University of Groningen

Abstraction and control by interconnection of linear systems: A geometric approach

Megawati, Noorma Yulia; van der Schaft, Arjan

Published in:
Systems & Control Letters

DOI:
[10.1016/j.sysconle.2017.04.010](https://doi.org/10.1016/j.sysconle.2017.04.010)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2017

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Megawati, N. Y., & van der Schaft, A. (2017). Abstraction and control by interconnection of linear systems: A geometric approach. *Systems & Control Letters*, 105, 27-33.
<https://doi.org/10.1016/j.sysconle.2017.04.010>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Abstraction and control by interconnection of linear systems: A geometric approach

Noorma Yulia Megawati^{a,b,*}, Arjan van der Schaft^a

^a Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9747AG Groningen, The Netherlands

^b Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia

ARTICLE INFO

Article history:

Received 9 December 2016

Received in revised form 14 March 2017

Accepted 23 April 2017

Available online 25 May 2017

Keywords:

Abstraction

Simulation relation

Control by interconnection

Canonical controller

Bisimilar

ABSTRACT

We consider the problem of constructing a controller achieving a desired linear specification, based on a linear abstraction system of the plant system. First, we extend the necessary and sufficient conditions for control by interconnection by bisimulation equivalence to the case of non-deterministic linear systems. Then we apply the controller constructed on the basis of the lower-dimensional abstraction system to the original plant system and show that the closed-loop system is simulated by the given specification system. We distinguish between two forms of abstraction of the plant system. In the first one, the set of variables available for controller interconnection remains the same. In the second, more general form, this is not anymore the case, and we show how an adapted form of interconnection of the controller system to the plant system yields the same result.

© 2017 Elsevier B.V. All rights reserved.

1. Introduction

A basic problem in system and control theory is to construct a controller such that the closed-loop system behaves exactly like a given specification system. This general problem has been explored in various frameworks, see e.g. [1–3] and the references quoted therein. When dealing with a large-scale plant system the controller system tends to become high-dimensional as well, posing severe problems for computation and implementation. One of the methods to address this complexity problem is to approximate the linear plant system by a *lower-dimensional* system. There are many methods for approximation. In this paper, we approximate the plant system in the sense that it is *simulated* by a lower-dimensional linear system. This is the idea of *abstraction* [4,5], and we call this lower-dimensional, but possibly non-deterministic, system an *abstraction system*. Loosely speaking, the abstraction system is a lower-dimensional linear system whose external behavior (with respect to a given set of input and output variables) *contains* the external behavior of the original system. Next, we want to apply the controller based on the abstraction system to the plant system in such a way that the closed-loop system *approximates* the specification system, where again approximation is formalized as simulation.

A similar problem setting was extensively studied in a number of papers, see e.g. [6,7] and the references quoted therein, with the main difference that the abstraction system in these papers is a *discrete transition system*. Instead, in the current paper the abstraction system is again a (lower-dimensional and non-deterministic) linear system. This allows us to remain completely within the framework of linear geometric control theory. In [6] the problem is studied of refining the controller for the discrete abstraction system in such a way that it can be applied to the plant system. Furthermore, the notion of *alternating simulation relation* is used to relate the plant system and the abstraction system. Examples in [7] show that the alternating simulation relations are not adequate for controller refinement whenever the controller has only quantized or symbolic state information, and the complexity of the refined controller exceeds the complexity of the controller for the abstraction system. Therefore, a novel notion of *feedback refinement relations* is proposed to resolve both issues. Moreover, [7] shows that feedback refinement relations are necessary and sufficient for controller refinement. The current paper shows that for linear abstraction systems the problem of applying the controller constructed for the abstraction system to the original plant system admits a direct and elegant solution within the framework of geometric control theory.

In this paper we consider linear plant systems with two types¹ of inputs f and u , and two types of outputs z and y . The first type of

* Corresponding author at: Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9747AG Groningen, The Netherlands.

E-mail addresses: n.y.megawati@rug.nl, noorma_yulia@ugm.ac.id (N.Y. Megawati), a.j.van.der.schaft@rug.nl (A. van der Schaft).

¹ Note that in [6,7] the manifest and control variables of the abstraction system coincide, leading to a less general (and simpler) scenario.

input f together with the output z describes the interaction of the system with its environment. We call (f, z) the *manifest variables*. The second type of input u and the output y are variables that are to be connected to the controller system. The variables (u, y) are called the *control variables*. Given a linear plant system we approximate it by a lower-dimensional linear abstraction system. A preliminary problem studied in this paper consists in finding a controller for the abstraction system such that the abstraction system interconnected with the controller system is bisimilar to a given specification system. This problem extends the ‘control by interconnection’ problem studied in [3] to linear systems involving *internal disturbances*. These internal disturbances are used for modeling the typical case of ‘non-determinism’ in the abstraction system. ‘Non-determinism’ means that the state of the system, starting from a given initial condition, and for a given input function may evolve into different time-trajectories.² Next we consider the problem of applying the controller system derived for the abstraction system to the original plant system. Here we make a distinction between the situation where the set of control variables of the abstraction system is equal to the set of control variables, and the more general situation where this is not anymore the case. In this last case we need to modify the interconnection of the controller to the original plant system. The main theorem consists of showing that the resulting interconnection of the original plant system and the controller system derived for the abstraction system is *simulated* by the specification system.

The paper uses throughout the framework of ‘control by interconnection’ and that of ‘(bi-)simulation’ equivalence. In the control by interconnection framework the plant system is interconnected to a controller system via shared control variables. This does *not* always correspond to dynamical output feedback, as usually studied in control theory. On the other hand, certain physical control mechanisms, as exemplified by the *door closing mechanism* given in [9], are not of dynamical output feedback type but do correspond to control by interconnection. Furthermore, we will use the *canonical controller* introduced in [2], and further employed in [3] for control by interconnection by bisimulation equivalence. We will give more details about control by interconnection and the canonical controller in Sections 2 and 3, respectively.

With regard to the use of (bi-)simulation we note that there are many ways to define *equivalence* between systems. In the behavioral approach, two systems are equivalent if their behaviors are equal. In a linear input–output context, two systems are called equivalent if their transfer matrices are equal. Furthermore, two state space systems can be called equivalent if there exists an invertible state space transformation matrix linking the two systems. The notion of (bi-)simulation relation was first introduced in computer science [10], and later extended to continuous dynamical systems in [5]; see also [1]. In [4] it was shown how the notion of bisimulation relation directly extends the classical notions of transfer matrix equality and state space equivalence. This was recently extended to linear differential–algebraic (DAE) systems in [11], which also provides the natural context for (bi-)simulation using the canonical controller. We will give more details in Section 2.

This paper is organized as follows. In Section 2, we recall the basic notions that we need for the rest of the paper such as the definition of interconnection between systems and the notion of (bi-)simulation relation using geometric control theory. In Section 3, we deal with the problem of finding a controller for linear systems with internal disturbances. Then we address the main problem of applying the controller constructed on the basis of the abstraction

system to the original plant system. We start with the case that the set of control variables of the abstraction system is equal to that of the plant system, and afterwards we show how this can be extended to the general case by making use of an adapted form of interconnection.

2. Preliminaries

In this section, we give definitions of the notions of *interconnection* and of *(bi-)simulation relations*.

2.1. Interconnection

Consider two linear systems

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i^u u_i + B_i^f f_i + G_i d_i, & x_i &\in \mathcal{X}_i, u_i \in \mathcal{U}_i, f_i \in \mathcal{F}, d_i \in \mathcal{D}_i \\ \Sigma_i : y_i &= C_i^y x_i, & y_i &\in \mathcal{Y}_i \\ z_i &= C_i^z x_i, & z_i &\in \mathcal{Z} \end{aligned} \quad (1)$$

where $A_i \in \mathbb{R}^{q_i \times n_i}$, $B_i^u \in \mathbb{R}^{q_i \times k_i}$, $B_i^f \in \mathbb{R}^{q_i \times l}$, $G_i \in \mathbb{R}^{q_i \times s_i}$, $C_i^y \in \mathbb{R}^{p_i \times n_i}$ and $C_i^z \in \mathbb{R}^{r \times n_i}$; $\mathcal{X}_i, \mathcal{U}_i, \mathcal{F}, \mathcal{D}_i, \mathcal{Y}_i$ and $\mathcal{Z}$ are finite dimensional linear spaces, of dimension, respectively n_i, k_i, l, s_i, p_i and r . Here x_i denotes the state of the system, u_i, f_i are inputs, d_i is ‘internal’ disturbance and y_i, z_i are outputs. The set of allowed time functions $x_i : \mathbb{R}^+ \rightarrow \mathcal{X}_i, u_i : \mathbb{R}^+ \rightarrow \mathcal{U}_i, f_i : \mathbb{R}^+ \rightarrow \mathcal{F}, d_i : \mathbb{R}^+ \rightarrow \mathcal{D}_i, y_i : \mathbb{R}^+ \rightarrow \mathcal{Y}_i$ and $z_i : \mathbb{R}^+ \rightarrow \mathcal{Z}$, with $\mathbb{R}^+ = [0, \infty)$, will be denoted by $\mathfrak{X}_i, \mathfrak{U}_i, \mathfrak{F}, \mathfrak{D}_i, \mathfrak{Y}_i$ and $\mathfrak{Z}$, respectively. For simplicity of the notation, we will denote these time functions by x_i, u_i, f_i, d_i, y_i and z_i . The exact choice of function classes is for the purpose of this paper not really important as long as the state trajectories $x(\cdot)$ are continuous. For example, we can take all the functions to be piecewise C^∞ .

Interconnection between two systems with respect to either manifest variables or control variables will be denoted by m and c , respectively. In order to allow for more general interconnections than the standard feedback one, we will use a permutation matrix Π as formalized in the following definition.

Definition 2.1. Let Σ_1 and Σ_2 be two systems of the form (1). Their interconnection through the manifest variables via a permutation matrix Π , denoted by $\Sigma_1 \parallel_m^\Pi \Sigma_2$, is defined

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i^u u_i + B_i^f f_i + G_i d_i, \\ y_i &= C_i^y x_i, \\ z_i &= C_i^z x_i, \quad i = 1, 2, \\ \begin{bmatrix} f_1 \\ z_1 \end{bmatrix} &= \Pi \begin{bmatrix} f_2 \\ z_2 \end{bmatrix}. \end{aligned}$$

The interconnection system $\Sigma_1 \parallel_m^\Pi \Sigma_2$ is a differential–algebraic system with algebraic constraints on the state variables (x_1, x_2) . The state space of this interconnected system, denoted by $\mathcal{X}_{\Sigma_1, \Sigma_2}$, is defined as

$\{(x_1, x_2) \in \mathcal{X}_{\Sigma_1} \times \mathcal{X}_{\Sigma_2} \mid \exists \text{ input functions } u_1, u_2, f_1, f_2, \text{ disturbance functions } d_1, d_2 \text{ and } \exists \text{ solution trajectory } (x_1(\cdot), x_2(\cdot)) \text{ such that}$

$$(x_1(0), x_2(0)) = (x_1, x_2) \text{ and } \begin{bmatrix} f_1(t) \\ z_1(t) \end{bmatrix} = \Pi \begin{bmatrix} f_2(t) \\ z_2(t) \end{bmatrix}, t \geq 0\},$$

Similarly, the interconnection through the control variables and a suitable permutation matrix Π is denoted by $\Sigma_1 \parallel_c^\Pi \Sigma_2$, where the first set of equations is as in Definition 2.1, while the state space of the interconnected system is $\{(x_1, x_2) \in \mathcal{X}_{\Sigma_1} \times \mathcal{X}_{\Sigma_2} \mid \exists \text{ input functions } u_1, u_2, f_1, f_2, \text{ disturbance functions } d_1, d_2 \text{ and } \exists \text{ solution trajectories } (x_1(\cdot), x_2(\cdot)) \text{ such that } (x_1(0), x_2(0)) = (x_1, x_2) \text{ and } \begin{bmatrix} u_1(t) \\ y_1(t) \end{bmatrix} = \Pi \begin{bmatrix} u_2(t) \\ y_2(t) \end{bmatrix}, t \geq 0\}$.

We use the notation $(x_1(0), u_1, y_1, f_1, z_1, d_1) \in \Sigma_1$ to indicate that starting from an initial condition $x_1(0)$, by applying input

² We want to emphasize that the problem studied here is fundamentally different from the problem studied in [8] where the problem of (behavioral) control by interconnection of a plant system with *external disturbances* was treated.

functions u_1 and f_1 and a disturbance function d_1 to the system Σ_1 , then the resulting output functions are y_1 and z_1 . Similarly, we shall denote a trajectory of interconnection system $\Sigma_1 \parallel_m^T \Sigma_2$ by the notation $(x_1(0), u_1, y_1, f_1, z_1, d_1) \times (x_2(0), u_2, y_2, f_2, z_2, d_2) \in \Sigma_1 \parallel_m^T \Sigma_2$ to indicate that $(x_i(0), u_i, y_i, f_i, z_i, d_i) \in \Sigma_i$ for $i = 1, 2$, while the functions (f_i, z_i) satisfy $\begin{bmatrix} f_i \\ z_i \end{bmatrix} = \Pi \begin{bmatrix} f_2 \\ z_2 \end{bmatrix}$. Further, when $\Pi = I$ (the identity matrix), then the interconnection system will be simply denoted by $\Sigma_1 \parallel_m \Sigma_2$.

2.2. (Bi)-simulation relation

Recall from [4] the definition of a simulation relation of Σ_1 by Σ_2 .

Definition 2.2. A simulation relation of Σ_1 by Σ_2 with respect to the variables f_i and z_i is a linear subspace $S_{f,z} \subset \mathcal{X}_1 \times \mathcal{X}_2$ with the following property. Take any $(x_{10}, x_{20}) \in S_{f,z}$ and any common input function $f_1 = f_2$. Then

- for any input function u_1 and disturbance function d_1 , there exist an input function u_2 and disturbance function d_2 such that the resulting trajectories $x_1(\cdot)$ with $x_1(0) = x_{10}$ and $x_2(\cdot)$ with $x_2(0) = x_{20}$ satisfy

$$(x_1(t), x_2(t)) \in S_{f,z}, \quad t \geq 0 \quad (2)$$

- for all $(x_1, x_2) \in S_{f,z}$

$$C_1^z x_1 = C_2^z x_2. \quad (3)$$

System Σ_1 is said to be *simulated* by system Σ_2 , denoted by $\Sigma_1 \preceq \Sigma_2$, if there exists a simulation relation $S_{f,z}$ of Σ_1 by Σ_2 such that $\pi_1(S_{f,z}) = \mathcal{X}_1$.

The algebraic characterization of a simulation relation is given as follows [4].

Proposition 2.3. A subspace $S_{f,z} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a simulation relation of Σ_1 by Σ_2 with respect to f_i and z_i if and only if

$$\begin{aligned} (a) \quad & S_{f,z} + \text{im} \begin{bmatrix} B_1^u & G_1 \\ 0 & 0 \end{bmatrix} \subset S_{f,z} + \text{im} \begin{bmatrix} 0 & 0 \\ B_2^u & G_2 \end{bmatrix}, \\ (b) \quad & \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} S_{f,z} \subset S_{f,z} + \text{im} \begin{bmatrix} 0 & 0 \\ B_2^u & G_2 \end{bmatrix}, \\ (c) \quad & \text{im} \begin{bmatrix} B_1^f \\ B_2^f \end{bmatrix} \subset S_{f,z} + \text{im} \begin{bmatrix} 0 & 0 \\ B_2^u & G_2 \end{bmatrix}, \\ (d) \quad & S_{f,z} \subset \ker \begin{bmatrix} C_1^z & \vdots & -C_2^z \end{bmatrix}. \end{aligned} \quad (4)$$

The following lemma shows that the relation $\preceq$ is transitive.

Lemma 2.4. Let Σ_1, Σ_2 and Σ_3 be three systems of the form (1). If $\Sigma_1 \preceq \Sigma_2$ and $\Sigma_2 \preceq \Sigma_3$, then $\Sigma_1 \preceq \Sigma_3$.

Proof. Let $\mathcal{R}_{1,2} \subset \mathcal{X}_1 \times \mathcal{X}_2$ and $\mathcal{R}_{2,3} \subset \mathcal{X}_2 \times \mathcal{X}_3$ be full simulation relations of Σ_1 by Σ_2 and Σ_2 by Σ_3 , respectively. Then, a full simulation relation of Σ_1 by Σ_3 is given by

$$\{(x_1, x_3) \in \mathcal{X}_1 \times \mathcal{X}_3 \mid \exists x_2 \in \mathcal{X}_2 \text{ such that } (x_1, x_2) \in \mathcal{R}_{1,2} \text{ and } (x_2, x_3) \in \mathcal{R}_{2,3}\}.$$

A notion of bisimulation relation between Σ_1 and Σ_2 can be given as follows. Recall the definition of the inverse relation $\mathcal{T}^{-1} := \{(x_a, x_b) \mid (x_b, x_a) \in \mathcal{T}\}$.

Proposition 2.5. Let $\mathcal{S} \subset \mathcal{X}_1 \times \mathcal{X}_2$ be a simulation relation of Σ_1 by Σ_2 and let $\mathcal{T} \subset \mathcal{X}_2 \times \mathcal{X}_1$ be a simulation relation of Σ_2 by Σ_1 . Then $\mathcal{R} := \mathcal{S} + \mathcal{T}^{-1}$ is a bisimulation relation between Σ_1 and Σ_2 .

For later use we note the obvious fact:

Proposition 2.6. The identity relation $\mathcal{R}_{id} = \{(x, x) \mid x \in \mathcal{X}\}$ is a bisimulation relation between Σ and itself.

3. Existence of controller for the abstraction system

Consider a plant P given by the equations

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p^u u_p + B_p^f f_p, \quad x_p \in \mathcal{X}_p \\ P : z_p &= C_p^z x_p, \\ y_p &= C_p^y x_p. \end{aligned} \quad (5)$$

A controller is a system that is interconnected to the plant system via the control variables. In general, we define a controller C as

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c^u u_c + G_c d_c, \quad x_c \in \mathcal{X}_c \\ C : y_c &= C_c^y x_c, \end{aligned} \quad (6)$$

with d_c again denoting possible non-determinism.

Furthermore, let S denote the specification system which is expressed in terms of the manifest variables as

$$\begin{aligned} S : \dot{x}_s &= A_s x_s + B_s^f f_s, \quad x_s \in \mathcal{X}_s \\ z_s &= C_s^z x_s. \end{aligned} \quad (7)$$

3.1. Case 1

Consider a plant system P together with a surjective map

$$H : \mathcal{X}_p \rightarrow \bar{\mathcal{X}}_p,$$

where $\bar{\mathcal{X}}_p$ denotes another linear space satisfying

$$\ker H \subset \ker C_p^y \quad \text{and} \quad \ker H \subset \ker C_p^z.$$

Recall from [4] that P defines the following dynamical system P_a on $\bar{\mathcal{X}}_p$

$$\begin{aligned} \dot{\bar{x}}_p &= \bar{A}_p \bar{x}_p + \bar{B}_p^u u_p + \bar{B}_p^f f_p + G_p d_p, \quad \bar{x}_p \in \bar{\mathcal{X}}_p \\ P_a : z_p &= \bar{C}_p^z \bar{x}_p, \\ y_p &= \bar{C}_p^y \bar{x}_p \end{aligned} \quad (8)$$

where

$$\begin{aligned} \bar{A}_p &= H A_p H^\dagger, \\ \bar{B}_p^u &= H B_p^u, \\ \bar{B}_p^f &= H B_p^f, \\ \bar{C}_p^y &= C_p^y H^\dagger, \\ \bar{C}_p^z &= C_p^z H^\dagger, \end{aligned}$$

$$G_p = \begin{bmatrix} H A_p v_1 & \cdots & H A_p v_k \end{bmatrix},$$

and the vectors $v_1, \dots, v_k$ span $\ker H$ with $H^\dagger$ a pseudoinverse of matrix H . The system P_a is called an *abstraction* of the plant system P [5], while P is simulated by P_a with simulation relation

$$\mathcal{R}_{PP_a} = \{(x_p, \bar{x}_p) \in \mathcal{X}_p \times \bar{\mathcal{X}}_p \mid \bar{x}_p = H x_p\}.$$

This means that for any $(x_p^0, \bar{x}_p^0) \in \mathcal{R}_{PP_a}$, any joint input function $u_p = u_{p_a}$ and $f_p = f_{p_a}$, for any disturbance function d_p the resulting trajectories $(x_p(\cdot), \bar{x}_p(\cdot))$ (with $x_p(0) = x_p^0, \bar{x}_p(0) = \bar{x}_p^0$) satisfy

- $(x_p(t), \bar{x}_p(t)) \in \mathcal{R}_{PP_a}, \quad \forall t \geq 0.$
- $C_p^y x_p = \bar{C}_p^y \bar{x}_p, \quad \forall (x_p, \bar{x}_p) \in \mathcal{R}_{PP_a}.$
- $C_p^z x_p = \bar{C}_p^z \bar{x}_p, \quad \forall (x_p, \bar{x}_p) \in \mathcal{R}_{PP_a}.$

Next, a particular subsystem of the abstraction system P_a is obtained by setting the control variables (u_p, y_p) to be zero. The resulting system, denoted by P_a^0 , is defined by the differential-algebraic (DAE) system

$$\begin{aligned} \dot{\bar{x}}_p &= \bar{A}_p \bar{x}_p + \bar{B}_p^f f_p + G_p d_p, \\ P_a^0 : 0 &= y_p = \bar{C}_p^y \bar{x}_p, \\ z_p &= \bar{C}_p^z \bar{x}_p. \end{aligned} \quad (9)$$

According to *geometric control theory*, see e.g. [12,13], the state space of P_a^0 , denoted by $\bar{\mathcal{X}}_{p_a^0}$, is the largest $(\bar{A}_p, [\bar{B}_p^f G_p])$ -invariant subspace contained in $\ker \bar{C}_p^y$. In the original system P_a , the input functions f_p and disturbance functions d_p are arbitrary. Now, since $y_p = 0$, every input and disturbance function can be written as

$$\begin{bmatrix} f_p \\ d_p \end{bmatrix} = \begin{bmatrix} F_f \\ F_d \end{bmatrix} \bar{x} + \begin{bmatrix} L_f \\ 0 \end{bmatrix} f' + \begin{bmatrix} 0 \\ L_d \end{bmatrix} d',$$

where $F = \begin{bmatrix} F_f \\ F_d \end{bmatrix}$ is a linear map such that $(\bar{A}_p + \bar{B}_p^f F_f + G_p F_d) \bar{\mathcal{X}}_{p_a^0} \subset \bar{\mathcal{X}}_{p_a^0}$, L_f is such that $\text{im } L_f = \bar{\mathcal{X}}_{p_a^0} \cap \text{im } \bar{B}_p^f$, L_d is such that $\text{im } L_d = \bar{\mathcal{X}}_{p_a^0} \cap \text{im } G_p$, and f', d' are arbitrary functions. This set of input functions f_p and disturbance functions d_p in P_a^0 will be denoted by $\mathcal{I}(f_p, P_a^0)$ and $\mathcal{I}(d_p, P_a^0)$, respectively.

Problem Statement 1. Given the abstraction system P_a as in (8), find necessary and sufficient conditions for the existence of a controller C^a such that $P_a \|_c^T C^a$ is bisimilar to S for some permutation matrix Π .

The necessary and sufficient conditions for the achievability of S is given in the following theorem, extending [3, Theorem 7] to systems with internal disturbances.

Theorem 3.1. S is achievable if and only if $P_a^0 \preceq S \preceq P_a$.

Before proving Theorem 3.1, we state some preliminary results. The condition $S \preceq P_a$ is necessary, since it guarantees that the trajectories in S can be generated by P_a . The necessary condition $P_a^0 \preceq S$ is more subtle. Since P_a^0 is the behavior that is present when $u_p = 0$ and $y_p = 0$, see (9), these trajectories of P_a^0 continue to exist in the controlled system whenever we attach any controller. Thus the trajectories of P_a^0 must be contained in S .

For proving the sufficiency of the condition $P_a^0 \preceq S \preceq P_a$, we show that the *canonical controller* [4], denoted by C_{can}^a and defined as $C_{can}^a := S \|_m P_a$, achieves S . The canonical controller C_{can}^a is given as

$$\begin{aligned} \begin{bmatrix} \dot{x}_s \\ \dot{x}_p \end{bmatrix} &= \begin{bmatrix} A_s & 0 \\ 0 & \bar{A}_p \end{bmatrix} \begin{bmatrix} x_s \\ \bar{x}_p \end{bmatrix} + \begin{bmatrix} B_s^f \\ \bar{B}_p^f \end{bmatrix} f + \begin{bmatrix} 0 \\ \bar{B}_p^u \end{bmatrix} u \\ &+ \begin{bmatrix} 0 \\ G_p \end{bmatrix} d_p, \\ 0 &= \begin{bmatrix} C_s^z & \vdots & -\bar{C}_p^z \end{bmatrix} \begin{bmatrix} x_s \\ \bar{x}_p \end{bmatrix}, \\ y &= \begin{bmatrix} 0 & \vdots & \bar{C}_p^y \end{bmatrix} \begin{bmatrix} x_s \\ \bar{x}_p \end{bmatrix}. \end{aligned} \quad (10)$$

The system (10) is a differential-algebraic system. According to [11], the consistent subset $\mathcal{V}^*$ for C_{can}^a is either empty or given as the maximal subspace $\mathcal{V} \subset \mathbb{R}^n$ satisfying

$$\begin{aligned} \begin{bmatrix} A_s & 0 \\ 0 & \bar{A}_p \\ C_s^z & -\bar{C}_p^z \end{bmatrix} \mathcal{V} &\subseteq \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \mathcal{V} + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \\ 0 \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \\ 0 \end{bmatrix}, \\ \text{im} \begin{bmatrix} B_s^f \\ \bar{B}_p^f \\ 0 \end{bmatrix} &\subseteq \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \mathcal{V} + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \\ 0 \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \\ 0 \end{bmatrix}. \end{aligned} \quad (11)$$

It follows that $\mathcal{V}^*$ is the set of all initial conditions x_0 for which for every piecewise-continuous input function $f_p = f_s = f$, there exist a piecewise-continuous input function u_p and a piecewise-continuous disturbance function d_p and also a piecewise-continuously differentiable solution trajectory $x(\cdot)$ of C_{can}^a with $x(0) = x_0$.

The maximal simulation relation of S by P_a is the largest subspace $\mathcal{R}_{f,z}^{SP_a} \subseteq \mathcal{X}_S \times \bar{\mathcal{X}}_p$ such that

$$\begin{aligned} (a) \quad \begin{bmatrix} A_s & 0 \\ 0 & \bar{A}_p \end{bmatrix} \mathcal{R}_{f,z}^{SP_a} &\subseteq \mathcal{R}_{f,z}^{SP_a} + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \end{bmatrix}, \\ (b) \quad \text{im} \begin{bmatrix} B_s^f \\ \bar{B}_p^f \end{bmatrix} &\subseteq \mathcal{R}_{f,z}^{SP_a} + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \end{bmatrix}, \\ (c) \quad \mathcal{R}_{f,z}^{SP_a} &\subseteq \ker \begin{bmatrix} C_s^z & \vdots & -\bar{C}_p^z \end{bmatrix}. \end{aligned} \quad (12)$$

Denote the maximal simulation relation of P_a^0 by S by $\mathcal{R}_{f,z}^{P_a^0 S}$, which is the largest subspace $\mathcal{R}_{f,z}^{P_a^0 S} \in \bar{\mathcal{X}}_{p_a^0} \times \mathcal{X}_S$ such that

$$\begin{aligned} (a) \quad \mathcal{R}_{f,z}^{P_a^0 S} + \text{im} \begin{bmatrix} G_p \\ 0 \end{bmatrix} &\subset \mathcal{R}_{p_a^0 S}, \\ (b) \quad \begin{bmatrix} \bar{A}_p & 0 \\ 0 & A_s \end{bmatrix} \mathcal{R}_{f,z}^{P_a^0 S} &\subset \mathcal{R}_{f,z}^{P_a^0 S}, \\ (c) \quad \text{im} \begin{bmatrix} \bar{B}_p^f \\ B_s^f \end{bmatrix} &\subset \mathcal{R}_{f,z}^{P_a^0 S}, \\ (d) \quad \mathcal{R}_{f,z}^{P_a^0 S} &\subset \ker \begin{bmatrix} \bar{C}_p^z & \vdots & -C_s^z \end{bmatrix}. \end{aligned} \quad (13)$$

Lemma 3.2. Given that $\mathcal{R}_{f,z}^{SP_a}$ exists, $\mathcal{R}_{f,z}^{SP_a} = \mathcal{V}^*$.

Proof. ($\mathcal{R}_{f,z}^{SP_a} \subseteq \mathcal{V}^*$). We know that $\mathcal{R}_{f,z}^{SP_a}$ satisfies (12). According to (12a)–(12c) we have

$$\begin{bmatrix} A_s & 0 \\ 0 & \bar{A}_p \\ C_s^z & -\bar{C}_p^z \end{bmatrix} \mathcal{R}_{f,z}^{SP_a} \subseteq \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \mathcal{R}_{f,z}^{SP_a} + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \\ 0 \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \\ 0 \end{bmatrix}.$$

Clearly, since (12b) holds, then

$$\text{im} \begin{bmatrix} B_s^f \\ \bar{B}_p^f \\ 0 \end{bmatrix} \subseteq \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \mathcal{R}_{f,z}^{SP_a} + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \\ 0 \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \\ 0 \end{bmatrix}$$

also holds. Hence, $\mathcal{R}_{f,z}^{SP_a} \subseteq \mathcal{V}^*$.

($\mathcal{V}^* \subseteq \mathcal{R}_{f,z}^{SP_a}$). According to (11b), clearly that

$$\text{im} \begin{bmatrix} B_s^f \\ \bar{B}_p^f \end{bmatrix} \subseteq \mathcal{V}^* + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \end{bmatrix}$$

holds. Since (11a) holds, we have

$$\begin{aligned} \begin{bmatrix} A_s & 0 \\ 0 & \bar{A}_p \end{bmatrix} \mathcal{V}^* &\subseteq \mathcal{V}^* + \text{im} \begin{bmatrix} 0 \\ \bar{B}_p^u \end{bmatrix} + \text{im} \begin{bmatrix} 0 \\ G_p \end{bmatrix}, \\ \mathcal{V}^* &\subseteq \ker \begin{bmatrix} C_s^z & \vdots & -\bar{C}_p^z \end{bmatrix}. \end{aligned}$$

Thus $\mathcal{V}^* \subseteq \mathcal{R}_{f,z}^{SP_a}$.

We need one more result about the abstraction system. Suppose $(\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \in P_a$. Assume u_p and y_p are fixed, then we

characterize the set of inputs f'_p , the disturbances d'_p and the set of states $\bar{x}'_p(0)$ such that $(\bar{x}'_p(0), u_p, y_p, f'_p, z'_p, d'_p) \in P_a$ for some z'_p as follows.

Lemma 3.3. *The two trajectories $(\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p)$ and $(\bar{x}'_p(0), u_p, y_p, f'_p, z'_p, d'_p)$ are both trajectories in P_a if and only if $\bar{x}'_p(0) - \bar{x}_p(0) \in \mathcal{X}_{p_a^0}$, $f'_p - f_p \in \mathcal{I}(f_p, P_a^0)$, and $d'_p - d_p \in \mathcal{I}(d_p, P_a^0)$ where $\mathcal{I}(f_p, P_a^0)$ is the set of allowed input f_p in P_a^0 and $\mathcal{I}(d_p, P_a^0)$ is the set of allowed disturbance in P_a^0 .*

Proof. The output function resulting from the initial condition $\bar{x}_p(0)$, input functions u_p, f_p and disturbance function d_p is

$$y_p(t) = \bar{C}_p^y e^{\bar{A}_p(t)} \bar{x}_p(0) + \bar{C}_p^y \int_0^t e^{\bar{A}_p(t-\tau)} \bar{B}_p^f f_p(\tau) d\tau + \bar{C}_p^y \int_0^t e^{\bar{A}_p(t-\tau)} \times \bar{B}_p^u u_p(\tau) d\tau + \bar{C}_p^y \int_0^t e^{\bar{A}_p(t-\tau)} G_p d_p(\tau) d\tau. \quad (14)$$

The output resulting from $\bar{x}'_p(0), f'_p$ and d'_p is obtained by replacing $\bar{x}_p(0), f_p, d_p$ by $\bar{x}'_p(0), f'_p, d'_p$. The proof is analogous to the proof of [3, Lemma 9].

We now prove Theorem 3.1. ($\Rightarrow$) Suppose $P_a \parallel_c^{\Pi} C^a \approx S$ for some Π . It is sufficient to show that $P_a \parallel_c^{\Pi} C^a$ is simulated by P_a and itself simulates P_a^0 . Let $(\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \times (x_c(0), u_c, y_c, d_c) \in P_a \parallel_c^{\Pi} C^a$ and $(\bar{x}_p(0), u_p, y'_p, f'_p, z'_p, d'_p) \in P_a$. Since $P_a \preceq_{id} P_a$ where the identity simulation relation is given by $R_{id}^{P_a} := \{(\bar{x}_p, \bar{x}_p) | \bar{x}_p \in \bar{\mathcal{X}}_p\}$, we have that $z_p = z'_p$ and $y_p = y'_p$. Thus

$$\{(a, b, c) \in \bar{\mathcal{X}}_p \times \mathcal{X}_c \times \bar{\mathcal{X}}_p | a = c, (a, c) \in \mathcal{R}_{id}^{P_a}, (a, b) \in \mathcal{X}_{p_a, c^a}\}$$

is a simulation relation of $P_a \parallel_c^{\Pi} C^a$ by P_a .

Now consider any $(\bar{x}_p(0), 0, 0, f_p, z_p, d_p) \in P_a^0$. Then take $(\bar{x}_p(0), 0, 0, f'_p, z'_p, d'_p) \times (0, 0, 0, 0) \in P_a \parallel_c^{\Pi} C^a$; this can be done since $(\bar{x}_p(0), 0)$ is always in $\mathcal{X}_{p_a, c^a}$. Then by the identity bisimulation relation $\mathcal{R}_{id}^{P_a^0}$, we have $z_p = z'_p$. Thus

$$\{(a, b, c) \in \bar{\mathcal{X}}_{p_a^0} \times \bar{\mathcal{X}}_p \times \mathcal{X}_c | a = b, (a, b) \in \mathcal{R}_{id}^{P_a^0} \text{ and } c = 0\}$$

is a simulation relation of P_a^0 by $P_a \parallel_c^{\Pi} C^a$.

($\Leftarrow$) Consider the canonical controller C_{can}^a . Under the conditions $P_a^0 \leq S \leq P_a$, we will show that $C_{can}^a \parallel_c P_a \approx S$. Proving this is equivalent to proving the following two statements.

1. $S \leq C_{can}^a \parallel_c P_a$.
Let $(x_s(0), f_s, z_s) \in S$. Since $S \leq P_a$, there exist $\bar{x}_p(0), u_p, y_p, d_p, f_p = f_s$ and $z_p = z_s$ such that $(x_s(0), f_s, z_s) \times (\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \in P_a$. Because of the identity bisimulation relation on P_a , we have $f_p = f_s$ and $z_p = z_s$. Thus the simulation relation $S \leq C_{can}^a \parallel_c P_a$ is given by

$$\{(a, b, c, d) \in \mathcal{X}_S \times \mathcal{X}_S \times \bar{\mathcal{X}}_p \times \bar{\mathcal{X}}_p | a = b, c = d, (b, c) \in \mathcal{R}_{f,z}^{SP_a}, (c, d) \in \mathcal{R}_{id}^{P_a}\}.$$

2. $C_{can}^a \parallel_c P_a \leq S$.
Let $(x_s(0), f_s = f_p, z_s = z_p) \times (\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \in C_{can}^a \parallel_c P_a$. Since $S \leq P_a$, then we have that $(x_s(0), \bar{x}_p(0)) \in \mathcal{R}_{SP_a}$. Furthermore, since $P_a^0 \leq S$, there exists a state $x_{p_a^0 S} \in \mathcal{X}_S$ such that $(\bar{x}'_p(0) - \bar{x}_p(0), 0, 0, f'_p - f_p, z'_p - z_p, d'_p - d_p) \in P_a^0$ and $(x_{NS}(0), f'_p - f_p, z'_p - z_p) \in S$ where $(\bar{x}'_p(0) - \bar{x}_p(0), x_{NS}(0)) \in \mathcal{R}_{p_a^0 S}$. To this trajectory in S , we can add the trajectory $(x_s(0), f_s, z_s)$ to get $(x_s(0) + x_{NS}(0), f'_p, z'_p) \in S$. Thus we found a state in S such that for the same input f'_p , the output of S and C_{can}^a are equal

to z'_p . Hence the simulation relation $C_{can}^a \parallel_c P_a \leq S$ is given by

$$\{(a, b, c, d) \in \mathcal{X}_S \times \bar{\mathcal{X}}_p \times \bar{\mathcal{X}}_p \times \mathcal{X}_S | (a, b) \in \mathcal{R}_{f,z}^{SP_a}, c - b \in \bar{\mathcal{X}}_{p_a^0}, ((c - b), (d - a)) \in \mathcal{R}_{f,z}^{P_a^0}\}.$$

Remark 3.4. Theorem 3.1 implies that any controller that achieves S is simulated by the canonical controller. Indeed, suppose C^a is a controller such that $P_a \parallel_c^{\Pi} C^a \approx S$ for some interconnection matrix Π . Based on Theorem 3.1, we have that $S \approx C_{can}^a \parallel_c P_a$. Now let $(\bar{x}_p(0), f_p, z_p, u_p, y_p, d_p) \times (x_c(0), u_c, y_c, d_c) \in P_a \parallel_c^{\Pi} C^a$. Since $P_a \parallel_c^{\Pi} C^a \approx S$, then there exists $(x_s(0), f_s, z_s) \in S$ such that $f_s = f_p$ and $z_s = z_p$. Hence, $(x_s(0), f_p, z_p) \times (\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \times (\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \in C_{can}^a \parallel_c P_a$. Thus $C^a \parallel_c^{\Pi} P_a \leq C_{can}^a \parallel_c P_a$ with simulation relation

$$\{(a, b, c, d, e) \in \mathcal{X}_c \times \bar{\mathcal{X}}_p \times \mathcal{X}_S \times \bar{\mathcal{X}}_p \times \bar{\mathcal{X}}_p | b = d = e, (d, e) \in \mathcal{R}_{id}^{P_a}, (b, a) \in \mathcal{X}_{p_a, c^a}, (c, d) \in \mathcal{R}_{f,z}^{SP_a}\},$$

where $\leq_c$ is a simulation relation with respect to the control variables.

Thus we have obtained necessary and sufficient conditions for the existence of a controller for the abstraction system. The next question is how this controller can be applied to the original plant system.

Problem Statement 2. Given the canonical controller solving Problem 1, then, if we interconnect this canonical controller to the plant P , what can we say about the relation of interconnected system $P \parallel_c C_{can}^a$ with the specification system S ?

The answer to the above problem is given in the following theorem.

Theorem 3.5. *If $P_a^0 \leq S \leq P_a$ then $P \parallel_c C_{can}^a \leq S$.*

Proof. Suppose C_{can}^a is a canonical controller such that $P_a \parallel_c C_{can}^a \approx S$, then we will prove that $P \parallel_c C_{can}^a \leq P_a \parallel_c C_{can}^a$. Take $(x_p(0), u, y, f', z') \times (x_s(0), f, z) \times (\bar{x}_p(0), u, y, f, z, d) \in P \parallel_c C_{can}^a$. Since $P \leq P_a$, then there exists a state $x_{pp_a} \in \bar{\mathcal{X}}_p$ such that $(\bar{x}'_p(0), u_p, y_p, f_p, z_p, d') \in P_a$ where $u_p = u, y_p = y, f_p = f'$ and $z_p = z'$. Hence, $(\bar{x}'_p(0), u, y, f', z', d') \times (x_s(0), f, z) \times (\bar{x}_p(0), u, y, f, z, d) \in P_a \parallel_c C_{can}^a$. Thus, $C_{can}^a \parallel_c P \leq P_a \parallel_c C_{can}^a$ with simulation relation

$$\{(a, b, c, d, e, f) \in \mathcal{X}_S \times \bar{\mathcal{X}}_p \times \mathcal{X}_p \times \bar{\mathcal{X}}_p \times \mathcal{X}_S \times \bar{\mathcal{X}}_p | a = e, b = f, (a, b) \in \mathcal{R}_{f,z}^{SP_a}, (c, a, b) \in \mathcal{X}_{p, c_{can}^a}, (c, d) \in \mathcal{R}_{pp_a}\}.$$

According to Theorem 3.5, if $P_a^0 \leq S \leq P_a$, then the canonical controller for the abstraction system can be applied directly to the plant system. Furthermore, the controlled system will be simulated by the specification system.

Remark 3.6. According to the Remark 3.4, any controller that achieves S is simulated by the canonical controller. Furthermore, for any controller C^a that achieves S the interconnected system $P \parallel_c^{\Pi} C^a$ is simulated by S .

Indeed, suppose C^a is a controller such that $P_a \parallel_c^{\Pi} C^a \approx S$ for some interconnection matrix Π . Take $(x_p(0), u_p, y_p, f_p, z_p) \times (x_c(0), u_c, y_c, d_c) \in P \parallel_c^{\Pi} C^a$. Since $P \leq P_a$, there exists $\bar{x}_p(0) \in \bar{\mathcal{X}}_p$ such that $(\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \in P_a$ satisfying $(x_p(0), \bar{x}_p(0)) \in \mathcal{R}_{pp_a}$. Hence $(\bar{x}_p(0), u_p, y_p, f_p, z_p, d_p) \times (x_c(0), u_c, y_c, d_c) \in P_a \parallel_c^{\Pi} C^a$. Thus $C^a \parallel_c^{\Pi} P \leq P_a \parallel_c^{\Pi} C^a$ with simulation relation

$$\{(a, b, c, d) \in \mathcal{X}_c \times \mathcal{X}_p \times \bar{\mathcal{X}}_p \times \mathcal{X}_c | a = d, (b, a) \in \mathcal{X}_{p, c^a}, (b, c) \in \mathcal{R}_{pp_a}\}.$$

Example 3.7. Consider a plant system

$$P : \begin{aligned} \dot{x}_p^1 &= x_p^1 + x_p^2 + x_p^4, \\ \dot{x}_p^2 &= x_p^2 + x_p^3 + u_p, \\ \dot{x}_p^3 &= x_p^2 + x_p^3 + f_p, \\ \dot{x}_p^4 &= x_p^4, \\ y_p &= x_p^3, \\ z_p &= x_p^2 \end{aligned}$$

and a specification system

$$S : \begin{aligned} \dot{x}_s^1 &= x_s^1 + x_s^2 + x_s^3, \\ \dot{x}_s^2 &= x_s^1 + x_s^2 + f_s, \\ \dot{x}_s^3 &= x_s^3, \\ z_s &= x_s^1. \end{aligned}$$

The abstraction of the plant P is taken to be

$$P_a : \begin{aligned} \dot{\bar{x}}_p^1 &= \bar{x}_p^1 + \bar{x}_p^2 + d_p, \\ \dot{\bar{x}}_p^2 &= \bar{x}_p^2 + \bar{x}_p^3 + u_p, \\ \dot{\bar{x}}_p^3 &= \bar{x}_p^2 + \bar{x}_p^3 + f_p, \\ y_p &= \bar{x}_p^3, \\ z_p &= \bar{x}_p^2. \end{aligned}$$

The state space of P_a^0 is found to be the span of $\{[1 \ 0 \ 0]^T, [0 \ 1 \ 0]^T\}$. Note that $\mathcal{X}_{P_a^0} \cap \text{im } \bar{B}_p^f = 0$ then, f is uniquely determined in P_a^0 , in fact $f_p = -\bar{x}_p^2$. Thus adapting the basis of $\bar{\mathcal{X}}_p$ to $\bar{\mathcal{X}}_{P_a^0}$ yields

$$P_a^0 : \begin{aligned} \dot{\bar{x}}_p^1 &= \bar{x}_p^1 + \bar{x}_p^2 + d_p, \\ \dot{\bar{x}}_p^2 &= \bar{x}_p^2, \\ z_p &= \bar{x}_p^2 \end{aligned}$$

where $(\bar{x}_p^1, \bar{x}_p^2, 0) \in \bar{\mathcal{X}}_{P_a^0}$. Let $((0, \bar{x}_p^2(0), 0), 0, 0, -\bar{x}_p^2, z_p, d_p) \in P_a^0$ and $((\bar{x}_p^1(0), 0, 0), -\bar{x}_p^2, z_s) \in S$. Then from the equation of P_a^0 and S it is clear that $z_p = z_s$. Thus $P_a^0 \preceq S$.

Now consider S such that $((x_s^1(0), x_s^2(0), x_s^3(0)), f_s, z_s) \in S$. Let $((\bar{x}_p^1(0), x_s^1(0), x_s^2(0)), \bar{x}_p^3, y_p, f_s, z_p, d_p) \in P_a$. Then we see that $z_p = z_s$. Thus $S \preceq P_a$. Hence by Theorem 3.1, there exists a controller which when interconnected with the plant yields a system which is bisimilar to S . One such controller for the abstraction system is

$$C^a : \begin{aligned} \dot{x}_s^1 &= x_s^1 + x_s^2 + e^t, \\ \dot{x}_s^2 &= x_s^1 + x_s^2, \\ y_c &= x_s^2 \end{aligned}$$

where $u_c = e^t$. Interconnecting C^a to P with constraint $u_p = u_c$ and $y_p = y_c$, we found that f is uniquely determined in $P \parallel_c C^a$; in fact $f_p = x_s^1 - x_p^2$. The interconnected system $P \parallel_c C^a$ can be given as follows

$$\begin{aligned} \dot{x}_p^1 &= x_p^1 + x_p^2 + x_p^4, \\ \dot{x}_p^2 &= x_p^2 + x_p^3 + e^t, \\ \dot{x}_p^3 &= x_p^3 + x_s^1, \\ \dot{x}_p^4 &= x_p^4, \\ \dot{x}_s^1 &= x_s^1 + x_p^3 + e^t, \\ z_p &= x_p^2. \end{aligned}$$

Further, let $((x_p^1(0), x_p^2(0), x_p^3(0), x_p^4(0)), e^t, x_s^2, x_s^1 - x_p^2, z_p) \times ((x_p^1(0), x_p^3(0)), e^t, x_s^2) \in P \parallel_c C^a$ and $((x_p^2(0), x_p^3(0), 1), x_s^1 - x_p^2, z_s) \in S$. Then from the equation of $P \parallel_c C^a$ and S we have that $z_p = z_s$. Thus $P \parallel_c C^a \preceq S$.

3.2. Case 2

In the second case, we consider the problem that $\ker H \not\subseteq \ker C_p^y$ and thus the output space $\mathcal{Y}$ of the abstraction system is different from the output space of the plant system. In this case the controller for the abstraction system cannot be directly applied to the original plant system.

Thus consider a plant P given in (1) together with surjective map $H : \mathcal{X}_p \rightarrow \bar{\mathcal{X}}_p$, only satisfying

$$\ker H \subset \ker C_p^z.$$

Since $C_p^y(\ker H) \subset \mathcal{Y}_p$, we define $\bar{\mathcal{Y}}_p := \mathcal{Y}_p / C_p^y(\ker H)$. Furthermore, define the linear map

$$\bar{C}_p^y : \bar{\mathcal{X}}_p \rightarrow \bar{\mathcal{Y}}_p,$$

where

$$\bar{C}_p^y : [x]_{\ker H} \mapsto [C_p^y x]_{C_p^y(\ker H)}. \quad (15)$$

Hence, we define the abstraction system given by

$$\begin{aligned} \dot{\bar{x}}_p &= \bar{A}_p \bar{x}_p + \bar{B}_p^u u_p + \bar{B}_p^f f_p + G_p d_p, \quad x_p \in \bar{\mathcal{X}}_p, \\ P_a : z_p &= \bar{C}_p^z \bar{x}_p, \\ \bar{y}_p &= \bar{C}_p^y \bar{x}_p \end{aligned} \quad (16)$$

where

$$\begin{aligned} \bar{A}_p &= H A_p H^\dagger, \\ \bar{B}_p^u &= H B_p^u, \\ \bar{B}_p^f &= H B_p^f, \\ \bar{C}_p^z &= C_p^z H^\dagger, \\ G_p &= \begin{bmatrix} H A v_1 & \dots & H A v_k \end{bmatrix}, \end{aligned}$$

and $\bar{C}_p^y$ defined in (15), $v_1, \dots, v_k$ span $\ker H$ and $H^\dagger$ denotes a pseudoinverse of H .

The simulation relation of P by P_a with respect to u and (f, z) is now given as the maximal subspace $\mathcal{R}_{u,f,z}^{PP_a} \subset \mathcal{X}_p \times \bar{\mathcal{X}}_p$ satisfying

$$\begin{aligned} \text{(i)} \quad & \begin{bmatrix} A_p & 0 \\ 0 & \bar{A}_p \end{bmatrix} \mathcal{R}_{u,f,z}^{PP_a} \subset \mathcal{R}_{u,f,z}^{PP_a} + \text{im} \begin{bmatrix} 0 \\ G_p \end{bmatrix}, \\ \text{(ii)} \quad & \text{im} \begin{bmatrix} B_p^u \\ \bar{B}_p^u \end{bmatrix} \subset \mathcal{R}_{u,f,z}^{PP_a} + \text{im} \begin{bmatrix} 0 \\ G_p \end{bmatrix}, \\ \text{(iii)} \quad & \text{im} \begin{bmatrix} B_p^f \\ \bar{B}_p^f \end{bmatrix} \subset \mathcal{R}_{u,f,z}^{PP_a} + \text{im} \begin{bmatrix} 0 \\ G_p \end{bmatrix}, \\ \text{(iv)} \quad & \mathcal{R}_{u,f,z}^{PP_a} \subset \ker \begin{bmatrix} C_p^z & \vdots & -\bar{C}_p^z \end{bmatrix}. \end{aligned} \quad (17)$$

Furthermore, according to (15)

$$\begin{array}{ccc} \mathcal{X}_p & \xrightarrow{H} & \bar{\mathcal{X}}_p \\ C_p^y \downarrow & & \downarrow \bar{C}_p^y \\ \mathcal{Y}_p & \xrightarrow{\pi} & \bar{\mathcal{Y}}_p \end{array} \quad (18)$$

and we can define a canonical projection $\pi : \mathcal{Y}_p \rightarrow \bar{\mathcal{Y}}_p$ such that $\bar{y}_p = \pi y_p$.

In this case we cannot directly interconnect the canonical controller anymore to the plant system. Instead, we modify the Problem Statement 2 as follows.

Problem Statement 3. Given P and S as in (5) and (7), respectively. By computing the abstraction system P^a given by (16), if $P_a^0 \preceq S \preceq P_a$, then prove that $P \parallel_c C_{can}^a \preceq S$ where $\Pi = \begin{bmatrix} I & 0 \\ 0 & \pi \end{bmatrix}$ (I is the identity matrix).

The answer for the above problem is given in the following theorem.

Theorem 3.8. *If $P_a^0 \preceq S \preceq P_a$, then $P \parallel_c^{II} C_{can}^a \preceq S$.*

Proof. Suppose C_{can}^a is a canonical controller such that $S \approx P_a \parallel_c C_{can}^a$. Then we will prove that $P \parallel_c^{II} C_{can}^a \preceq P_a \parallel_c C_{can}^a$.

Take $(x_p(0), u, y, f', z') \times (x_s(0), f, z) \times (\bar{x}_p(0), u, \pi y, f, z, d) \in P \parallel_c^{II} C_{can}^a$. Since $P \preceq P_a$, then there exist a state $x_{pp_a} \in \bar{x}_p$ and disturbance d' such that $(\bar{x}_p'(0), u_p, y_p, f_p, z_p, d') \in P_a$ satisfying $u_p = u, y_p = \pi y, f_p = f'$, and $z_p = z'$. Thus $(\bar{x}_p'(0), u, \pi y, f', z', d') \times (x_s(0), f, z) \times (\bar{x}_p(0), u, \pi y, f, z, d) \in P_a \parallel_c^{II} C_{can}^a$. Therefore, $C_{can}^a \parallel_c^{II} P \preceq P_a \parallel_c C_{can}^a$ with simulation relation

$$\{(a, b, c, d, e, f) \in \mathcal{X}_S \times \bar{\mathcal{X}}_p \times \mathcal{X}_p \times \bar{\mathcal{X}}_p \times \mathcal{X}_S \times \bar{\mathcal{X}}_p \mid a = e, \\ b = f, (a, b) \in \mathcal{R}_{f,z}^{SP_a}, (c, a, b) \in \mathcal{X}_{P,C_{can}^a}, (c, d) \in \mathcal{R}_{u,f,z}^{PP_a}\}.$$

4. Conclusions

We have derived necessary and sufficient conditions for the existence of a controller C^a for the abstraction system P_a such that the interconnection $P_a \parallel_c^{II} C^a$ is bisimilar to the specification system S , thereby generalizing the results of [3] to systems with internal disturbances. In order to apply the controller obtained for the abstraction system to the original system we considered two cases. In the first case the abstraction system is such that the set of control variables of P_a is equal to that of P . In this case we show that we may directly apply the controller C^a constructed for the abstraction system to the plant system P with the result that the interconnected system $P \parallel_c^{II} C^a$ is simulated by specification system S . In the second, more general, case, the output space $\mathcal{Y}$ of the abstraction system is different from that of the plant system. This leads to a naturally adapted form of interconnection of C^a to the plant system, which is shown to yield the same result.

An important problem for further research concerns the characterization of the existence of *feedback controllers* (as opposed to the general control by interconnection set-up; see also Remark 3.4).

Acknowledgment

The work of the first author is supported by the Directorate General of Resources for Science, Technology and Higher Education, The Ministry of Research, Technology, and Higher Education of Indonesia.

References

- [1] P. Tabuada, G. Pappas, Bisimilar control affine systems, *Syst. Control Lett.* 52 (1) (2004) 49–58.
- [2] A. van der Schaft, Achievable behavior of general systems, *Syst. Control Lett.* 49 (2003) 141–149.
- [3] H. Vinjamoer, A. van der Schaft, The achievable dynamics via control by interconnection, *IEEE Trans. Automat. Control* 56 (5) (2011) 1110–1117.
- [4] A. van der Schaft, Equivalence of dynamical systems by bisimulation, *IEEE Trans. Automat. Control* 49 (12) (2004) 2160–2172.
- [5] G.J. Pappas, Bisimilar linear systems, *Automatica* 39 (12) (2003) 2035–2047.
- [6] P. Tabuada, *Verification and Control of Hybrid Systems: A Symbolic Approach*, first ed., Springer Publishing Company, Incorporated, 2009.
- [7] G. Reißig, A. Weber, M. Rungger, Feedback refinement relations for the synthesis of symbolic controllers, *IEEE Trans. Automat. Control* 62 (4) (2017) 1781–1796.
- [8] A.A. Julius, M. Belur, Behavioral control in the presence of disturbance, in: 44th IEEE Conference on Decision and Control and the European Control Conference, Seville, Spain, December 12–15, 2005, pp. 155–160.
- [9] M. Kuijper, Why do stabilizing controllers stabilize? *Automatica* 31 (4) (1995) 621–625.
- [10] E.M. Clarke Jr., O. Grumberg, D. Peled, *Model Checking*, MIT Press, Cambridge, MA, USA, 1999.
- [11] N. Megawati, A. van der Schaft, Bisimulation equivalence of differential-algebraic systems, *Internat. J. Control* (2016) in press. <http://dx.doi.org/10.1080/00207179.2016.1266519>.
- [12] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, third ed., Springer-Verlag, New York, 1985.
- [13] H. Trentelman, A. Stoorvogel, M. Hautus, *Control Theory for Linear Systems*, Springer-Verlag, London, 2001.