad (22)$$

We consider two cases, corresponding to the underlying graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ being undirected or directed, starting with the undirected case. We introduce the following assumption on the functions g_{ij} .

Assumption 13. For all $e_{ji} \in \mathcal{E}$, the right and left limits of g_{ij} and g_{ji} at the origin exist, and satisfy $g_{ij}(0^-) = -g_{ji}(0^+)$.

To illustrate the need of [Assumption 13](#), we give the following example.

Example 14. If $g_{ij}(0^-) \neq -g_{ji}(0^+)$, then sliding consensus may occur. For instance, consider the system (22) defined on the undirected graph in [Fig. 2a](#) given by

$$\begin{aligned} \dot{x}_1(t) &= g_{12}(x_2(t) - x_1(t)) + g_{13}(x_3(t) - x_1(t)) \\ \dot{x}_2(t) &= g_{21}(x_1(t) - x_2(t)) + g_{23}(x_3(t) - x_2(t)) \\ \dot{x}_3(t) &= g_{31}(x_1(t) - x_3(t)) + g_{32}(x_2(t) - x_3(t)) \end{aligned}$$

where

$$g_{ij}(x) = \begin{cases} 1.5 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -0.5 & \text{if } x < 0. \end{cases} \quad \forall e_{ji} \in \mathcal{E},$$

Suppose that at time t_0 the state $x(t_0) \in \text{span}\{\mathbf{1}\}$, then $\mathcal{F}[h](x(t_0))$ is the closed convex hull spanned by $[-1, 1, 3]^T$, $[-1, 3, 1]^T$, $[1, -1, 3]^T$, $[3, -1, 1]^T$, $[1, 3, -1]^T$ and $[3, 1, -1]^T$. Hence, $\mathbf{1} \in \mathcal{F}[h](x(t_0))$ and thus $x(t) = t\mathbf{1} + x(t_0)$ is a Filippov solution for $t > t_0$.

Next, we present the main result of this section.

Theorem 15. Consider the dynamics (22) defined on a connected undirected graph. Suppose the functions g_{ij} satisfy [Assumptions 2](#) and [13](#). Then the trajectories of the system (22) achieve consensus asymptotically.

Proof. Consider the Lyapunov candidate functions V and W as defined in (14). We use the same notations as in the proof of [Theorem 7](#). Similarly, as the proof of [Theorem 7](#), we only prove that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \leq 0$ for all $t \in \Psi$ where $\mu(\mathbb{R}_{\geq 0} \setminus \Psi) = 0$ and the set $\tilde{\mathcal{L}}_{\mathcal{F}[h]}V(x(t)) \neq \emptyset$ for all $t \in \Psi$.

By introducing the functions $\varphi_{ji}(x) = (e_j - e_i)^T x$ for $i, j \in \mathcal{I}$, the function $h_i(x)$ in (22) can be rewritten as

$$h_i(x) = \sum_{j=1}^n a_{ij}(g_{ij} \circ \varphi_{ji})(x). \quad (23)$$

Then, using Theorem 1 in [Paden and Sastry \(1987\)](#), we see that the Filippov set-valued map $\mathcal{F}[h](x)$ satisfies

$$\mathcal{F}[h](x) \subset \bigcap_{i=1}^n \mathcal{F}[h_i](x) \quad (24)$$

$$\subset \bigcap_{i=1}^n \sum_{j=1}^n a_{ij} \mathcal{F}[g_{ij}](\varphi_{ji}(x)). \quad (25)$$

By Proposition 1 in Theorem 1 in [Paden and Sastry \(1987\)](#), for each g_{ij} there exists a set $N_{g_{ij}} \subset \mathbb{R}^n$ with $\mu(N_{g_{ij}}) = 0$ such that

$$\mathcal{F}[g_{ij}](z) = \text{co}\left\{ \lim_{k \rightarrow \infty} h(z^k) \mid \lim_{k \rightarrow \infty} z^k = z \text{ and } z^k \notin N_{g_{ij}} \cup N' \right\} \quad (26)$$

for any set N' with $\mu(N') = 0$. Similarly, there exists a set $N_h \subset \mathbb{R}^n$ with $\cup_{e_{ij} \in \mathcal{E}} N_{g_{ij}} \subset N_h$ and $\mu(N_h) = 0$ such that

$$\mathcal{F}[h](x(t)) = \text{co}\left\{ \lim_{k \rightarrow \infty} h(y^k) \mid \lim_{k \rightarrow \infty} y^k = x(t), y^k \notin N_h \cup S \right\}, \quad (27)$$

where $S = \{x \in \mathbb{R}^n \mid \exists i, j \in \mathcal{I} \text{ such that } x_i = x_j\}$, which has measure zero in $\mathbb{R}^n$. Notice that $\mathbb{R}^n \setminus S$ admits a partition $\mathbb{R}^n \setminus S = S_1 \cup S_2 \cup \dots \cup S_{2^n}$, with $S_1, S_2, \dots, S_{2^n}$ open sets satisfying $S_i \cap S_j = \emptyset$ for all $i \neq j$, such that within a fixed open set S_i , the components $y_1, y_2, \dots, y_n$ of each vector $y \in S_i$ are all different and have the same fixed order.

Now, to study the right-hand side of (27), let t be a given time and let (y^k) be a sequence in $\mathbb{R}^n \setminus (N_h \cup S)$ that converges to $x(t)$ for which the limit $\tilde{h} := \lim_{k \rightarrow \infty} h(y^k)$ exists. Note that the existence of $\lim_{k \rightarrow \infty} h(y^k)$ means that all the components $\tilde{h}_i := h_i(y^k)$ have a limit. We will study the term $\sum_{i \in \alpha(t)} \tilde{h}_i$ in order to derive that $\sum_{i \in \alpha(t)} v_i \leq 0$ for each $v \in \mathcal{F}[h](x(t))$. For this, we first define two sets of edges, namely

$$\mathcal{E}_1(t) = \{e_{ij} \in \mathcal{E} \mid i, j \in \alpha(t)\}, \quad (28)$$

$$\mathcal{E}_2(t) = \{e_{ij} \in \mathcal{E} \mid i \in \alpha(t), j \notin \alpha(t)\}. \quad (29)$$

The sequence (y^k) has a subsequence (y^{k_ℓ}) such that $y^{k_\ell} \in S_r$ for all ℓ for a fixed $r \in \{1, 2, \dots, 2^n\}$. For an edge $e_{ij} \in \mathcal{E}_1$, we have $y_i^{k_\ell} - y_j^{k_\ell} \uparrow 0$ or $y_i^{k_\ell} - y_j^{k_\ell} \downarrow 0$, depending on the set S_r . Therefore, we have

$$\begin{aligned} \lim_{\ell \rightarrow \infty} [g_{ij}(\varphi_{ji}(y^{k_\ell})), g_{ji}(\varphi_{ij}(y^{k_\ell}))] &= [g_{ij}(0^-), g_{ji}(0^+)] \text{ or} \\ \lim_{\ell \rightarrow \infty} [g_{ij}(\varphi_{ji}(y^{k_\ell})), g_{ji}(\varphi_{ij}(y^{k_\ell}))] &= [g_{ij}(0^+), g_{ji}(0^-)]. \end{aligned}$$

Using [Assumption 13](#), we see that in both cases we have

$$\lim_{\ell \rightarrow \infty} g_{ij}(\varphi_{ji}(y^{k_\ell})) + g_{ji}(\varphi_{ij}(y^{k_\ell})) = 0. \quad (30)$$

Now, we can write

$$\begin{aligned} &\sum_{i \in \alpha(t)} \lim_{k \rightarrow \infty} h_i(y^k) \\ &= \lim_{\ell \rightarrow \infty} \sum_{i \in \alpha(t)} \sum_{j=1}^n a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) \\ &= \lim_{\ell \rightarrow \infty} \left[\sum_{e_{ij} \in \mathcal{E}_1} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) + \sum_{e_{ij} \in \mathcal{E}_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) \right] \\ &= \frac{1}{2} \sum_{e_{ij} \in \mathcal{E}_1} \lim_{\ell \rightarrow \infty} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) + a_{ji} g_{ji}(\varphi_{ij}(y^{k_\ell})) \\ &\quad + \lim_{\ell \rightarrow \infty} \sum_{e_{ij} \in \mathcal{E}_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) \\ &= \lim_{\ell \rightarrow \infty} \sum_{e_{ij} \in \mathcal{E}_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})), \end{aligned} \quad (31)$$

where the last two equalities are implied by the fact that the graph $\mathcal{G}$ is undirected and by Eq. (30). Next, we consider two possible cases: $x(t) \notin \text{span}\{\mathbf{1}\}$, and $x(t) \in \text{span}\{\mathbf{1}\}$.

First, we look at the case that $x(t) \notin \text{span}\{\mathbf{1}\}$, in which case $\mathcal{E}_2 \neq \emptyset$. For an edge $e_{ij} \in \mathcal{E}_2$ we have $x_j < x_i$, and since g_{ij} is a sign-preserving function, this implies that $\mathcal{F}[g_{ij}](x_j - x_i) \subset \mathbb{R}_-$. As $y^{k_\ell} \notin N_{g_{ij}}$, all the accumulation points of the sequence $\{g_{ij}(\varphi_{ji}(y^{k_\ell}))\}$ belong to $\mathcal{F}[g_{ij}](x_j - x_i)$. Therefore, we have that $\sum_{i \in \alpha(t)} \lim_{k \rightarrow \infty} h_i(y^k) = \lim_{\ell \rightarrow \infty} \sum_{e_{ij} \in \mathcal{E}_2} a_{ij} g_{ij}(\varphi_{ji}(y^{k_\ell})) < 0$, i.e., $\sum_{i \in \alpha(t)} \tilde{h}_i < 0$. By Eq. (27), we can conclude that $\sum_{i \in \alpha(t)} v_i < 0$ for any $v \in \mathcal{F}[h](x(t))$. Hence, by observation (18), we have $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \subset \mathbb{R}_-$. By the fact that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is closed (see e.g. page 63 in [Cortés \(2008\)](#)), we have $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) < 0$.

Second, we consider the case that $x(t) \in \text{span}\{\mathbf{1}\}$, in which case $\mathcal{E}_1(t) = \mathcal{E}$ and $\mathcal{E}_2(t) = \emptyset$. In this case, Eq. (31) boils down to

$$\sum_{i \in \alpha(t)} \tilde{h}_i = \sum_{i \in \mathcal{I}} \lim_{k \rightarrow \infty} h_i(y^k) = 0. \quad (32)$$

By Eq. (27), we can conclude that $\sum_{i \in \mathcal{I}} v_i = 0$ for any $v \in \mathcal{F}[h](x(t))$. This implies that $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) = \{0\}$ since $\frac{1}{n} \mathbf{1} \in \partial V(x(t))$.

By using the same arguments as above, we can prove that

- (i) $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) < 0$ if $x(t) \notin \text{span}\{\mathbf{1}\}$,
- (ii) $\tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) = \{0\}$ if $x(t) \in \text{span}\{\mathbf{1}\}$.

The above analysis implies that all trajectories are bounded. Indeed for any $N \in \mathbb{R}_+$ the set $S_N = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq N\}$ is strongly invariant. By Theorem 2 in [Cortés \(2008\)](#), the conclusion follows. $\square$

Remark 16. A stronger assumption is to assume that $g_{ij}(y) = -g_{ji}(-y)$ for all $e_{ij} \in \mathcal{E}$ and all $y \in \mathbb{R}$. This would imply that $\sum_{i \in \mathcal{I}} h_i(x) = 0$ for any $x \in \mathbb{R}^n$. Hence for any $x \in \mathbb{R}^n$ and for any $v \in \mathcal{F}[h](x)$, we have $\mathbf{1}^\top v = 0$. Then any Filippov solution $x(t)$ of system (22) satisfies $\mathbf{1}^\top \dot{x}(t) = 0$. Under the same assumption as in [Theorem 15](#), the trajectories of (22) converge to a consensus value defined by the average of the initial condition.

For the rest of this section, we consider *directed* graphs. In this case, [Assumption 13](#) is not sufficient to guarantee convergence to consensus as shown by the following example.

Example 17. Consider system (22) on the directed graph as in [Fig. 2b](#), where the functions g_{ij} are the signum function. Suppose that at time t_0 , the state satisfies $x(t_0) \in \text{span}\{\mathbf{1}\}$. Then the Filippov set-valued map $\mathcal{F}[h](x(t_0))$ is the same as in (13). Hence, by the same argument as in [Example 5](#), there are Filippov solutions that exhibit sliding consensus.

For digraphs, we quote the following result from [Pachristodoulou et al. \(2010\)](#) for the case that the functions g_{ij} are continuous.

Theorem 18. Consider the system (22) with continuous functions g_{ij} . If the underlying graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ contains a directed spanning tree, then the trajectories of (22) achieve consensus asymptotically.

Extension of [Theorem 18](#) to the case of discontinuous functions g_{ij} is a topic for further research.

3.4. Combining node and edge nonlinearities

The multi-agent system given in (6) can be seen as a combination of system (11) and system (22). For this system, we have the following result.

Theorem 19. Consider system (6) defined on a digraph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with continuous functions g_{ij} . If one of the following three conditions holds, i.e.,

- (i) $\mathcal{I}_r \cap \mathcal{I}_c$ is not empty,
- (ii) $|\mathcal{I}_r| = 1$,
- (iii) $|\mathcal{I}_r| = 2, f_i(0^-)$ and $f_i(0^+)$ exist, and $f_i(0^-) = -f_i(0^+)$ for $i \in \mathcal{I}_r$,

then all Filippov solutions of system (11) achieve consensus asymptotically, for all initial conditions.

Proof. Since the proof is similar to the proof of Theorem 7, we only provide a sketch of the proof. Recall that $\alpha(t) = \{i \in \mathcal{I} \mid x_i(t) = V(x(t))\}$ and $\beta(t) = \{i \in \mathcal{I} \mid x_i(t) = -W(x(t))\}$. Let V and W be candidate Lyapunov functions. We will show that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V \leq 0$ by considering two cases: $\mathcal{I}_r \not\subseteq \alpha(t)$ and $\mathcal{I}_r \subseteq \alpha(t)$.

When $\mathcal{I}_r \not\subseteq \alpha(t)$, there exists at least one $k \in \alpha(t)$ with $\sum_{j=1}^n a_{kj} g_{kj}(x_j - x_k) < 0$, which implies the k th component of $\mathcal{F}[h](x(t))$ is in $\mathbb{R}_-$. Hence, $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V < 0$.

When $\mathcal{I}_r \subseteq \alpha(t)$, we can use similar arguments as in the proof of Theorem 7 to see that the set-valued Lie derivative $\tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t))$ is either $\{0\}$ or $\emptyset$ if one of the conditions (i), (ii) and (iii) holds. Hence $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} V(x(t)) \leq 0$.

Similarly, we have that $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) < 0$ if $\mathcal{I}_r \not\subseteq \beta(t)$, and $\max \tilde{\mathcal{L}}_{\mathcal{F}[h]} W(x(t)) \leq 0$ if $\mathcal{I}_r \subseteq \beta(t)$. Based on Theorem 2 in Cortés (2008), the conclusion follows. $\square$

4. A port-Hamiltonian perspective on consensus error dynamics

An alternative approach to consensus analysis is to consider the dynamics of the ‘error’ vector $z = -Lx$. In many cases, the convergence of x to ordinary, static, consensus is equivalent to the convergence of z to the origin. On the other hand, in the previous section it was shown that for differential inclusions this equivalence does not necessarily hold. In the present section we first provide sufficient conditions which guarantee asymptotic stability of the origin for the error dynamics for compatible initial conditions. Combining this with Theorem 7, we then formulate sufficient conditions for the equivalence between convergence of z to the origin and of x to static consensus.

Theorem 20. Given system (12), the error $z = -Lx$ satisfies

$$\dot{z} \in -L\mathcal{F}[f](z). \quad (33)$$

If $\mathcal{G}$ is strongly connected and the functions f_i are sign-preserving and non-decreasing, then all Filippov solutions of (33) for $z(0) \in \text{im}L$ converge to the origin.

Proof. By almost everywhere differentiability of $z(\cdot)$

$$\begin{aligned} \dot{z}(t) &= -L\dot{x}(t) \\ &\in -L\mathcal{F}[f(-Lx)](x(t)) \\ &\subset -L\mathcal{F}[f](z(t)), \end{aligned}$$

where the last inclusion holds by Padén and Sastry (1987, Theorem 1 (1)) for any $z(t) = -Lx(t)$. Next, we prove asymptotic stability of the origin for any $z(0) \in \text{im}L$. We only provide the sketch of the proof. First denote $F(y) = [F_1(y_1), F_2(y_2), \dots, F_n(y_n)]^\top$ for any $y \in \mathbb{R}^n$, where $F_i(y_i) := \int_0^{y_i} f_i(s)ds$ is convex and radially unbounded, since f_i is sign-preserving and non-decreasing. Furthermore, since $\mathcal{G}$ is strongly connected, there exists $\sigma \in \mathbb{R}_+^n$ such that $\sigma^\top L = 0$ Bollobas (1998, Theorem 14). Consider $V_1(z) = \sigma^\top F(z)$ as Lyapunov function, which is convex and hence regular. Then since

F_i is differentiable almost everywhere Theorem 1(1) in Padén and Sastry (1987) implies that the generalized gradient of V_1 is

$$\partial V_1(z) = \bigcap_{i=1}^n [\sigma_i f_i(z_i^-), \sigma_i f_i(z_i^+)] \quad (34)$$

$$= \Sigma \mathcal{F}[f](z), \quad (35)$$

where $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$. Then for any $a \in \tilde{\mathcal{L}}_{-L\mathcal{F}[f]} V_1(z)$ there exists $v \in \mathcal{F}[f](z)$ such that $a = -v^\top \Sigma L v$. Since ΣL is the Laplacian matrix of a balanced graph, and the symmetric part of ΣL is positive semidefinite Bullo, Cortés, and Martínez (2009, Theorem 1.37), we have $a \leq 0$. Thus we have shown that if $\tilde{\mathcal{L}}_{-L\mathcal{F}[f]} V_1(z) \neq \emptyset$, then it belongs to $\mathbb{R}_{\leq 0}$. Furthermore, by Theorem 2 in Cortés (2008) all solutions of (33) converge to $\Omega = \{z \mid 0 \in \tilde{\mathcal{L}}_{-L\mathcal{F}[f]} V_1(z)\}$ asymptotically. By computing $\tilde{\mathcal{L}}_{-L\mathcal{F}[f]} V_1(z)$ it follows that $\Omega = \{z \mid \text{span}\{1\} \subset \mathcal{F}[f](z)\}$. Since $z(t) \in \text{im}L$ for all t and the functions f_i are sign-preserving, we obtain $\Omega = \{0\}$. $\square$

Remark 21. The stability of the system (33) can be approached from the following point of view. By using the new coordinates $w = \Sigma z$ we can write

$$\dot{w} \in -\Sigma L \frac{\partial H}{\partial w}(w) = -(J + R) \frac{\partial H}{\partial w}(w), \quad (36)$$

where $H(w) := \sigma^\top F(\Sigma^{-1}w)$, and J and R are the skew-symmetric and symmetric parts of ΣL . The system (36) is a generalized (differential inclusion) port-Hamiltonian system (van der Schaft and Jeltsema, 2014). Thus the Lyapunov function V_1 in the new coordinates w is nothing else than the Hamiltonian of this port-Hamiltonian system.

Example 22. The system (11) and the error dynamics (33) resulting from $z = -Lx$ can be illustrated as follows. Consider a hydraulic network with x_i being the pressure at the i th node, where the flow through the pipe from node i to node j is linearly dependent on $x_i - x_j$. Then the flow extracted/injected at the i th node equals the i th component of $z = -Lx$, where L is a symmetric Laplacian matrix. System (11) is obtained by assuming that the rate of increase of x_i depends on z_i through the function f_i . Obviously the pressures will converge to consensus if and only if the vector of flows z at the nodes converges to zero. H is the storage function resulting from summing the integrals of f_i . Non-symmetric Laplacian matrices L may occur in other types of transportation networks (van der Schaft, 2015).

Example 23. Consider the system in Example 5. Its error dynamics is given as $\dot{z} \in -L\mathcal{F}[\text{sign}](z)$. Hence $\lim_{t \rightarrow \infty} z(t) = 0, \forall z(0) \in \text{im}L$. Indeed, following the proof of Theorem 20 the Lyapunov function is the 1-norm $V_1(z) = \|z\|_1$.

As shown by Examples 5 and 23 the convergence of the error vector z to zero does not necessarily imply the convergence of x to consensus, since ‘sliding’ consensus may occur. Hence, one must be careful to derive the convergence of x to consensus by analyzing the error vector z . In the following remark, we combine Theorems 7 and 20 to obtain sufficient conditions for when convergence of the error vector z to the origin guarantees (static) consensus of x .

Remark 24. Consider systems (12) and (33) defined on a strongly connected digraph. Suppose the functions f_i are sign-preserving and non-decreasing. Then if one of the following conditions hold

- (i) $\mathcal{I}_c$ is non-empty,
- (ii) $|\mathcal{I}| = 2, f_i(0^-)$ and $f_i(0^+)$ exist and $f_i(0^-) = -f_i(0^+)$ for $i \in \mathcal{I}$,

then $z(t)$ converges to the origin for any $z(0) = -Lx(0)$ and $x(t)$ converges to consensus for any $x(0)$.

5. Conclusion

In this paper, we considered a very general model of multi-agent systems defined on a directed graph, with nonlinear discontinuous functions defined on the nodes and edges. Since the right-hand sides of the differential equations are discontinuous, we interpreted the solutions in the Filippov sense. Under the crucial assumptions of (i) the graph containing a directed spanning tree, (ii) all nonlinear functions to be sign-preserving, we provided sufficient conditions for all Filippov solutions of the consensus protocol to achieve consensus asymptotically. Furthermore, a common approach in the study of consensus, namely the error dynamics, was analyzed. A sufficient condition was given to guarantee the convergence of the error to zero. For differential inclusions, however, it was shown that convergence of the error to zero is not equivalent to the convergence of the original states to consensus, and that one has to be careful in order to avoid ‘sliding’ consensus.

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