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Bisimulation Equivalence of Discrete-Time Stochastic Linear Control Systems

Giordano Pola¹, Senior Member, IEEE, Costanzo Manes², Arjan J. van der Schaft³, Fellow, IEEE, and Maria Domenica Di Benedetto⁴, Fellow, IEEE

Abstract—In this paper, we propose a definition of equivalence via stochastic bisimulation for the class of discrete-time stochastic linear control systems with possibly degenerate normally distributed disturbances. The notion is inspired by the notion of probabilistic bisimulation for probabilistic chains. Geometric necessary and sufficient conditions for checking this notion are derived. Model reduction via Kalman-like decomposition is also proposed. Connections with stochastic reachability are discussed and with finite horizon stochastic safety problems established. A discussion on the use of stochastic bisimulation equivalence for control design is given, and an application to optimal control problems with stochastic reachability specifications is finally presented.

Index Terms—Bisimulation equivalence, geometric control theory, stochastic linear systems.

I. INTRODUCTION

ATHEME widely studied in the community of computer science is the characterization of equivalent models of computation. Several equivalence notions have been proposed in the literature; see, e.g., [41] and the references therein. Among these notions, bisimulation [23], [25] and trace equivalences play a prominent role. As discussed in [41], trace equivalence is the weakest equivalence notion, while bisimulation equivalence is the strongest one, apart from the notion of equivalence via isomorphism. Bisimulation equivalence is extensively used in the community of computer science as an effective tool to mitigate software verification. In the last 30 years, many researchers in the control systems and computer science communities were attracted by this research topic with the aim of reducing the complexity of real-world complex systems for formal verification/analysis and control design purposes. The research in this

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TABLE I
RELATED LITERATURE ON EQUIVALENCES-BASED REDUCTION OF
DISCRETE/CONTINUOUS/HYBRID
DETERMINISTIC/NONDETERMINISTIC/STOCHASTIC SYSTEMS

	(D1)	(D2)
(R1)–(E1)	[9]	[17], [21]
(R1)–(E2)	[8], [10], [26]	[5], [12]
(R2)–(E1)	[3], [19], [24], [35], [39], [43], [44]	[22], [36]
(R2)–(E2)	[13], [15], [27], [39]	[1], [45], [46]
(R3)–(E1)	[16], [34], [40]	[4], [7], [37], [38]
(R3)–(E2)	[14], [47]	[18], [48], [49]

field is very broad and can be roughly categorized along the following directions:

- 1) (D) type of dynamics considered: deterministic/nondeterministic (D1), versus, stochastic (D2);
- 2) (R) type of reduction obtained: reduction of a finite states model to a smaller finite states model (R1), versus, reduction of a continuous/hybrid (infinite states) model to a finite states model (R2), versus, reduction of a continuous/hybrid (infinite states) model to a smaller (with lower dimensional state space) continuous/hybrid (infinite states) model (R3);
- 3) (E) type of equivalence notions employed: exact simulation/bisimulation/trace equivalence notions (E1), versus, approximate simulation/bisimulation/trace equivalence notions (E2).

A (nonexhaustive) list of literature relevant in this research topic is reported in Table I. The present paper is along the research line (D2)–(R3)–(E1) and aims at extending the theory of bisimulation equivalence given for nondeterministic control systems in [40] to a stochastic setting. As discussed later on in this paper, the proposed notions have been inspired by the corresponding notions given in the finite systems domain (D2)–(R1)–(E1). We briefly recall that within (D2)–(R1)–(E1), bisimulation equivalence for probabilistic chains has been introduced in [21]; a generalization of this notion to labeled Markov processes has been studied in [12], and to interactive Markov chains, mixing transitions due to interaction with spontaneous probabilistic transitions, in [17]. Within the research line (D2)–(R3)–(E1) where the present paper is placed, a notion of bisimulation for general stochastic hybrid systems (GSHSs) (with no inputs and outputs) has been proposed in [7] and for communicating piecewise deterministic Markov processes (CPDMPs) in [37] and

[38]. However, given the generality of the models considered in [7], [37], and [38], checkable conditions for verifying bisimulation equivalence are difficult to find. For this reason in this paper, we consider a simpler class of stochastic control systems and propose equivalence notions that can be effectively checked.

We consider the class of discrete-time stochastic linear control systems with possibly degenerate normally distributed disturbances and propose the notion of equivalence via stochastic bisimulation. Comparisons of this notion with the ones proposed in [7], [12], [21], [37], and [38] are discussed in this paper. Necessary and sufficient conditions to check this notion in terms of geometric control theory are derived and model reduction discussed. The concepts of stochastic reachability and safety (see, e.g., [2], [6], [28]) are related to the notion of stochastic bisimulation. As discussed in this paper, the proposed notion of equivalence via stochastic bisimulation preserves stochastic reachability and finite horizon stochastic safety properties of the systems involved. This feature extends well-known facts for (finite states) concurrent processes (see, e.g., [9]) to stochastic systems with an infinite number of states. A discussion on the use of stochastic bisimulation for control design is given, and an application to optimal control problems with stochastic reachability specifications is presented.

A characterization of the notion of equivalence of stochastic external behavior that follows the approach taken in this paper can be found in [30] that also includes some technical proofs of the present paper. The problem addressed in this paper has been recently investigated in the continuous-time domain in [31]; see also [29]. A preliminary version of this paper appeared in the conference publication [32]. The present paper extends the work [32] by introducing novel results on model reduction and finite horizon stochastic safety problems, a discussion on control design via bisimulation and by presenting an application to optimal control for stochastic reachability problems. Finally, it also includes proofs of the results proposed.

This paper is organized as follows. In Section II, we introduce the notation employed and recall preliminary definitions. In Section III, we present the notion of equivalence via stochastic bisimulation; connections with stochastic reachability properties are also discussed. In Section IV, we provide geometric conditions for checking the proposed notion; proofs of some technical results are reported in the Appendix; connections with finite horizon stochastic safety properties are also established. Model reduction is discussed in Section V. In Section VI, we discuss the use of stochastic bisimulation for control design and present an application to optimal control problems for stochastic reachability. In Section VII, we discuss connections with notions of bisimulation for probabilistic chains and labeled Markov processes. Finally, Section VIII offers some concluding remarks and outlook.

II. NOTATION AND PRELIMINARY DEFINITIONS

Given a pair of sets S_1 and S_2 and a relation $\mathcal{R} \subseteq S_1 \times S_2$, we define for any sets $X_1 \subseteq S_1$ and $X_2 \subseteq S_2$, $\mathcal{R}(X_1) = \{x_2 \in S_2 \mid \exists x_1 \in X_1 \text{ s.t. } (x_1, x_2) \in \mathcal{R}\}$ and $\mathcal{R}^{-1}(X_2) = \{x_1 \in S_1 \mid \exists x_2 \in X_2 \text{ s.t. } (x_1, x_2) \in \mathcal{R}\}$. Relation $\mathcal{R}$ is total if $\mathcal{R}(S_1) =$

S_2 and $\mathcal{R}^{-1}(S_2) = S_1$. The standard symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the sets of nonnegative integer, real, and positive real numbers, respectively. Given a vector $x \in \mathbb{R}^n$, the symbol $x[i]$ denotes the i th component of x . Given a matrix $M \in \mathbb{R}^{m \times n}$, the symbols M^T , $\text{rank}(M)$, $\text{Im}(M)$, and $\ker(M)$ denote the transpose, the rank, the image, and the kernel of M , respectively. If M is square, $\det(M)$ denotes the determinant of M . Given a subset X of $\mathbb{R}^n$, we denote by MX the image of X through M , i.e., the set $\{y \in \mathbb{R}^m \mid \exists x \in X \text{ s.t. } y = Mx\}$. Given a subset Y of $\mathbb{R}^m$, we denote by $M^{-1}Y$ the inverse image of Y through M , i.e., the set $\{x \in \mathbb{R}^n \mid \exists y \in Y \text{ s.t. } y = Mx\}$. The symbols I_n and $0_{n \times m}$ denote the (n, n) -identity matrix and the (n, m) -null matrix, respectively; the symbol 0_n denotes the null vector in $\mathbb{R}^n$. Given a collection of square matrices $M_1, M_2, \dots, M_N$, we denote by $\text{diag}(M_1, M_2, \dots, M_N)$ the block-diagonal matrix with block-entries M_i . The symbol $\oplus$ denotes the direct sum operator between subspaces. Given $f_i : \mathbb{N} \rightarrow \mathbb{R}^n$, $i = 1, 2$, we write $f_1 = f_2$ instead of $f_1(t) = f_2(t)$ for all $t \in \mathbb{N}$ and also $f = 0$ instead of $f(t) = 0$ for all $t \in \mathbb{N}$.

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. $\mathbf{P}(S_1|S_2)$ denotes the conditional probability of event S_1 given event S_2 . Given a random variable $x : \Omega \rightarrow \mathbb{R}^n$ and a measurable set $X \subseteq \mathbb{R}^n$, we use standard shorthand notation $\mathbf{P}(x \in X)$ instead of $\mathbf{P}(\{\omega \in \Omega : x(\omega) \in X\})$; we denote by $\text{supp}(x)$ the support of x ; we recall that

$$\text{supp}(x) = \{z \in \mathbb{R}^n \mid \mathbf{P}(x \in \mathcal{B}_\rho(z)) > 0, \text{ for any } \rho \in \mathbb{R}^+\}$$

where $\mathcal{B}_\rho(z) = \{z \in \mathbb{R}^n \mid \|z\| < \rho\}$. In this paper, we consider random variables whose support is a manifold. Random variable x is degenerate if $\dim(\text{supp}(x)) < n$ and nondegenerate, otherwise. Of course

$$\mathbf{P}(x \in X) = \mathbf{P}(x \in X \cap \text{supp}(x)). \quad (1)$$

In general, the random variables considered in this paper are degenerate and, therefore, do not admit a probability density function. $E(x)$ denotes the expected value of the random variable x , and $\text{cov}(x, y)$ denotes the covariance between two random vectors x and y , i.e., $\text{cov}(x, y) = E[(x - E(x))(y - E(y))^T]$. The following standard definition will be used in this paper.

Definition 2.1: Two stochastic processes $x_1 : \mathbb{N} \times \Omega \rightarrow \mathbb{R}^n$ and $x_2 : \mathbb{N} \times \Omega \rightarrow \mathbb{R}^n$ are stochastically equivalent, denoted $x_1 \sim x_2$, if the probability distributions of the two vectors $(x_1(t_1), x_1(t_2), \dots, x_1(t_k))$ and $(x_2(t_1), x_2(t_2), \dots, x_2(t_k))$ are equal for any choice of $k \in \mathbb{N}$ and any choice of $\{t_1, t_2, \dots, t_k\} \subset \mathbb{N}$.

The standard notation $x \sim \mathcal{N}(\mu, \Psi)$ indicates that x is a random variable with normal distribution with mean vector μ and covariance matrix Ψ ; we recall that Ψ is symmetric and positive semidefinite, $\text{supp}(x) = \mu + \text{Im}(\Psi)$ and x is degenerate if $\det(\Psi) = 0$ and nondegenerate, otherwise. Moreover, we recall that if $x \sim \mathcal{N}(\mu, \Psi)$, then $y = \alpha x + \beta \sim \mathcal{N}(\beta + \alpha\mu, \alpha\Psi\alpha^T)$.

III. BISIMULATION EQUIVALENCE

In this section, we propose the notion of equivalence via stochastic bisimulation for a pair of discrete-time stochastic

linear control systems Σ_1 and Σ_2 described, for $t \in \mathbb{N}$, by

$$\Sigma_i : \begin{cases} x_i(t+1) = A_i x_i(t) + B_i u_i(t) + G_i w_i(t), \\ y_i(t) = C_i x_i(t) + \nu_i(t), \\ x_i(t) \in \mathbb{R}^{n_i}, u_i(t) \in \mathbb{R}^m, w_i(t) \in \mathbb{R}^{l_i}, \\ y_i(t) \in \mathbb{R}^p, \nu_i(t) \in \mathbb{R}^p, \end{cases} \quad (2)$$

where $x_i(t)$ is the state, $u_i(t)$ is the deterministic control input, $y_i(t)$ is the output, and $w_i(t)$ and $\nu_i(t)$ are random disturbances, at time $t \in \mathbb{N}$. We assume that $\nu_i(t) \sim \mathcal{N}(0, \Psi_i)$, with $\Psi_i \in \mathbb{R}^{p \times p}$, and $w_i(t) \sim \mathcal{N}(\mu_i, W_i)$ with $\mu_i \in \mathbb{R}^{l_i}$ and $W_i \in \mathbb{R}^{l_i \times l_i}$. We also assume that both sequences $w_i(t)$ and $\nu_i(t)$ are white and mutually independent. Without loss of generality, we assume in the following that $W_i = I_{l_i}$, so that the resulting random vector $v_i = G_i w_i$ is $v_i(t) \sim \mathcal{N}(G_i \mu_i, G_i G_i^T)$. Note that $v_i(t)$ is degenerate if and only if $\text{rank}(G_i) < n_i$.

Remark 3.1: For notational simplicity, we introduce notions and results for the case of control inputs u_i that are deterministic; extensions to the case of white Gaussian control processes are discussed in Remarks 3.8, 4.8, and 5.9. Moreover, stochastic control design via bisimulation is discussed in Section VI.

In the following, the boldface symbols $\mathbf{u}_i$, $\mathbf{w}_i$, and $\boldsymbol{\nu}_i$ will be used to denote the whole sequences of deterministic inputs $u_i(t)$ and random noises $w_i(t)$ and $\nu_i(t)$, $t \geq 0$, i.e., $\mathbf{u}_i : \mathbb{N} \rightarrow \mathbb{R}^m$, $\mathbf{w}_i : \mathbb{N} \times \Omega \rightarrow \mathbb{R}^{l_i}$, and $\boldsymbol{\nu}_i : \mathbb{N} \times \Omega \rightarrow \mathbb{R}^p$. The state and output values of Σ_i at times $t \in \mathbb{N}$ are computed as

$$\begin{aligned} \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \mathbf{w}_i) &= A_i^t x_i^0 + \sum_{\tau=0}^{t-1} A_i^{t-1-\tau} (B_i u_i(\tau) + G_i w_i(\tau)) \\ \mathbf{y}_i(t, x_i^0, \mathbf{u}_i, \mathbf{w}_i, \boldsymbol{\nu}_i) &= C_i \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \mathbf{w}_i) + \nu_i(t) \\ &= C_i A_i^t x_i^0 + \sum_{\tau=0}^{t-1} C_i A_i^{t-1-\tau} (B_i u_i(\tau) + G_i w_i(\tau)) + \nu_i(t). \end{aligned}$$

For a given initial condition $x_i^0 \in \mathbb{R}^{n_i}$ and deterministic input $\mathbf{u}_i$, let us denote by $\mathbf{x}_i|_{x_i^0, \mathbf{u}_i} : \mathbb{N} \times \Omega \rightarrow \mathbb{R}^{n_i}$ and $\mathbf{y}_i|_{x_i^0, \mathbf{u}_i} : \mathbb{N} \times \Omega \rightarrow \mathbb{R}^p$ the state and output stochastic processes generated by system Σ_i driven by the stochastic sequences $\mathbf{w}_i$ and $\boldsymbol{\nu}_i$:

$$\begin{aligned} \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) &= \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \mathbf{w}_i) \\ \mathbf{y}_i|_{x_i^0, \mathbf{u}_i}(t) &= \mathbf{y}_i(t, x_i^0, \mathbf{u}_i, \mathbf{w}_i, \boldsymbol{\nu}_i). \end{aligned}$$

For later purposes, let us define the sequences $\boldsymbol{\mu}_i = E(\mathbf{w}_i)$ of expected values of the random sequences $\mathbf{w}_i$, $i = 1, 2$, and the sequence $\tilde{\mathbf{w}}_i = \mathbf{w}_i - \boldsymbol{\mu}_i$, that is the centered (i.e., zero mean) version of the disturbance $\mathbf{w}_i$. Then, by linearity

$$\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) = \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) + \mathbf{x}_i(t, 0, \mathbf{0}, \tilde{\mathbf{w}}_i) \quad (3)$$

$$\mathbf{y}_i|_{x_i^0, \mathbf{u}_i}(t) = \mathbf{y}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i, \mathbf{0}) + \mathbf{y}_i(t, 0, \mathbf{0}, \tilde{\mathbf{w}}_i, \boldsymbol{\nu}_i). \quad (4)$$

If the initial state x_i^0 in Σ_i is deterministic, then the first terms of the right-hand sides of (3) and (4) are the expected values of the state and output processes at time t . However, for the sake of generality, we look at these terms as expectations conditional to x_i^0 , considered as a random variable independent of both $\mathbf{w}_i$

and $\boldsymbol{\nu}_i$. Explicit expressions of these terms are

$$\begin{aligned} \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) &= E\{\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) | x_i^0\} \\ &= A_i^t x_i^0 + \sum_{\tau=0}^{t-1} A_i^{t-1-\tau} (B_i u_i(\tau) + G_i \boldsymbol{\mu}_i) \\ \mathbf{y}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i, \mathbf{0}) &= E\{\mathbf{y}_i|_{x_i^0, \mathbf{u}_i}(t) | x_i^0\} = C_i \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) \\ &= C_i A_i^t x_i^0 + \sum_{\tau=0}^{t-1} C_i A_i^{t-1-\tau} (B_i u_i(\tau) + G_i \boldsymbol{\mu}_i). \end{aligned}$$

By defining, for any time $t \in \mathbb{N}$ and $i = 1, 2$, the vectors

$$\mathbf{w}_i|_{0:t} = \begin{bmatrix} \mathbf{w}_i(t) \\ \vdots \\ \mathbf{w}_i(1) \\ \mathbf{w}_i(0) \end{bmatrix}, \quad \boldsymbol{\mu}_i|_{0:t} = \begin{bmatrix} \boldsymbol{\mu}_i \\ \vdots \\ \boldsymbol{\mu}_i(1) \\ \boldsymbol{\mu}_i(0) \end{bmatrix}, \quad \mathbf{u}_i|_{0:t} = \begin{bmatrix} \mathbf{u}_i(t) \\ \vdots \\ \mathbf{u}_i(1) \\ \mathbf{u}_i(0) \end{bmatrix} \quad (5)$$

where $\boldsymbol{\mu}_i|_{0:t} = E\{\mathbf{w}_i|_{0:t}\}$, the zero mean random vector

$$\tilde{\mathbf{w}}_i|_{0:t} = \mathbf{w}_i|_{0:t} - \boldsymbol{\mu}_i|_{0:t} \quad (6)$$

and the matrices

$$\begin{aligned} \text{Reach}_t(A_i, G_i) &= [G_i \ A_i G_i \ \dots \ A_i^{t-1} G_i] \\ \text{Obs}_t(A_i, C_i) &= (\text{Reach}_t(A_i^T, C_i^T))^T \end{aligned}$$

(3) and (4) can be rewritten as

$$\begin{aligned} \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) &= \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) \\ &\quad + \text{Reach}_t(A_i, G_i) \tilde{\mathbf{w}}_i|_{0:t-1} \\ &= A_i^t x_i^0 + \text{Reach}_t(A_i, B_i) \mathbf{u}_i|_{0:t-1} \\ &\quad + \text{Reach}_t(A_i, G_i) \boldsymbol{\mu}_i|_{0:t-1} \\ &\quad + \text{Reach}_t(A_i, G_i) \tilde{\mathbf{w}}_i|_{0:t-1} \\ \mathbf{y}_i|_{x_i^0, \mathbf{u}_i}(t) &= \mathbf{y}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i, \boldsymbol{\nu}_i) \\ &\quad + C_i \text{Reach}_t(A_i, G_i) \tilde{\mathbf{w}}_i|_{0:t-1} + \nu_i(t). \end{aligned} \quad (7)$$

From (7), it is easily understood that the support of the random variable $\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t)$ conditional to x_i^0 is

$$\text{supp}(\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t)) = \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) + \text{Im}(\text{Reach}_t(A_i, G_i)).$$

It is clear that the conditional random variables $\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t)$ and $\mathbf{y}_i|_{x_i^0, \mathbf{u}_i}(t)$ can be degenerate or not, depending on the rank of matrices $\text{Reach}_t(A_i, G_i)$, $C_i \text{Reach}_t(A_i, G_i)$, and Ψ_i . We set $\text{Reach}(A_i, G_i) = \text{Reach}_{n_i}(A_i, G_i)$ (by the Cayley–Hamilton theorem, $\text{Im}(\text{Reach}_t(A_i, G_i)) = \text{Im}(\text{Reach}(A_i, G_i))$ for any time $t \geq n_i$). Using (7) and (8), the conditional covariances are computed as

$$\begin{aligned} \text{cov}(\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t), \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(\tau)) &= \sum_{h=0}^{\tau-1} A_i^{t-\tau+h} G_i G_i^T (A_i^h)^T \\ \text{cov}(\mathbf{y}_i|_{x_i^0, \mathbf{u}_i}(t), \mathbf{y}_i|_{x_i^0, \mathbf{u}_i}(\tau)) &= \\ \Psi_i + \sum_{h=0}^{\tau-1} C_i A_i^{t-\tau+h} G_i G_i^T (A_i^h)^T C_i^T \end{aligned} \quad (9)$$

where $t \geq \tau$. We also need to recall the notion of linear equivalence of stochastic linear control systems from, e.g., [20].

Definition 3.2: Two stochastic linear control systems Σ_1 and Σ_2 as in (2) are linearly equivalent, denoted

$$\Sigma_1 \cong_l \Sigma_2$$

if $\Psi_1 = \Psi_2$, $n_1 = n_2$, and there exists an invertible matrix $\mathbb{T} \in \mathbb{R}^{n_1 \times n_1}$, called transformation matrix, such that

$$A_2 = \mathbb{T}^{-1} A_1 \mathbb{T}, \quad B_2 = \mathbb{T}^{-1} B_1, \quad C_2 = C_1 \mathbb{T}, \quad G_2 = \mathbb{T}^{-1} G_1.$$

From the definition above, Σ_1 and Σ_2 are linearly equivalent if the dynamics of Σ_2 can be obtained from those of Σ_1 via a linear and nonsingular transformation $x_2 = \mathbb{T}x_1$ of the state space of Σ_1 (and vice versa). The notion of linear equivalence is an equivalence relation on the class of linear systems. We now proceed further and propose a notion of stochastic bisimulation equivalence. We start with the case of linear systems with nondegenerate disturbances.

Definition 3.3: Given a pair of stochastic control systems Σ_1 and Σ_2 , as in (2) with $\text{rank}(G_i) = n_i$, a subspace $\mathcal{R} \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a stochastic bisimulation relation between Σ_1 and Σ_2 if for any pair $(x_1^0, x_2^0) \in \mathcal{R}$ and any input $\mathbf{u}$, the following conditions hold for all times $t \in \mathbb{N}$.

i) For any measurable set $X_1 \subseteq \mathcal{R}^{-1}(\mathbb{R}^{n_2})$

$$\mathbf{P}(x_1|_{x_1^0, \mathbf{u}}(t) \in X_1 | x_1^0) = \mathbf{P}(x_2|_{x_2^0, \mathbf{u}}(t) \in \mathcal{R}(X_1) | x_2^0).$$

ii) For any measurable set $X_2 \subseteq \mathcal{R}(\mathbb{R}^{n_1})$

$$\mathbf{P}(x_2|_{x_2^0, \mathbf{u}}(t) \in X_2 | x_2^0) = \mathbf{P}(x_1|_{x_1^0, \mathbf{u}}(t) \in \mathcal{R}^{-1}(X_2) | x_1^0).$$

iii) $\mathbf{y}_1|_{x_1^0, \mathbf{u}} \sim \mathbf{y}_2|_{x_2^0, \mathbf{u}}$.

Systems Σ_1 and Σ_2 are equivalent via stochastic bisimulation, if there exists a total stochastic bisimulation relation between them.

Note that in conditions (i) and (ii) of Definition 3.3, we consider measurable sets $X_1 \subseteq \mathcal{R}^{-1}(\mathbb{R}^{n_2})$ and $X_2 \subseteq \mathcal{R}(\mathbb{R}^{n_1})$ rather than all measurable sets $X_1 \subseteq \mathbb{R}^{n_1}$ and $X_2 \subseteq \mathbb{R}^{n_2}$. This choice is motivated by the fact that since relation $\mathcal{R}$ may be not total, sets $\mathcal{R}(X_1)$ and $\mathcal{R}^{-1}(X_2)$ may be not defined for some sets $X_1 \subseteq \mathbb{R}^{n_1}$ and $X_2 \subseteq \mathbb{R}^{n_2}$, while they are defined for all sets $X_1 \subseteq \mathcal{R}^{-1}(\mathbb{R}^{n_2})$ and $X_2 \subseteq \mathcal{R}(\mathbb{R}^{n_1})$. When $\mathcal{R}$ is total, all measurable sets $X_1 \subseteq \mathbb{R}^{n_1}$ and $X_2 \subseteq \mathbb{R}^{n_2}$ are clearly considered. Definition 3.3 has been inspired by similar notions given for probabilistic chains and Markov processes; see, e.g., [12], [17], and [21]. A detailed discussion in this regard is reported in Section VII.

Remark 3.4: As stressed in the introduction, this paper is within the research line (D2)–(R3)–(E1), where notions of stochastic bisimulation have been also proposed for GSHSs with no inputs and outputs in [7] and for CPDMPs in [37] and [38]. A comparison of the proposed Definition 3.3 with the ones given in [7] and [37], [38] follows. The notion proposed in [7] generalizes the one of stochastic bisimulation given for labeled Markov processes with countable sets of states in [4] to the GSHS. Since the definition given in [4] generalizes the one given in [21] for probabilistic chains, then both Definition 3.3

Fig. 1. Illustration of the sets involved in Example 3.5.

and the one given in [7] are based on the same ideas as in the seminal work [21]. Regarding the comparison with the definitions of stochastic bisimulation given in [37] and [38] for CPDMP, we recall that the semantic of the CPDMP is characterized by no stochasticity in the continuous-state flow; stochasticity only appears in the discrete-state dynamics, via spontaneous Poisson-type transitions and in the reset of both continuous and discrete variables. Since the systems in (2) present stochasticity in the continuous-state flow and have no discrete-state dynamics, Definition 3.3 and the one given in [37] and [38] are not comparable. However, we mention that the definitions proposed in [37] and [38] are inspired by the one given in [17] for interactive Markov chains, which combines the classical definition of bisimulation for concurrent (nonstochastic) processes (see, e.g., [23], [25]) with the one given in [21] for probabilistic chains. Hence, also in this case, the seminal work [21] is a common denominator in inspiring Definition 3.3 and the one given in [37] and [38].

As already pointed out at the beginning of this section, in this paper, we consider linear systems with possibly degenerate disturbance distribution. The following example shows that Definition 3.3 is not appropriate to deal with linear systems with disturbances with degenerate distributions.

Example 3.5: Consider a pair of stochastic control systems Σ_1 and Σ_2 as in (2), where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_1 = G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$A_2 = B_2 = G_2 = C_2 = 1$, and $w_i(t) \sim \mathcal{N}(0, 1)$, without output noise ν_i . The dynamics of Σ_1 and Σ_2 suggest that $\Sigma_1 \cong_b \Sigma_2$ with stochastic bisimulation relation $\mathcal{R}$ defined by $(x_1^0, x_2^0) \in \mathcal{R}$ if and only if $x_1^0[1] = x_2^0$; indeed, the dynamics of x_2 coincide with the dynamics of $x_1[1]$, $y_2(t) = x_2(t)$, and $y_1(t) = x_1[1](t)$. We now apply Definition 3.3 only at time $t = 1$. We consider $(x_1^0, x_2^0) = (0, 0) \in \mathcal{R}$, $\mathbf{u} = \mathbf{0}$, and the two measurable sets $X'_1, X''_1 \subseteq \mathbb{R}^{n_1}$ depicted in Fig. 1. We first note that $\mathcal{R}(X'_1) = \mathcal{R}(X''_1) = X'_2$, with $X'_2 \subseteq \mathbb{R}^{n_2}$, as depicted in Fig. 1. Hence, according to condition (i) of Definition 3.3, a necessary condition for $\Sigma_1 \cong_b \Sigma_2$ is that

$$\mathbf{P}(x_1|_{0,0}(1) \in X'_1 | 0) = \mathbf{P}(x_1|_{0,0}(1) \in X''_1 | 0). \quad (10)$$

However, since $\text{supp}(x_1|_{0,0}(1)) = \text{supp}(G_1 w_1(0)) = \text{Im}(G_1)$, by (1), we get $\mathbf{P}(x_1|_{0,0}(1) \in X'_1 | 0) = 0$ and $\mathbf{P}(x_1|_{0,0}(1) \in$

$X_1''(0) \neq 0$, thus contradicting (10). Hence, Σ_1 and Σ_2 are not equivalent via stochastic bisimulation according to Definition 3.3.

The example above motivates us to extend Definition 3.3 to linear systems with possibly degenerate disturbances, as follows.

Definition 3.6: Given a pair of stochastic control systems Σ_1 and Σ_2 , as in (2), a subspace $\mathcal{R} \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a stochastic bisimulation relation between Σ_1 and Σ_2 if for any pair $(x_1^0, x_2^0) \in \mathcal{R}$ and any input $\mathbf{u}$, the following conditions hold for all times $t \in \mathbb{N}$:

i) For any measurable set $X_1 \subseteq \mathcal{R}^{-1}(\mathbb{R}^{n_2})$

$$\begin{aligned} \mathbf{P}(x_1|_{x_1^0, \mathbf{u}}(t) \in X_1 | x_1^0) = \\ \mathbf{P}(x_2|_{x_2^0, \mathbf{u}}(t) \in \mathcal{R}(X_1 \cap \text{supp}(x_1|_{x_1^0, \mathbf{u}}(t))) | x_2^0). \end{aligned}$$

ii) For any measurable set $X_2 \subseteq \mathcal{R}(\mathbb{R}^{n_1})$

$$\begin{aligned} \mathbf{P}(x_2|_{x_2^0, \mathbf{u}}(t) \in X_2 | x_2^0) = \\ \mathbf{P}(x_1|_{x_1^0, \mathbf{u}}(t) \in \mathcal{R}^{-1}(X_2 \cap \text{supp}(x_2|_{x_2^0, \mathbf{u}}(t))) | x_1^0). \end{aligned}$$

iii) $\mathbf{y}_1|_{x_1^0, \mathbf{u}} \sim \mathbf{y}_2|_{x_2^0, \mathbf{u}}$.

Systems Σ_1 and Σ_2 are equivalent via stochastic bisimulation, denoted

$$\Sigma_1 \cong_b \Sigma_2$$

if there exists a total stochastic bisimulation relation between them.

Remark 3.7: Conditions (i) and (ii) of Definition 3.6, more complicated than the corresponding ones in Definition 3.3, are introduced to guarantee the consistency of the bisimulation notion when the state trajectories starting from a given initial state live in affine subspaces of the state space, as it happens when the state noise distribution is degenerate (see the geometric characterization in (19) of the supports $\text{supp}(x_i|_{x_i^0, \mathbf{u}}(t))$, $i = 1, 2$, appearing in conditions (i) and (ii) of Definition 3.6).

Example 3.5: (Continued) When conditions (i) and (ii) of Definition 3.3 are replaced by conditions (i) and (ii) of Definition 3.6, one gets $\mathcal{R}(X_1' \cap \text{supp}(x_1(1))) = \mathcal{R}(\emptyset) = \emptyset$ and $\mathcal{R}(X_1'' \cap \text{supp}(x_1(1))) = X_2'' \neq \emptyset$. Therefore, condition (i) of Definition 3.6 correctly distinguishes between sets X_1' and X_1'' , whereas condition (i) of Definition 3.3 does not. A straightforward computation reveals indeed that systems Σ_1 and Σ_2 are equivalent via stochastic bisimulation according to Definition 3.6, while we showed they are not according to Definition 3.3.

In the following, if not stated explicitly, when referring to equivalence via stochastic bisimulation, we consider Definition 3.6.

Remark 3.8: It is easy to see that Definition 3.6 applies also to systems Σ_i in (2), where deterministic control inputs are replaced by stochastic control processes (neither necessarily Gaussian nor white) possibly correlated with the state stochastic processes generated by Σ_i .

The notion of equivalence via stochastic bisimulation is an equivalence relation on the class of linear systems. By comparing Definitions 3.2 and 3.6, we obtain the following.

Proposition 3.9: If $\Sigma_1 \cong_l \Sigma_2$, then $\Sigma_1 \cong_b \Sigma_2$.

Remark 3.10: While it is easy to see that converse implication in the result above is not true in general, we will show in the next section that this is the case when state noises of Σ_i are nondegenerate.

Classical notions of bisimulation equivalences given for deterministic, nondeterministic, and stochastic discrete (concurrent) processes preserve reachability properties of equivalent states, together with, e.g., linear temporal logic properties; see, e.g., [9], [17]. These notions only involve next states of equivalent states, rather than all states involved in runs originating from equivalent states. Definition 3.6 clearly preserves reachability properties of states related by relation $\mathcal{R}$. In contrast to the case of discrete processes, however, it requires properties (i)–(iii) to hold for all times $t \in \mathbb{N}$, rather than only for time $t = 1$. We now show by a simple example that if the notion of stochastic bisimulation equivalence is defined in one step (i.e., only for $t = 1$), stochastic reachability properties of states related by $\mathcal{R}$ may be not preserved.

Example 3.11: Consider a pair of stochastic control systems Σ_1 and Σ_2 as in (2), where

$$A_1 = \begin{bmatrix} b & 1 \\ 0 & a \end{bmatrix}, B_1 = 0_{2 \times 1}, G_1 = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}, C_1 = [0 \quad 1]$$

$$A_2 = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}, B_2 = 0_{2 \times 1}, G_2 = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}, C_2 = [0 \quad 1]$$

without output noise ν_i , with $a, b, \sigma \neq 0$. We first show that properties (i)–(iii) of Definition 3.6 are satisfied for time $t = 1$, that is to say that systems Σ_1 and Σ_2 are equivalent via stochastic bisimulation “in one step”, as typically required for discrete processes. Consider the relation $\mathcal{R}$ defined by $(x_1^0, x_2^0) \in \mathcal{R}$ if and only if $x_1^0[2] = x_2^0[2]$ (recall that the argument within square brackets denotes the component of a vector). We start with condition (i). Consider any $(x_1^0, x_2^0) \in \mathcal{R}$, any input $\mathbf{u}$, any measurable set $X_1 \subseteq \mathbb{R}^{n_1}$, and define $X_{1,2} = \{x_1[2] \in \mathbb{R} | \exists x_1[1] \in \mathbb{R} \text{ s.t. } x_1 \in X_1 \cap \text{supp}(x_1|_{x_1^0, \mathbf{u}}(1))\}$. Then, one gets

$$\begin{aligned} \mathbf{P}(x_1|_{x_1^0, \mathbf{u}}(1) \in X_1 | x_1^0) &= \mathbf{P}(ax_1^0[2] + \sigma w_1(0) \in X_{1,2}) \\ &= \mathbf{P}(ax_2^0[2] + \sigma w_2(0) \in X_{1,2}) \\ &= \mathbf{P}(x_2|_{x_2^0, \mathbf{u}}(1) \in \mathcal{R}(X_1 \cap \text{supp}(x_1|_{x_1^0, \mathbf{u}}(1))) | x_2^0) \end{aligned}$$

which is condition (i) of Definition 3.6 for $t = 1$. Condition (ii) can be shown similarly and condition (iii) is trivially satisfied because $C_i x_i|_{x_i^0, \mathbf{u}}(1) = ax_i^0[2] + \sigma w_i(0)$ in this example. We now show that $\mathcal{R}$ does not satisfy those conditions for time $t = 2$. We consider $(x_i^0, \mathbf{u}_i) = (0, \mathbf{0})$ and obtain

$$x_1|_{0, \mathbf{0}}(2) = \begin{bmatrix} 0 & \sigma \\ \sigma & a\sigma \end{bmatrix} \begin{bmatrix} w_1(1) \\ w_1(0) \end{bmatrix}$$

$$x_2|_{0, \mathbf{0}}(2) = \begin{bmatrix} 0 & 0 \\ \sigma & a\sigma \end{bmatrix} \begin{bmatrix} w_2(1) \\ w_2(0) \end{bmatrix}.$$

Since $\mathbf{x}_1|_{0,0}(2)$ is nondegenerate, there exists a pair of sets in form of $X_1 = Z_1 \times Z_2$ and $X'_1 = Z'_1 \times Z_2$ with $Z_1 \subset Z'_1$ such that

$$\mathbf{P}(\mathbf{x}_1|_{0,0}(2) \in X_1|0) < \mathbf{P}(\mathbf{x}_1|_{0,0}(2) \in X'_1|0). \quad (11)$$

Considering that $\text{supp}(\mathbf{x}_1|_{0,0}(2)) = \mathbb{R}^2$, condition (i) of Definition 3.6 for set X_1 rewrites as

$$\mathbf{P}(\mathbf{x}_1|_{0,0}(2) \in Z_1 \times Z_2|0) = \mathbf{P}(\mathbf{x}_2|_{0,0}(2) \in \mathcal{R}(Z_1 \times Z_2)|0).$$

Note that $\mathcal{R}(Z_1 \times Z_2) = \mathbb{R} \times Z_2$, because $(x_1, x_2) \in \mathcal{R}$ if $x_1[2] = x_2[2]$. Thus

$$\mathbf{P}(\mathbf{x}_1|_{0,0}(2) \in Z_1 \times Z_2|0) = \mathbf{P}(\mathbf{x}_2|_{0,0}(2) \in \mathbb{R} \times Z_2|0).$$

Analogously, condition (i) of Definition 3.6 for set $X'_1 = Z'_1 \times Z_2$ rewrites as

$$\mathbf{P}(\mathbf{x}_1|_{0,0}(2) \in Z'_1 \times Z_2|0) = \mathbf{P}(\mathbf{x}_2|_{0,0}(2) \in \mathbb{R} \times Z_2|0).$$

From these, it follows that

$$\mathbf{P}(\mathbf{x}_1|_{0,0}(2) \in Z_1 \times Z_2|0) = \mathbf{P}(\mathbf{x}_1|_{0,0}(2) \in Z'_1 \times Z_2|0)$$

which contradicts the inequality in (11). Hence, condition (i) of Definition 3.6 is not satisfied for time $t = 2$, although it is satisfied at $t = 1$.

The example above motivated us to propose a definition of stochastic bisimulation equivalence in t steps rather than in one step, as instead commonly done for discrete processes.

IV. GEOMETRIC CONDITIONS

In this section, we derive geometric conditions characterizing the equivalence notion proposed in Definition 3.6. Without loss of generality, we consider subspaces $\mathcal{R}$ in Definition 3.6 in the form of

$$\mathcal{R} = \ker([R_1 - R_2]) \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \quad (12)$$

where $R_i \in \mathbb{R}^{r \times n_i}$, $i = 1, 2$, so that

$$(x_1, x_2) \in \mathcal{R} \iff R_1 x_1 = R_2 x_2.$$

We first introduce some technical results that are instrumental to prove the main result of this section. For any subsets $X_1 \subseteq \mathbb{R}^{n_1}$ and $X_2 \subseteq \mathbb{R}^{n_2}$, we have

$$\mathcal{R}(X_1) = R_2^{-1}(R_1 X_1), \quad \mathcal{R}^{-1}(X_2) = R_1^{-1}(R_2 X_2) \quad (13)$$

as it easily follows from the following identities:

$$\begin{aligned} \mathcal{R}(X_1) &= \{x_2 \in \mathbb{R}^{n_2} \mid \exists x_1 \in X_1 \text{ s.t. } R_1 x_1 = R_2 x_2\} \\ &= \{x_2 \in \mathbb{R}^{n_2} \mid \exists y \in R_1 X_1 \text{ s.t. } y = R_2 x_2\} \\ &= R_2^{-1}(R_1 X_1) \end{aligned}$$

and similarly for $\mathcal{R}^{-1}(X_2)$. Note that for some nonempty $X_1 \subseteq \mathbb{R}^{n_1}$ (or $X_2 \subseteq \mathbb{R}^{n_2}$), the set $\mathcal{R}(X_1)$ (or $\mathcal{R}^{-1}(X_2)$) can be empty, unless $\mathcal{R}$ is a total relation.

Proposition 4.1: Relation $\mathcal{R}$ as in (12) is total if and only if $\text{Im}(R_1) = \text{Im}(R_2)$ or equivalently

$$\text{rank}(R_1) = \text{rank}(R_2) = \text{rank}([R_1 - R_2]). \quad (14)$$

Lemma 4.2: Let $\mathcal{R}$ be a relation as in (12) and consider any sets $X_i \subseteq \mathbb{R}^{n_i}$, $i = 1, 2$. Then, we have the following.

- i) If $\mathcal{R}$ is total, then $R_1 X_1 = R_2 \mathcal{R}(X_1)$.
- ii) If $\mathcal{R}$ is total, then $R_1 \mathcal{R}^{-1}(X_2) = R_2 X_2$.
- iii) $\mathcal{R}(X_1) = \mathcal{R}(X_1) + \ker(R_2)$.
- iv) $\mathcal{R}^{-1}(X_2) = \mathcal{R}^{-1}(X_2) + \ker(R_1)$.

Proof: See the Appendix. ■

For later purposes, we need to point out that $R_i \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}$, for $i = 1, 2$, are Gaussian processes whose conditional means at time $t \in \mathbb{N}$ are

$$\begin{aligned} E(R_i \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) | x_i^0) &= R_i \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) \\ &= R_i A_i^t x_i^0 + \sum_{h=0}^{t-1} R_i A_i^{t-1-h} (B_i u_i(h) + G_i \mu_i) \end{aligned} \quad (15)$$

and conditional covariances, for $t \geq \tau$,

$$\begin{aligned} \text{cov}(R_i \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t), R_i \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(\tau)) \\ = \sum_{h=0}^{\tau-1} R_i A_i^{t-\tau+h} G_i G_i^T (A_i^h)^T R_i^T. \end{aligned} \quad (16)$$

Lemma 4.3: Let $\mathcal{R} = \ker([R_1 - R_2]) \subseteq \mathbb{R}^{n_1+n_2}$ be a total stochastic bisimulation relation for two stochastic control systems Σ_1 and Σ_2 as in (2). Then, $\forall (x_1^0, x_2^0) \in \mathcal{R}$, and for any input $\mathbf{u}$, we have

$$R_1 \mathbf{x}_1|_{x_1^0, \mathbf{u}}(t) \sim R_2 \mathbf{x}_2|_{x_2^0, \mathbf{u}}(t) \quad \forall t \geq 0. \quad (17)$$

Proof: See the Appendix. ■

Recalling (7), it is useful to define the following zero mean processes, for $i = 1, 2$:

$$\begin{aligned} \xi_i(t) &= \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) - \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) \\ &= \text{Reach}_t(A_i, G_i) \tilde{\mathbf{w}}_i|_{0:t-1} \end{aligned} \quad (18)$$

which has the same covariances of $\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}$ reported in (9), i.e., for $t \geq \tau$

$$\text{cov}(\xi_i(t), \xi_i(\tau)) = \sum_{h=0}^{\tau-1} A_i^{t-\tau+h} G_i G_i^T (A_i^h)^T.$$

We also have, for any $x_i^0 \in \mathbb{R}^{n_i}$, $\mathbf{u}$, and $t \geq 0$, the following:

$$\begin{aligned} \text{supp}(\xi_i(t)) &= \text{Im}(\text{Reach}_t(A_i, G_i)) \\ \text{supp}(\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t)) &= \mathbf{x}_i(t, x_i^0, \mathbf{u}_i, \boldsymbol{\mu}_i) + \text{Im}(\text{Reach}_t(A_i, G_i)). \end{aligned} \quad (19)$$

Lemma 4.4: Consider a stochastic linear control system Σ_i as in (2) and a matrix $R_i \in \mathbb{R}^{r \times n_i}$. Let A_i, G_i , and R_i be such that

$$\text{Im}(\text{Reach}(A_i, G_i)) \cap \ker(R_i) = \{0\}. \quad (20)$$

Then

$$\mathbf{P}(R_i \mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) \in R_i \bar{X}_i | x_i^0) = \mathbf{P}(\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t) \in \bar{X}_i | x_i^0)$$

for any time $t \in \mathbb{N}$, initial condition $x_i^0 \in \mathbb{R}^{n_i}$, control input sequence $\mathbf{u}_i$, and any measurable set

$$\bar{X}_i \subseteq \text{supp}(\mathbf{x}_i|_{x_i^0, \mathbf{u}_i}(t)). \quad (21)$$

Proof: See the Appendix. ■

We now have all the ingredients to present the main result of this paper that provides geometric conditions for characterizing stochastic bisimulation relations.

Theorem 4.5: Consider two stochastic systems Σ_1 and Σ_2 , as in (2), such that $\Psi_1 = \Psi_2$ and a subspace total relation $\mathcal{R}$ as in (12). Then, $\mathcal{R}$ is a stochastic bisimulation relation between Σ_1 and Σ_2 if and only if the following conditions are satisfied:

$$h_0) \text{diag}(A_1, A_2) \mathcal{R} \subseteq \mathcal{R}$$

$$(\text{i.e., } R_1 x_1^0 = R_2 x_2^0 \Rightarrow R_1 A_1 x_1^0 = R_2 A_2 x_2^0)$$

$$h_1) \text{Im}(\text{col}(B_1, B_2)) \subseteq \mathcal{R} \text{ (i.e., } R_1 B_1 = R_2 B_2)$$

$$h_2) \text{Im}(\text{col}(G_1 \mu_1, G_2 \mu_2)) \subseteq \mathcal{R} \text{ (i.e., } R_1 G_1 \mu_1 = R_2 G_2 \mu_2)$$

$$h_3) R_1 G_1 G_1^T R_1^T = R_2 G_2 G_2^T R_2^T$$

$$h_4) \mathcal{R} \subseteq \ker([C_1 \ -C_2])$$

$$h_5) \text{Im}(\text{Reach}(A_i, G_i)) \cap \ker(R_i) = \{0\}, \ i = 1, 2.$$

Proof: (Sufficiency) First of all, note that the properties (h_0) , (h_1) and (h_2) imply the following:

$$h'_0) R_1 A_1^t x_1^0 = R_2 A_2^t x_2^0 \quad \forall t \in \mathbb{N}, \forall (x_1^0, x_2^0) \in \mathcal{R}$$

$$h'_1) R_1 A_1^t B_1 = R_2 A_2^t B_2 \quad \forall t \in \mathbb{N}$$

$$h'_2) R_1 A_1^t G_1 \mu_1 = R_2 A_2^t G_2 \mu_2 \quad \forall t \in \mathbb{N}.$$

Property (h_3) is equivalent to the existence of a matrix H such that $R_1 G_1 H = R_2 G_2$. Then, by the invariance property (h_0) , it follows that $R_1 A_1^k G_1 H = R_2 A_2^k G_2$, for any $t \in \mathbb{N}$, and therefore, for all $k, h \in \mathbb{N}$, we have

$$h'_3) R_1 A_1^k G_1 G_1^T (A_1^h)^T R_1^T = R_2 A_2^k G_2 G_2^T (A_2^h)^T R_2^T.$$

From (h'_0) – (h'_3) , it easily follows that the Gaussian processes $R_1 x_1|_{x_1^0, u_1}$ and $R_2 x_2|_{x_2^0, u_1}$, when $(x_1^0, x_2^0) \in \mathcal{R}$ and $u_1 = u_2$, have the same means (15) and covariances (16); therefore

$$R_1 x_1|_{x_1^0, u} \sim R_2 x_2|_{x_2^0, u} \quad \forall (x_1^0, x_2^0) \in \mathcal{R}. \quad (22)$$

For any measurable set $X_1 \subseteq \mathbb{R}^{n_1}$ and any $t \in \mathbb{N}$, define

$$\overline{X}_1 = X_1 \cap \text{supp}(x_1|_{x_1^0, u_1}(t))$$

$$X_2 = \mathcal{R}(\overline{X}_1).$$

By definition of sets X_1 and $\overline{X}_1$ and by (1), for all $t \in \mathbb{N}$, we get

$$\mathbf{P}(x_1|_{x_1^0, u_1}(t) \in X_1|x_1^0) = \mathbf{P}(x_1|_{x_1^0, u_1}(t) \in \overline{X}_1|x_1^0). \quad (23)$$

Since by construction, set $\overline{X}_1$ satisfies condition (21), by condition (h_5) in the statement and Lemma 4.4, for all $t \in \mathbb{N}$, we get

$$\mathbf{P}(x_1|_{x_1^0, u_1}(t) \in \overline{X}_1|x_1^0) = \mathbf{P}(R_1 x_1|_{x_1^0, u_1}(t) \in R_1 \overline{X}_1|x_1^0). \quad (24)$$

Since $\mathcal{R}$ is total, by Lemma 4.2(i), we get

$$R_1 \overline{X}_1 = R_2 X_2. \quad (25)$$

By combining (22) and (25), for all $t \in \mathbb{N}$, we get

$$\mathbf{P}(R_1 x_1|_{x_1^0, u_1}(t) \in R_1 \overline{X}_1|x_1^0) = \mathbf{P}(R_2 x_2|_{x_2^0, u_2}(t) \in R_2 X_2|x_2^0). \quad (26)$$

By definition of set X_2 and by Lemma 4.2(iii), for all $t \in \mathbb{N}$, we get

$$\begin{aligned} &\mathbf{P}(R_2 x_2|_{x_2^0, u_2}(t) \in R_2 X_2|x_2^0) \\ &= \mathbf{P}(x_2|_{x_2^0, u_2}(t) \in (X_2 + \ker(R_2))|x_2^0) \\ &= \mathbf{P}(x_2|_{x_2^0, u_2}(t) \in X_2|x_2^0). \end{aligned} \quad (27)$$

By combining the equalities in (23), (24), (26), and (27), we get condition (i) of Definition 3.6. Condition (ii) of Definition 3.6 can be shown by using a symmetric reasoning. In particular, while the proof of condition (i) makes use of Lemma 4.4, Lemma 4.2(i) and Lemma 4.2(iii), the proof of condition (ii) makes use of Lemma 4.4, Lemma 4.2(ii), and Lemma 4.2(iv). Regarding condition (iii) of Definition 3.6, note that condition (h_4) implies existence of a matrix S such that $S[R_1 - R_2] = [C_1 - C_2]$, i.e., $SR_1 = C_1$ and $SR_2 = C_2$. Thus, from condition (22), considering that $\Psi_1 = \Psi_2$ implies $\nu_1 \sim \nu_2$, by the independence of x_i and ν_i , for any $(x_1^0, x_2^0) \in \mathcal{R}$ and input u , we have

$$SR_1 x_1|_{x_1^0, u} + \nu_1 \sim SR_2 x_2|_{x_2^0, u} + \nu_2.$$

Since $y_i|_{x_i^0, u_i} = SR_i x_i|_{x_i^0, u_i} + \nu_i$, condition (iii) of Definition 3.6 is satisfied.

(Necessity) By Lemma 4.3, if $\mathcal{R}$ is a stochastic bisimulation relation, then for any $(x_1^0, x_2^0) \in \mathcal{R}$, and input u , we have $R_1 x_1|_{x_1^0, u}(t) \sim R_2 x_2|_{x_2^0, u}(t), \forall t \geq 0$. Then, necessarily $E(R_1 x_1|_{x_1^0, u}(1)|x_1^0) = E(R_2 x_2|_{x_2^0, u}(1)|x_2^0)$, that means $R_1 A_1 x_1^0 + R_1 B_1 u_1(0) + R_1 G_1 \mu_1 = R_2 A_2 x_2^0 + R_2 B_2 u_2(0) + R_2 G_2 \mu_2$ for any $(x_1^0, x_2^0) \in \mathcal{R}$, and input u . This easily implies conditions (h_0) , (h_1) , and (h_2) . Moreover, equating the covariances of $R_1 x_1|_{x_1^0, u}(1)$ and $R_2 x_2|_{x_2^0, u}(1)$, condition (h_3) follows. Condition (h_4) follows from condition (iii) of Definition 3.6: comparing the expected values of $y_1|_{x_1^0, u}(0)$ and $y_2|_{x_2^0, u}(0)$, it follows that if $(x_1^0, x_2^0) \in \mathcal{R}$, then $C_1 x_1^0 = C_2 x_2^0$, that is the inclusion (h_4) . It remains to show that if $\mathcal{R}$ is a total stochastic bisimulation relation between Σ_1 and Σ_2 , then necessarily (h_5) is verified. The proof is obtained by contradiction. Let $\mathcal{R} = \ker([R_1 - R_2])$ be a total stochastic bisimulation relation between Σ_1 and Σ_2 . We will prove that for any pair $(x_1^0, x_2^0) \in \mathcal{R}$, i.e., such that $R_1 x_1^0 = R_2 x_2^0$, if $\text{Im}(\text{Reach}(A_i, G_i)) \cap \ker(R_i) \neq \{0\}$ for $i = 1$ or $i = 2$, then property (i) or (ii) of Definition 3.6 is not verified for some set X_1 such that $R_1 X_1 = R_2 \mathcal{R}(X_1)$, or X_2 such that $R_2 X_2 = R_1 \mathcal{R}^{-1}(X_2)$, and hence, $\mathcal{R}$ is not a stochastic bisimulation relation. We only give the proof for $i = 1$, since the case $i = 2$ follows a symmetric reasoning. Note first that if $\text{Im}(\text{Reach}(A_1, G_1)) \cap \ker(R_1) \neq \{0\}$, then there exists $v \in \mathbb{R}^{n_1}$, with $v \neq 0_{n_1}$ such that $v \in \text{Im}(\text{Reach}_t(A_1, G_1))$ and $v \in \ker(R_1)$. Consider now the random vector $\xi_1(t)$ defined in (18) with $i = 1$, and consider any time $t \geq n_1$. We can always take a pair of sets $H_1 \subset \text{Im}(\text{Reach}(A_1, G_1))$ and $V_1 \subseteq \text{Im}(v)$ (so that $V_1 \subseteq \text{Im}(\text{Reach}(A_1, G_1)) \cap \ker(R_1)$) such that

$$\mathbf{P}(\xi_1(t) \in H_1) < \mathbf{P}(\xi_1(t) \in (H_1 + V_1)). \quad (28)$$

By defining the set $X_1 = H_1 + \mathbf{x}_1(t, x_1^0, \mathbf{u}_1, \boldsymbol{\mu}_1)$, we have

$$\mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in X_1 | x_1^0) = \mathbf{P}(\xi_1(t) \in H_1)$$

$$\mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in (X_1 + V_1) | x_1^0) = \mathbf{P}(\xi_1(t) \in (H_1 + V_1))$$

so that from (28)

$$\mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in X_1 | x_1^0) < \mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in (X_1 + V_1) | x_1^0). \quad (29)$$

Since both H_1 and V_1 belong to $\text{Im}(\text{Reach}(A_1, G_1))$, we have

$$X_1 \subset \text{supp}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t))$$

$$X_1 + V_1 \subset \text{supp}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t))$$

so that

$$\mathcal{R}(X_1 \cap \text{supp}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t))) = \mathcal{R}(X_1)$$

$$\mathcal{R}((X_1 + V_1) \cap \text{supp}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t))) = \mathcal{R}(X_1 + V_1). \quad (30)$$

By assumption, the pair (x_1^0, x_2^0) belongs to the total relation $\mathcal{R} = \ker([R_1 \ -R_2])$, so that by property (i) of Definition 3.6 and by (30), we have

$$\begin{aligned} \mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in X_1 | x_1^0) &= \mathbf{P}(\mathbf{x}_2|_{x_2^0, \mathbf{u}_2}(t) \in \mathcal{R}(X_1) | x_2^0) \\ \mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in X_1 + V_1 | x_1^0) &= \mathbf{P}(\mathbf{x}_2|_{x_2^0, \mathbf{u}_2}(t) \in \mathcal{R}(X_1 + V_1) | x_2^0). \end{aligned} \quad (31)$$

Since, by construction, $V_1 \subseteq \ker(R_1)$, i.e., $R_1 V_1 = 0$, it follows that $R_1 X_1 = R_1(X_1 + V_1)$, and from this, $\mathcal{R}(X_1) = \mathcal{R}(X_1 + V_1)$. Thus, from (31), we get

$$\mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in X_1 | x_1^0) = \mathbf{P}(\mathbf{x}_1|_{x_1^0, \mathbf{u}_1}(t) \in X_1 + V_1 | x_1^0)$$

which contradicts the inequality (29). Hence, condition (h_5) must necessarily be satisfied, and the theorem is proved. ■

Remark 4.6: As pointed out in Section I, in this paper, we extend the work [40] from nondeterministic linear systems to stochastic linear systems. The connections with the equivalence notions given in [40] are discussed in [29, Sec. VII],¹ which is summarized as follows. Let Σ'_i be the nondeterministic (and nonstochastic) version of control systems Σ_i , where stochastic description of disturbance w_i is suppressed. By combining [29, Propositions 7.2 and 7.5], we get that if $\Sigma_1 \cong_b \Sigma_2$, then Σ'_1 and Σ'_2 are bisimilar in the sense of [40]; on the other hand, [29, Example 7.3] shows that the converse implication does not hold, in general.

In the following, we give a version of Theorem 4.5 specialized to systems with nondegenerate state noises.

Corollary 4.7: Consider systems Σ_1 and Σ_2 as in (2), and suppose that $\text{rank}(\text{Reach}(A_i, G_i)) = n_i$, $i = 1, 2$ (i.e., the probability measure on both the state spaces is nondegenerate after n_i steps). Then, Σ_1 and Σ_2 are equivalent via stochastic bisimulation if and only if they are linearly equivalent.

¹Although the connections discussed in [29] pertain continuous-time linear systems, the same reasoning applies to discrete-time linear systems, as considered in this paper.

Proof: The sufficiency comes from Proposition 3.9. As far as the necessity, since $\text{rank}(\text{Reach}(A_i, G_i)) = n_i$, then condition (h_5) of Theorem 4.5 boils down to $\text{rank}(R_i) = n_i$. Then, necessarily, $n_i \leq r$, where r is the number of rows of R_1 and R_2 . Moreover, by assumption, $\text{rank}(R_1) = \text{rank}(R_2)$ because $\mathcal{R}$ is total, and therefore, necessarily, Σ_1 and Σ_2 have the same dimension ($n_1 = n_2$). Hence, the result follows by defining the nonsingular transformation matrix $\mathbb{T} = R_2^+ R_1$, where $R_2^+ = (R_2^T R_2)^{-1} R_2^T$ is the Moore–Penrose pseudoinverse matrix of R_2 , and verifying that $(x_1, x_2) \in \mathcal{R} \Leftrightarrow x_2 = \mathbb{T} x_1$. ■

Remark 4.8: When deterministic control inputs u_i of Σ_i in (2) are replaced by white Gaussian control processes, it is possible to show that the necessary and sufficient geometric conditions in Theorem 4.5 remain the same except for condition (h_5) that needs to be replaced by the stronger condition

$$h'_5) \text{Im}(\text{Reach}(A_i, [B_i \ G_i])) \cap \ker(R_i) = \{0\}, \quad i = 1, 2.$$

It is also easy to see that conditions (h_1) , (h_2) , (h_3) , (h_4) , and (h'_5) given for matrices A_i , B_i , C_i , and G_i , are the same as conditions (h_1) , (h_2) , (h_3) , (h_4) , and (h_5) given for matrices A_i , B_i , and C_i and where matrices G_i are replaced by $[B_i \ G_i]$.

We conclude this section by showing that the proposed definition of stochastic bisimulation not only preserves stochastic reachability properties but also finite horizon stochastic safety properties; see, e.g., [2], [28].

Proposition 4.9: Suppose that $\Sigma_1 \cong_b \Sigma_2$. Let $\mathcal{R} = \ker([R_1 \ -R_2])$ be a total stochastic bisimulation relation between them that satisfies the (A_1, A_2) -invariance condition (h_0) of Theorem 4.5. Then, for any pair $(x_1^0, x_2^0) \in \mathcal{R}$ and any input $\mathbf{u}$, the following conditions hold for all times $t \in \mathbb{N}$:

i) For any sequence $\{X_1^\tau\}_{\tau \in [0; t]}$ of measurable sets $X_1^\tau \subseteq \mathbb{R}^{n_1}$, $\tau \in [0, t]$,

$$\begin{aligned} \mathbf{P}\left(\bigwedge_{\tau \in [0; t]} \{\mathbf{x}_1|_{x_1^0, \mathbf{u}}(\tau) \in X_1^\tau\} \middle| x_1^0\right) &= \\ \mathbf{P}\left(\bigwedge_{\tau \in [0; t]} \{\mathbf{x}_2|_{x_2^0, \mathbf{u}}(\tau) \in \mathcal{R}(X_1^\tau \cap \text{supp}(\mathbf{x}_1|_{x_1^0, \mathbf{u}}(\tau)))\} \middle| x_2^0\right). \end{aligned}$$

ii) For any sequence $\{X_2^\tau\}_{\tau \in [0; t]}$ of measurable sets $X_2^\tau \subseteq \mathbb{R}^{n_2}$, $\tau \in [0, t]$,

$$\begin{aligned} \mathbf{P}\left(\bigwedge_{\tau \in [0; t]} \{\mathbf{x}_2|_{x_2^0, \mathbf{u}}(\tau) \in X_2^\tau\} \middle| x_2^0\right) &= \\ \mathbf{P}\left(\bigwedge_{\tau \in [0; t]} \{\mathbf{x}_1|_{x_1^0, \mathbf{u}}(\tau) \in \mathcal{R}^{-1}(X_2^\tau \cap \text{supp}(\mathbf{x}_2|_{x_2^0, \mathbf{u}}(\tau)))\} \middle| x_1^0\right). \end{aligned}$$

Proof: We only prove that (i) since (ii) follows a symmetric reasoning. The proof relies on the equivalence of processes $R_1 \mathbf{x}_1|_{x_1^0, \mathbf{u}}$ and $R_2 \mathbf{x}_2|_{x_2^0, \mathbf{u}}$, $\forall (x_1^0, x_2^0) \in \mathcal{R}$, i.e., the property (22) proved and used in Theorem 4.5. Let

$$\bar{X}_i^\tau = X_i^\tau \cap \text{supp}(\mathbf{x}_i|_{x_i^0, \mathbf{u}}(\tau)), \quad i = 1, 2.$$

By necessary condition (h_5) of Theorem 4.5, we have

$$\{\mathbf{x}_i|_{x_i^0}, \mathbf{u}(\tau) \in X_i^\tau | x_i^0\} = \{R_i \mathbf{x}_i|_{x_i^0}, \mathbf{u}(\tau) \in R_i \bar{X}_i^\tau | x_i^0\}$$

$i = 1, 2$, already used in Theorem 4.5 [see (24)]. Thus, taking also into account the equivalence $R_1 \mathbf{x}_1|_{x_1^0}, \mathbf{u} \sim R_2 \mathbf{x}_2|_{x_2^0}, \mathbf{u}$, we get

$$\begin{aligned} & \mathbf{P} \left(\bigwedge_{\tau \in [0;t]} \{\mathbf{x}_1|_{x_1^0}, \mathbf{u}(\tau) \in X_1^\tau\} \middle| x_1^0 \right) \\ &= \mathbf{P} \left(\bigwedge_{\tau \in [0;t]} \{R_1 \mathbf{x}_1|_{x_1^0}, \mathbf{u}(\tau) \in R_1 \bar{X}_1^\tau\} \middle| x_1^0 \right) \\ &= \mathbf{P} \left(\bigwedge_{\tau \in [0;t]} \{R_2 \mathbf{x}_2|_{x_2^0}, \mathbf{u}(\tau) \in R_1 \bar{X}_1^\tau\} \middle| x_2^0 \right) \\ &= \mathbf{P} \left(\bigwedge_{\tau \in [0;t]} \{\mathbf{x}_2|_{x_2^0}, \mathbf{u}(\tau) \in R_2^{-1}(R_1 \bar{X}_1^\tau)\} \middle| x_2^0 \right) \\ &= \mathbf{P} \left(\bigwedge_{\tau \in [0;t]} \{\mathbf{x}_2|_{x_2^0}, \mathbf{u}(\tau) \in \mathcal{R}(\bar{X}_1^\tau)\} \middle| x_2^0 \right). \end{aligned}$$

V. MODEL REDUCTION

In this section, we consider a linear system Σ , described for $t \in \mathbb{N}$, by

$$\Sigma : \begin{cases} x(t+1) = Ax(t) + Bu(t) + Gw(t), \\ y(t) = Cx(t) + \nu(t), \\ x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, w(t) \in \mathbb{R}^l, y(t), \nu(t) \in \mathbb{R}^p \end{cases} \quad (32)$$

with $w(t) \sim \mathcal{N}(\mu, I_l)$, and we investigate the construction of a linear system of smaller, possibly minimal, dimension in the state space, which is equivalent via stochastic bisimulation to Σ . In the following, we follow standard practice (see, e.g., [9] for concurrent processes, [40] for control systems, and [34] for switching control systems) and consider relations involved in Definition 3.6 that are also equivalence relations on the set of states of Σ , so that it is possible to define the quotient of system Σ induced by these equivalence relations. To this purpose, the following result is useful.

Proposition 5.1: A total relation $\mathcal{R} = \ker([R_1 - R_2])$ is an equivalence relation on $\mathbb{R}^n$ if and only if $R_1 = R_2$.

The following results specialize the geometric conditions derived in the previous section to equivalence relations.

Proposition 5.2: A total equivalence relation

$$\mathcal{R}_b = \ker([R_b - R_b]) \quad (33)$$

satisfies conditions of Definition 3.6 (i.e., it is a stochastic bisimulation with $\Sigma_1 = \Sigma_2 = \Sigma$ and $R_1 = R_2 = R_b$) if and only if

$$A \ker(R_b) \subseteq \ker(R_b) \subseteq \ker(C) \quad (34)$$

$$\ker(R_b) \cap \text{Reach}(A, G) = \{0\}. \quad (35)$$

The proof of the result above can be found in [30]. Consider any total equivalence relation $\mathcal{R}_b$ as in (33) with $\text{rank}(R_b) = n_b$ and satisfying (34) and (35) and consider any invertible square matrix $\mathbb{T}_b = [\mathbb{T}_b^1 \ \mathbb{T}_b^2] \in \mathbb{R}^{n \times n}$ such that $\text{Im}(\mathbb{T}_b^2) = \ker(R_b)$ and $\text{Im}(\text{Reach}(A, G)) \subseteq \text{Im}(\mathbb{T}_b^1)$. By (34) and (35), we get the following structure for the system matrices:

$$\begin{aligned} \mathbb{T}_b^{-1} A \mathbb{T}_b &= \begin{bmatrix} A_b^{11} & 0 \\ 0 & A_b^{22} \end{bmatrix}, \quad \mathbb{T}_b^{-1} B = \begin{bmatrix} B_b^1 \\ B_b^2 \end{bmatrix} \\ C \mathbb{T}_b &= [C_b^1 \ 0], \quad \mathbb{T}_b^{-1} G = \begin{bmatrix} G_b^1 \\ 0 \end{bmatrix}. \end{aligned} \quad (36)$$

We now have all the ingredients to define the quotient $\Sigma/\mathcal{R}_b$ induced by relation $\mathcal{R}_b$ as follows, for $t \in \mathbb{N}$:

$$\Sigma/\mathcal{R}_b : \begin{cases} x_b(t+1) = A_b^{11} x_b(t) + B_b^1 u(t) + G_b^1 w(t), \\ y(t) = C_b^1 x_b(t) + \nu(t), \\ x_b(t) \in \mathbb{R}^{n_b}, u(t) \in \mathbb{R}^m, \\ w(t) \in \mathbb{R}^l, y(t), \nu(t) \in \mathbb{R}^p, \end{cases} \quad (37)$$

with $w(t) \sim \mathcal{N}(\mu, I_l)$ and $\nu(t) \sim \mathcal{N}(0, \Psi)$. The following result then holds as a direct consequence of Theorem 4.5.

Theorem 5.3: $\Sigma/\mathcal{R}_b \cong_b \Sigma$.

We now proceed with a further step by discussing minimal model reduction. Given Σ , we denote by Σ_b a linear system of minimal dimension in the state space such that $\Sigma_b \cong_b \Sigma$. It is readily seen that system Σ_b is unique up to linear transformations. In the following, we characterize Σ_b . Let

$$\mathcal{R}_b^* = \ker([R_{b,1}^* - R_{b,2}^*])$$

be total relation such that $\mathcal{R}_b^*$ satisfies also condition (h_0) – (h_5) of Theorem 4.5 and is of maximal dimension. It is worth mentioning that such a relation is not unique in general, as shown in the following simple example.

Example 5.4: Consider system Σ as in (32) with $A = G = C = I_2$ and any matrix B of compatible dimensions. Then, the following two relations in the form of (12)

$$\ker \left(\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \right), \quad \ker \left(\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \right)$$

are both total and of maximal dimension. Moreover, they both satisfy conditions of Theorem 4.5 with $\Sigma_1 = \Sigma_2 = \Sigma$.

The example above also shows that in general, $R_{b,1}^* \neq R_{b,2}^*$. In this case, by Proposition 5.1, $\mathcal{R}_b^*$ is not an equivalence relation. However, it is readily seen that relation $\ker([R_{b,i}^* - R_{b,i}^*])$ with $i = 1, 2$ still satisfy conditions of Theorem 4.5, (with $\Sigma_1 = \Sigma_2 = \Sigma$), and by (14), it has the same dimensions of $\mathcal{R}_b^*$ (and therefore it is still of maximal dimension). Moreover, it is an equivalence relation on the set of states of Σ . Hence, without

loss of generality, we consider in the following a total relation $\mathcal{R}_b^*$ that is also an equivalence relation, i.e., in the form of

$$\mathcal{R}_b^* = \ker \left([R_b^* - R_b^*] \right). \quad (38)$$

We now have all the ingredients to present the main results of this section. We start with the following.

Theorem 5.5: $\Sigma_b \cong_l \Sigma / \mathcal{R}_b^*$.

The proof of the result above can be found in [30]. Provided that one can compute relation $\mathcal{R}_b^*$, by using the result above, the system Σ_b is completely specified. The computation of the maximal bisimulation relation is generally done through fixed-point operators; see, e.g., [9], [40]. In particular, in [40], an algorithm is proposed, which converges in finite steps to the desired maximal bisimulation relation. Crucial in this approach is the property of closedness of bisimulation relations with respect to sum of subspaces. Unfortunately, this approach cannot be used here because such closedness properties do not hold. The following example clarifies this issue.

Example 5.6: Consider a linear system Σ as in (32), where

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = [0 \quad 0 \quad 1]$$

with $\alpha, \beta \in \mathbb{R}$ and matrix B of compatible dimensions. By applying Theorem 4.5 with $\Sigma_1 = \Sigma_2 = \Sigma$, it is possible to show that the equivalence relations

$$\mathcal{R}_b = \ker \left(\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \end{bmatrix} \right)$$

$$\mathcal{R}_b' = \ker \left(\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \right)$$

are total stochastic bisimulation relations between Σ and itself. By a straightforward computation, we get

$$\mathcal{R}_b + \mathcal{R}_b' = \ker \left(\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \right)$$

which is not a stochastic bisimulation relation between Σ and itself because condition (h_5) of Theorem 4.5 (with $\Sigma_1 = \Sigma_2 = \Sigma$) is violated.

Nonclosedness of stochastic bisimulation relations with respect to sum of subspaces poses serious limitations to the use of fixed-point approaches to compute $\mathcal{R}_b^*$ in the general case. However, Proposition 5.2 provides a characterization of $\mathcal{R}_b^*$ that can be helpful for its computation and for the consequent computation of $\Sigma / \mathcal{R}_b^* \cong_b \Sigma_b$. Indeed, by Theorem 5.3, a system Σ_b of minimal dimension that is equivalent via stochastic bisimulation to Σ can be computed by finding a total relation $\mathcal{R}_b^*$ of maximal dimension, and this can be computed by finding an A -invariant subspace $\mathcal{U} \subseteq \mathbb{R}^n$ of maximal dimension such that

$$\mathcal{U} \subseteq \ker(\text{Obs}(A, C)), \text{ and } \mathcal{U} \cap \text{Im}(\text{Reach}(A, G)) = \{0\}.$$

(see Proposition 5.2). Under few additional assumptions, we can provide an explicit expression for a relation $\mathcal{R}_b^*$ of maximal dimension and, therefore, of a system Σ_b of minimal dimension that is equivalent via stochastic bisimulation to Σ . Let $\mathcal{G} = \text{Im}(\text{Reach}(A, G))$, $\mathcal{Q} = \ker(\text{Obs}(A, C))$. Let $\lambda_k \in \mathbb{C}$, $k \in \{1, \dots, \delta\}$, $\delta \leq n$, denote the eigenvalues of A , and let

$\mathcal{S}_k \subseteq \mathbb{R}^n$ denote the generalized real eigenspaces associated to λ_k (subspaces $\mathcal{S}_k$ are A -invariant). Eigenspace $\mathcal{S}_k$ is said to be:

- 1) totally reachable (from the noise), if $\mathcal{S}_k \subseteq \mathcal{G}$;
- 2) totally unreachable (from the noise), if $\mathcal{S}_k \cap \mathcal{G} = \{0\}$;
- 3) totally observable, if $\mathcal{S}_k \cap \mathcal{Q} = \{0\}$;
- 4) totally unobservable, if $\mathcal{S}_k \subseteq \mathcal{Q}$.

Theorem 5.7: Suppose that any generalized eigenspace $\mathcal{S}_k$ of matrix A is either totally reachable or totally unreachable, and it is either totally unobservable or totally observable, i.e.,

$$[[\mathcal{S}_k \subseteq \mathcal{G}] \vee [\mathcal{S}_k \cap \mathcal{G} = \{0\}]] \wedge [[\mathcal{S}_k \subseteq \mathcal{Q}] \vee [\mathcal{S}_k \cap \mathcal{Q} = \{0\}]]. \quad (39)$$

Let $\mathcal{I} = \{k_1, \dots, k_{\bar{\delta}}\} \subset \{1, \dots, \delta\}$ be the subset of all indexes such that the eigenspaces $\mathcal{S}_{k_i}$, $k_i \in \mathcal{I}$, are totally unreachable and totally unobservable (i.e., $\mathcal{S}_{k_i} \subseteq \mathcal{Q}$ and $\mathcal{S}_{k_i} \cap \mathcal{G} = \{0\}$), and define $\mathcal{S}_{\Sigma} = \mathcal{S}_{k_1} \oplus \mathcal{S}_{k_2} \oplus \dots \oplus \mathcal{S}_{k_{\bar{\delta}}} \subseteq \mathbb{R}^n$. Then, a total relation $\mathcal{R}_b^*$ of maximal dimension takes the form (38), where the (nonunique) matrix R_b^* is chosen such that $\mathcal{S}_{\Sigma} = \ker(R_b^*)$.

Proof: Subspace $\mathcal{S}_{\Sigma}$, given as the sum of A -invariant subspaces is itself A -invariant. Moreover, $\mathcal{S}_{k_i} \subseteq \mathcal{Q}$ and $\mathcal{S}_{k_i} \cap \mathcal{G} = \{0\} \forall k_i \in \mathcal{I}$ implies that $\mathcal{S}_{\Sigma} \subseteq \mathcal{Q}$ and $\mathcal{S}_{\Sigma} \cap \mathcal{G} = \{0\}$. Hence, $\mathcal{S}_{\Sigma}$ satisfies (34) and (35). Moreover, by its definition, $\mathcal{S}_{\Sigma}$ is of maximal dimension. A straightforward consequence of Proposition 5.2 is that a total stochastic bisimulation and equivalence relation $\mathcal{R}_b$ as in (33) between Σ and itself is of maximal dimension if and only if $\ker(R_b)$ satisfies (34) and (35) and is of maximal dimension. Hence, the result follows. ■

By combining Theorems 5.5 and 5.7, we get that if eigenspaces $\mathcal{S}_k$ of matrix A in Σ satisfy (39), then Σ_b can be chosen as the subsystem of Σ with all and only modes of Σ that are either reachable from the noise or observable. Such a subsystem can be easily computed via the Kalman decomposition [20] with the additional requirement of choosing a complementary space to $\mathcal{G} \cap \mathcal{Q}$ in $\mathcal{Q}$ that is A -invariant. For the computation of Σ_b , it is sufficient to choose an invertible matrix $T_b = [T_b^1 \ T_b^2] \in \mathbb{R}^{n \times n}$ such that $\text{Im}(T_b^2) = \mathcal{S}_{\Sigma}$ and $\text{Im}(T_b^1) = \mathcal{S}_{h_1} \oplus \mathcal{S}_{h_2} \oplus \dots \oplus \mathcal{S}_{h_{\delta-\bar{\delta}}}$, where $\{h_1, \dots, h_{\delta-\bar{\delta}}\} = \{1, \dots, \delta\} \setminus \mathcal{I}$, use matrix T_b to change the coordinates to Σ as in (36) and select the subsystem (37).

Although the assumption of Theorem 5.7 is not demanding (for instance, it is trivially fulfilled for all systems Σ such that the matrix A has distinct eigenvalues), it can be weakened as follows.

Theorem 5.8: Suppose that any generalized eigenspace $\mathcal{S}_k$ of matrix A can be decomposed as $\mathcal{S}_k = \mathcal{S}_k^1 \oplus \mathcal{S}_k^2$ with both $\mathcal{S}_k^1$ and $\mathcal{S}_k^2$ A -invariant and $\mathcal{S}_k^1$ such that

$$\mathcal{S}_k^1 \subseteq \mathcal{Q}, \quad \mathcal{S}_k^1 \cap \mathcal{G} = \{0\} \quad (40)$$

(note that $\mathcal{S}_k^1$ or $\mathcal{S}_k^2$ can vanish for some k).

Define

$$\mathcal{S}_{\Sigma}^1 = \mathcal{S}_1^1 \oplus \mathcal{S}_2^1 \oplus \dots \oplus \mathcal{S}_{\delta}^1.$$

Then, a total relation $\mathcal{R}_b^*$ of maximal dimension takes the form (38), where the (nonunique) matrix R_b^* is chosen such that $\mathcal{S}_{\Sigma}^1 = \ker(R_b^*)$.

Proof: As in the proof of Theorem 5.7, $\mathcal{S}_{\Sigma}^1$ is a sum of A -invariant subspaces satisfying (40) and, therefore, is itself

TABLE II
MODAL ANALYSIS OF MATRIX A IN THE EXAMPLE OF SECTION VI

	B reachable	G reachable	C observable
$\lambda = 1$	Y	Y	N
$\lambda = 2$	Y	Y	Y
$\lambda = 3$	N	Y	N
$\lambda = 4$	N	Y	Y
$\lambda = 5$	Y	N	N
$\lambda = 6$	Y	N	Y
$\lambda = 7$	N	N	N
$\lambda = 8$	N	N	Y

A -invariant and satisfies (40), i.e., $\mathcal{S}_\Sigma \subseteq \mathcal{Q}$ and $\mathcal{S}_\Sigma^1 \cap \mathcal{G} = \{0\}$, and hence satisfies (34) and (35) and is of maximal dimension. The same discussion made in the proof of Theorem 5.7 leads to the thesis. ■

Note that the assumption of Theorem 5.8 is weaker than the one of Theorem 5.7, in that the assumption (39) coincides with the assumption made in Theorem 5.8 with the additional condition that for each $k \in \{1, \dots, \delta\}$ either $\mathcal{S}_k^1 = \{0\}$ or $\mathcal{S}_k^2 = \{0\}$ (for $k_i \in \mathcal{I}$, we have $\mathcal{S}_{k_i}^1 = \mathcal{S}_{k_i}$ and $\mathcal{S}_{k_i}^2 = \{0\}$).

We conclude this section with a simple example illustrating the results presented above. Consider the linear system Σ in (32), where

$$A = \begin{bmatrix} 1 & * & * & * & * & * & * & * \\ 0 & 2 & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 3 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 5 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 5 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0 \ 2 \ 0 \ 4 \ 0 \ 6 \ 0 \ 8].$$

The symbol $*$ denotes any real entry, whose value is inessential for the following analysis. The Kalman canonical form of matrices A , B , G , and C allows us to identify the following subspaces:

$$\begin{aligned} B\text{-reachable subspace:} \quad \mathcal{S}_B &= \{e_1, e_2, e_5, e_6\} \\ G\text{-reachable subspace:} \quad \mathcal{S}_G &= \{e_1, e_2, e_3, e_4\} \\ C\text{-unobservable subspace:} \quad \mathcal{S}_C &= \{e_1, e_3, e_5, e_7\} \end{aligned}$$

where $e_k \in \mathbb{R}^8$ denotes the k th vector of the canonical basis of $\mathbb{R}^8$. We report in Table II the modal analysis of matrix A . Each row of the table is referred to as an eigenvalue, and the entries corresponding to each column are as follows:

- 1) [B reachable] is “Y” if the eigenspace corresponding to the eigenvalue $\lambda = k$ is reachable from the control input u (i.e., $\mathcal{S}_k \subseteq \mathcal{S}_B$), and “N” otherwise.
- 2) [G reachable] is “Y” if the eigenspace corresponding to the eigenvalue $\lambda = k$ is reachable from the disturbance input w (i.e., $\mathcal{S}_k \subseteq \mathcal{S}_G$), and “N” otherwise.
- 3) [C observable], is “N” if the eigenspace corresponding to the eigenvalue $\lambda = k$ is unobservable (i.e., $\mathcal{S}_k \subseteq \mathcal{S}_C$), and “Y” otherwise.

By applying Theorem 5.7, we get that dynamical matrices of Σ_b can be chosen as

$$A_b = \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 2 & 0 & * & * & * \\ 0 & 0 & 3 & * & 0 & * \\ 0 & 0 & 0 & 4 & 0 & * \\ 0 & 0 & 0 & 0 & 6 & * \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \quad B_b = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \quad G_b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$C_b = [0 \ 2 \ 0 \ 4 \ 6 \ 8].$$

Remark 5.9: In view of Remark 3.8, it is possible to show that all the results presented in this section can be easily modified to consider the case of systems Σ in (32) with deterministic control inputs replaced by white Gaussian control processes, by replacing matrix G by matrix $[B \ G]$.

VI. CONTROL DESIGN VIA BISIMULATION AND APPLICATION TO OPTIMAL CONTROL FOR STOCHASTIC REACHABILITY

A key application of bisimulation theory in concurrent processes is control design; see, e.g., [9]. In this section, we first discuss the use of stochastic bisimulation equivalence for control design and then present an application to optimal control problems with stochastic reachability specifications. Consider a general stochastic control problem for system Σ_1 in (2) that consists in finding a stochastic control process u_1 enforcing some specification $\text{Spec}(\Sigma_1)$ on Σ_1 . We consider the general case where no assumptions are placed on the stochastic control process u_1 ; in particular, u_1 needs to be neither white nor Gaussian and can be also correlated with the state stochastic process generated by Σ_1 . Suppose now one wants to use stochastic bisimulation equivalence to simplify the design of u_1 . Let system Σ_2 in (2) be with $n_2 < n_1$ (possibly $n_2 \ll n_1$) and such that $\Sigma_1 \cong_b \Sigma_2$ with total stochastic bisimulation relation $\mathcal{R}$. The basic idea is to solve an adjoint control problem on Σ_2 to then use the information gained to derive u_1 . To this purpose, a specification $\text{Spec}(\Sigma_2)$ on Σ_2 is first defined, which is related to $\text{Spec}(\Sigma_1)$ via relation $\mathcal{R}$. Then, one first finds a stochastic control process u_2 enforcing $\text{Spec}(\Sigma_2)$ on Σ_2 and then derives u_1 from u_2 via relation $\mathcal{R}$. Of course, this control design procedure is correct as long as the notion of stochastic bisimulation equivalence we propose preserves specification $\text{Spec}(\Sigma_1)$. Consider a specification $\text{Spec}(\Sigma_1)$ that involves a set $X_1 \subseteq \mathbb{R}^{n_1}$ that, for example, needs to be reached or avoided. When defining $\text{Spec}(\Sigma_2)$, by following condition i) of Definition 3.6, set X_1 is mapped into set

$$X_2 = \mathcal{R}(X_1 \cap \text{supp}(x_1|_{x_1^0, u_1}(t))). \quad (41)$$

Here is the central part of our discussion: *from (41), in order to compute X_2 that is instrumental to derive u_2 , from which to finally derive u_1 , one needs to know u_1 in advance, which is not available at this stage. As a consequence, it is possible to use equivalence via stochastic bisimulation for control design purposes only if $\text{supp}(x_1|_{x_1^0, u_1}(t))$ is independent of u_1 .* In the next lemma, necessary and sufficient conditions are given that ensure that the support of a state process x of a system Σ as

in (32) is independent of the input process $\mathbf{u}$. The following standard causality assumption of the input process is required.

Definition 6.1: A stochastic input process $\mathbf{u}$ of system (32) is *causal* if $u(t)$ is independent of $w(\tau)$ for all $\tau \geq t$.

Note that any process $\mathbf{u}$ that is a state feedback is causal.

Lemma 6.2: Consider system Σ in (32) where the control input u is a causal stochastic process, possibly correlated with the state process. Then

$$\text{supp}(\mathbf{x}|_{x^0, \mathbf{u}}(t)) = \text{supp}(\mathbf{x}|_{x^0, \mathbf{u}'}(t)) \quad (42)$$

for any x^0 , for any causal processes $\mathbf{u}, \mathbf{u}'$, and for any $t \geq 0$, if and only if

$$\text{Im}(B) \subseteq \text{Im}(G). \quad (43)$$

Proof: Recall first that the supports in (42) are conditional to the initial state x^0 . For a given $t > 0$, consider the random vectors $\mathbf{w}|_{0:t-1}$, $\mathbf{u}|_{0:t-1}$ and $\tilde{\mathbf{w}}|_{0:t-1}$, as defined in (5) and (6), and the zero-mean vector $\tilde{\mathbf{u}}|_{0:t-1} = \mathbf{u}|_{0:t-1} - E\{\mathbf{u}|_{0:t-1}\}$. Thus

$$\begin{aligned} \mathbf{x}|_{x^0, \mathbf{u}}(t) &= E\{\mathbf{x}|_{x^0, \mathbf{u}}(t)\} + v_t \\ E\{\mathbf{x}|_{x^0, \mathbf{u}}(t)\} &= A^t x^0 + \text{Reach}_t(A, B)E\{\mathbf{u}|_{0:t-1}\} \\ &\quad + \text{Reach}_t(A, G)\boldsymbol{\mu}|_{0:t-1} \end{aligned} \quad (44)$$

where v_t is the zero-mean random vector defined as

$$v_t = \text{Reach}_t(A, B)\tilde{\mathbf{u}}|_{0:t-1} + \text{Reach}_t(A, G)\tilde{\mathbf{w}}|_{0:t-1}. \quad (45)$$

Then, $\text{supp}(\mathbf{x}|_{x^0, \mathbf{u}}(t)) = E\{\mathbf{x}|_{x^0, \mathbf{u}}(t)\} + \text{supp}(v_t)$.

(Sufficiency) Condition (43) implies that there exists $S \in \mathbb{R}^{m \times l}$ such that $B = GS$. From this, $A^k B = A^k GS$ and, for any $t \geq 0$,

$$\text{Reach}_t(A, B) = \text{Reach}_t(A, G)\bar{S}, \quad \bar{S} = I_t \otimes S \in \mathbb{R}^{mt \times lt}$$

where $\otimes$ is the standard Kronecker product, and $\bar{S}$ is block-diagonal. Then, $v_t = \text{Reach}_t(A, G)(\bar{S}\tilde{\mathbf{u}}|_{0:t-1} + \tilde{\mathbf{w}}|_{0:t-1})$, and by denoting $\xi_t = \bar{S}\tilde{\mathbf{u}}|_{0:t-1} + \tilde{\mathbf{w}}|_{0:t-1}$, we get $\text{supp}(v_t) = \text{Reach}_t(A, G)\text{supp}(\xi_t)$. Under assumption on causality of the input process, it can be proven that the covariance $\text{cov}(\xi_t)$ is positive definite, and $\text{supp}(\xi_t) = \text{Im}(\text{cov}(\xi_t)) = \mathbb{R}^{lt}$, and therefore, $\text{supp}(v_t) = \text{Im}(\text{Reach}_t(A, G))$. Since, by (43), $\text{Im}(\text{Reach}_t(A, B)) \subseteq \text{Im}(\text{Reach}_t(A, G))$, we have

$$\begin{aligned} \text{Im}(\text{Reach}_t(A, G)) + \text{Reach}_t(A, B)E\{\mathbf{u}|_{0:t-1}\} \\ + \text{Reach}_t(A, G)\boldsymbol{\mu}|_{0:t-1} = \text{Im}(\text{Reach}_t(A, G)) \end{aligned}$$

then

$$\text{supp}(\mathbf{x}|_{x^0, \mathbf{u}}(t)) = A^t x^0 + \text{Im}(\text{Reach}_t(A, G)). \quad (46)$$

Equation (46) proves that under (43), the support of $\mathbf{x}|_{x^0, \mathbf{u}}(t)$ is independent of the input process, i.e., (42) holds true.

(Necessity) Proved by contradiction. Assume that (42) holds true and (43) is not satisfied, i.e., $\text{Im}(G) \subset \text{Im}([B \ G])$. Take $x^0 = 0$ and $\mathbf{u}$ and $\mathbf{u}'$ such that $u(0) \sim \mathcal{N}(0, I_m)$ and $u'(0) = 0$. Then

$$\begin{aligned} \text{supp}(\mathbf{x}|_{0, \mathbf{u}}(1)) &= \text{supp}(G\boldsymbol{\mu} + Bu(0) + Gw_0) \\ &= G\boldsymbol{\mu} + \text{Im}([B \ G]) \end{aligned}$$

$$\text{supp}(\mathbf{x}|_{0, \mathbf{u}'}(1)) = \text{supp}(G\boldsymbol{\mu} + Gw_0) = G\boldsymbol{\mu} + \text{Im}(G).$$

Since $\text{Im}(G) \subset \text{Im}([B \ G])$, the identity (42) is violated. ■ Some remarks on condition (43) follow.

Remark 6.3: From a modeling perspective, condition (43) naturally arises when the control input $u(t)$ is subject to an additive random disturbance $d(t)$ that accounts for uncertainties in the actuation of the control input. In this case, system Σ in (32) can be rewritten as the following system:

$$\Sigma' : \begin{cases} x(t+1) = Ax(t) + Bu(t) + [B \ G] \begin{bmatrix} d(t) \\ w(t) \end{bmatrix}, \\ x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, d(t) \in \mathbb{R}^m, w(t) \in \mathbb{R}^l \end{cases}$$

that indeed satisfies (43) with $[B \ G]$ replacing G .

Remark 6.4: It is interesting to note that condition (43) combined with condition h_5) in Theorem 4.5 implies condition h'_5) in Remark 4.8. This is consistent with the fact that condition (43) allows considering control inputs that are general stochastic processes, possibly correlated with state stochastic processes generated by Σ_i , whereas condition h'_5) only allows considering white Gaussian control processes.

We conclude this section by presenting an application to optimal control with stochastic reachability specifications, as studied in [28] for nonlinear systems and in [2] for hybrid systems. Consider the following optimal control problem:

Problem 6.5: Consider a stochastic linear control system

$$\Sigma : \begin{cases} x(t+1) = Ax(t) + Bu(t) + Gw(t), \\ x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, w(t) \in \mathbb{R}^l \end{cases} \quad (47)$$

with $w(t) \sim \mathcal{N}(\mu, I_l)$ and satisfying (43). Given an initial state x^0 , a finite time horizon $T \in \mathbb{N}$, and a target (measurable) set $\mathcal{T} \subset \mathbb{R}^n$, find an optimal Markov control policy $u^*(x, t)$ ($u^*: \mathbb{R}^n \times [0; T-1] \rightarrow \mathbb{R}^m$) that maximizes $\mathbf{P}(x(T), x^0, \mathbf{u}, \mathbf{w}) \in \mathcal{T})$.

The solution u^* to the problem above can be found in [28]. We stress that u^* is not unique in general. An application of the optimal control problem above to quantitative finance can be found in [33]. Definition 3.6 naturally applies to systems in the form of (47) by removing condition iii), because no output is considered in (47). Theorem 4.5, providing geometric conditions to check stochastic bisimulation equivalence, adapts to systems in the form of (47), by removing condition h_4). In view of Lemma 6.2, since system Σ in (47) satisfies (43), the support $\text{supp}(\mathbf{x}|_{x^0, \mathbf{u}}(t))$ of the state $\mathbf{x}|_{x^0, \mathbf{u}}(t)$ generated by Σ is independent of $\mathbf{u}$. For this reason, in the rest of this section, we denote it by $\text{supp}_{x^0, t}$. The following result shows that the solution to Problem 6.5 for system Σ can be easily derived on the basis of a solution to Problem 6.5 for a control system Σ_b of minimum dimension² that is equivalent via stochastic bisimulation to Σ .

Proposition 6.6: Consider the system Σ in (47) that satisfies condition (43). Let $\mathcal{R}^* = \ker([R_1 - R_2]) \subset \mathbb{R}^n \times \mathbb{R}^{n_b}$ be a maximal stochastic bisimulation relation³ between Σ and

²System Σ_b can be computed by using the results in Section V for the system in (32) with $C = 0$.

³Relation $\mathcal{R}^*$ can be easily derived by using the results in Section V for the system in (32) with $C = 0$.

Σ_b , with Σ_b of dimension n_b (note that, by minimality of Σ_b , $R_2 \in \mathbb{R}^{n_b \times n_b}$ is nonsingular). Consider the control Problem 6.5 with given x^0 , T and $\mathcal{T}$, and let

$$\mathcal{T}_b = \mathcal{R}^*(\mathcal{T} \cap \text{supp}_{x^0, T}). \quad (48)$$

Let $x_b \in \mathbb{R}^{n_b}$ denote the state of Σ_b and let $u_b^*(x_b, t)$ be an optimal Markov control policy solving Problem 6.5 with Σ and $\mathcal{T}$ replaced by Σ_b and $\mathcal{T}_b$, respectively. Then, the Markov policy for Σ defined as

$$u(x, t) = u_b^*(R_2^{-1}R_1x, t), \quad t \in [0; T-1] \quad (49)$$

is an optimal Markov policy for Σ .

Proof: Consider Problem 6.5 formulated in the coordinate system (36) introduced in Section V. Let $z = \mathbb{T}_b^{-1}x$, where $\mathbb{T}_b = [\mathbb{T}_b^1 \ \mathbb{T}_b^2]$, with $\text{Im}(\mathbb{T}_b^1) = \text{Im}(\text{Reach}(A, G))$ of dimension n_b and $\text{Im}(\mathbb{T}_b^2) = \ker(R_2)$, of dimension $n - n_b$. In the new coordinates, the state z is naturally partitioned as (z_1, z_2) , with $z_1 \in \mathbb{R}^{n_b}$ and $z_2 \in \mathbb{R}^{n-n_b}$. By the A -invariance of both $\text{Im}(\mathbb{T}_b^1)$ and $\text{Im}(\mathbb{T}_b^2)$ and by property (43), the system Σ in z -coordinates takes the form

$$\begin{bmatrix} z_1(t+1) \\ z_2(t+1) \end{bmatrix} = \begin{bmatrix} A_b^{11} & 0 \\ 0 & A_b^{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} B_b^1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} G_b^1 \\ 0 \end{bmatrix} w(t) \quad (50)$$

with $z^0 = \mathbb{T}_b^{-1}x^0$ (note that in (36) a term B_b^2 is present, while $B_b^2 = 0$ in (50), because $\text{Im}(B) \subseteq \text{Im}(G)$). The support of the state $z(T)$ is independent of u and is easily computed

$$\text{supp}_{z^0, T} = \begin{bmatrix} (A_b^{11})^T z_1^0 \\ (A_b^{22})^T z_2^0 \end{bmatrix} + \text{Im} \left(\begin{bmatrix} \text{Reach}_T(A_b^{11}, G_b^1) \\ 0 \end{bmatrix} \right). \quad (51)$$

As discussed in Section V, a system of minimal dimension that is bisimilar to Σ , and to its linear equivalent (50), is obtained by projecting the state of the system in the new coordinates along the direction of z_1 :

$$z_1(t+1) = A_b^{11}z_1(t) + B_b^1u(t) + G_b^1w(t) \quad (52)$$

with $z_1^0 = [I_{n_b} \ 0] \mathbb{T}_b^{-1}x^0$. Since any system of minimal dimension bisimilar to Σ is linearly equivalent to (52), we can take as Σ_b exactly the system (52), and therefore, in the following, we set $x_b(t) = z_1(t)$, while the matrices R_i defining the maximal relation $\mathcal{R}^*$ are

$$R_1 = [I_{n_b} \ 0] \mathbb{T}_b^{-1}, \quad R_2 = I_{n_b}. \quad (53)$$

With this choice, the set $\mathcal{T}_b$ defined in (48) is computed as

$$\begin{aligned} \mathcal{T}_b &= [I_{n_b} \ 0] \mathbb{T}_b^{-1} (\mathcal{T} \cap \text{supp}_{x^0, T}) \\ &= [I_{n_b} \ 0] ((\mathbb{T}_b^{-1}\mathcal{T}) \cap \text{supp}_{z^0, T}). \end{aligned} \quad (54)$$

The structure of $\text{supp}_{z^0, T}$ revealed by (51) shows that for any given feedback law $u(z, t)$, we have

$$\mathbf{P}(z(T, z^0, u, w) \in \mathbb{T}_b \mathcal{T} | z^0) = \mathbf{P}(z_1(T, z_1^0, u, w) \in \mathcal{T}_b | z_1^0). \quad (55)$$

Let $\bar{u}(z, t)$ and $u_b^*(z_1, t)$ denote the optimal Markov policies for (50) and (52), respectively. Since the dynamics of $z_1(t)$ and $z_2(t)$ in (50) are independent and $z_2(t)$ is deterministic, conditionally to z^0 , the optimal Markov policy $\bar{u}(z, t)$ is independent of z_2 . Moreover, necessarily, $u_b^*(z_1, t)$ is an optimal Markov policy

for (50); otherwise, thanks to identity (55), a contradiction can be easily shown. When coming back to the original coordinates, we have that $u_b^*([I_{n_b} \ 0] \mathbb{T}_b^{-1}x, t)$ is an optimal Markov policy for system (47), which is exactly (49) (recall from (53) that $R_2^{-1}R_1 = [I_{n_b} \ 0] \mathbb{T}_b^{-1}$). ■

VII. CONNECTION WITH RELATED LITERATURE

Definition 3.3 has been inspired by the notion of probabilistic bisimulation given for probabilistic chains in [21]. The notion of [21] (corresponding to [17, Def. 3.5.3]) has been extended in [12, Def. 2.5] (see also [11, Def. 2.6]) to labeled Markov processes featuring continuous state space. [12, Def. 2.5] coincides with [17, Def. 3.5.3] except for the fact that it applies not to equivalence classes but to measurable $\mathcal{R}$ -closed sets; we recall that a set X is $\mathcal{R}$ -closed if $\mathcal{R}(X) \subseteq X$; if $\mathcal{R}$ is reflexive, this becomes $\mathcal{R}(X) = X$. As also pointed out in [11], if $\mathcal{R}$ is an equivalence relation, a set is $\mathcal{R}$ -closed if and only if it is a union of equivalence classes. The key difference between [12, Def. 2.5] and Definition 3.3 is that while the former considers only $\mathcal{R}$ -closed sets measurable sets, the latter considers any measurable set. By Corollary 4.7, linear systems with nondegenerate disturbances cannot be reduced to smaller ones while preserving equivalence via stochastic bisimulation. Reduction is possible in the case of linear systems with degenerate disturbances, as shown in Theorem 5.5. [12, Def. 2.5] of bisimulation for labeled Markov processes, instead, allows finding equivalent states even in the case of nondegenerate disturbances. This is a consequence of the fact that conditions in Definition 3.3 are required to hold for all measurable sets and not only for $\mathcal{R}$ -closed sets as in [12]. On the other hand, when [12, Def. 2.5] is used for reduction purposes, one gets that the original labeled Markov process and the labeled Markov process obtained by aggregating equivalent states are not bisimilar (in the sense of [12, Def. 2.5]), whereas in our framework, linear Σ and its reduced one $\Sigma/\mathcal{R}$ are equivalent via stochastic bisimulation, as formally shown in Theorem 5.3. When relaxing conditions in Definitions 3.3 and 3.6 to hold not for all measurable sets but only for $\mathcal{R}$ -closed sets (as in [12, Def. 2.5]), meaning in our framework that conditions in (i) and (ii) are requested to hold only for measurable sets X_1 and X_2 satisfying $\mathcal{R}^{-1}(\mathcal{R}(X_1)) = X_1$ and $\mathcal{R}(\mathcal{R}^{-1}(X_2)) = X_2$, Definitions 3.3 and 3.6 coincide. Consequently, geometric conditions in Theorem 4.5 change. More specifically, conditions (h_0) – (h_4) are still needed, while condition (h_5) is not. As a matter of fact, if X_1 is $\mathcal{R}$ -closed, equality in (24) would be true independently from condition (h_5) (and Lemma 4.4). As a consequence, in this case, it is possible to find *also* in the nondegenerate case two linear systems that are stochastically bisimilar in the sense of the coinciding Definitions 3.3 and 3.6 but not linearly equivalent.

VIII. CONCLUSION AND OUTLOOK

In this paper, we proposed a novel definition of equivalence via stochastic bisimulation for the class of discrete-time stochastic linear control systems with possibly degenerate disturbance distributions. Necessary and sufficient conditions based on geometric control theory to check these notions were derived and model reduction addressed. Connections with stochastic reach-

ability were discussed and with finite horizon stochastic safety problems established. Finally, a discussion on the use of stochastic bisimulation equivalence for control design was given, and an application of the proposed results to optimal control for stochastic reachability is presented.

In many real-world applications, complex systems are given as the composition of several subsystems. In our future work, we plan to extend the results presented in this paper to compositional stochastic systems. Useful insights in this regard are reported in the last paper by Jan Camiel Willems [42].

APPENDIX

In this section, we include the proofs of some technical results of Section IV.

Proof of Lemma 4.2.

Proof of (i): We first show $R_1 X_1 \subseteq R_2 \mathcal{R}(X_1)$. Consider any $y \in R_1 X_1$. Then, there exists $x_1 \in X_1$ such that $y = R_1 x_1$. Since $\mathcal{R}$ is total, then there exists $x_2 \in \mathbb{R}^{n_2}$ such that $(x_1, x_2) \in \mathcal{R}$. Note that $x_2 \in \mathcal{R}(X_1)$. Since $(x_1, x_2) \in \mathcal{R}$, then, by (12), $R_1 x_1 = R_2 x_2$, which, in turn, implies $y = R_2 x_2$. Since $R_2 x_2 \in R_2 \mathcal{R}(X_1)$, we get $y \in R_2 \mathcal{R}(X_1)$. We now show $R_2 \mathcal{R}(X_1) \subseteq R_1 X_1$. Pick any $y \in R_2 \mathcal{R}(X_1)$. Then, there exists $x_2 \in \mathcal{R}(X_1)$ such that $y = R_2 x_2$. Since $x_2 \in \mathcal{R}(X_1)$, there exists $x_1 \in X_1$ such that $(x_1, x_2) \in \mathcal{R}$, which implies $y = R_2 x_2 = R_1 x_1 \in R_1 X_1$.

Proof of (ii): This proof follows the same steps as those in the proof of (i).

Proof of (iii): Since $\mathcal{R}(X_1) \subseteq \mathcal{R}(X_1) + \ker(R_2)$, we only need to show $\mathcal{R}(X_1) + \ker(R_2) \subseteq \mathcal{R}(X_1)$ or equivalently

$$x_2 + z_2 \subseteq \mathcal{R}(X_1), \forall x_2 \in \mathcal{R}(X_1), \forall z_2 \in \ker(R_2).$$

Consider any $x_2 \in \mathcal{R}(X_1)$ and $z_2 \in \ker(R_2)$. There exists $x_1 \in X_1$ such that $(x_1, x_2) \in \mathcal{R}$, or equivalently, $R_1 x_1 = R_2 x_2$. Since $R_1 x_1 = R_2 x_2$ and $R_2 z_2 = 0$, then $R_1 x_1 = R_2 x_2 + R_2 z_2 = R_2(x_2 + z_2)$ or equivalently $(x_1, x_2 + z_2) \in \mathcal{R}$ from which, $x_2 + z_2 \in \mathcal{R}(X_1)$.

Proof of (iv): This proof follows the same steps as those in the proof of (iii). ■

Proof of Lemma 4.3: Let r denote the number of rows of matrices R_i . Condition (17) is equivalent to claim that for any measurable $S \subset \mathbb{R}^r$ it is $P(R_1 x_1|_{x_1^0, u}(t) \in S) = P(R_2 x_2|_{x_2^0, u}(t) \in S)$ for any $(x_1^0, x_2^0) \in \mathcal{R}$, input u , and $t \geq 0$. The necessity of (17) is proven by contradiction, by showing that if $\mathcal{R}$ is a stochastic bisimulation relation, if for some $(x_1^0, x_2^0) \in \mathcal{R}$, input u , and $t \geq 0$, there exists a subset $Q \subset \mathbb{R}^r$ such that $P(R_1 x_1|_{x_1^0, u}(t) \in Q) \neq P(R_2 x_2|_{x_2^0, u}(t) \in Q)$, then we arrive at a contradiction. In particular, using the shorthand notation $v_i = x_i|_{x_i^0, u}$, $i = 1, 2$, we will consider the case

$$P(R_1 v_1 \in Q) > P(R_2 v_2 \in Q)$$

(the case where the inequality is reversed can be handled with a symmetric reasoning). Of course, $P(R_i v_i \in Q) = P(v_i \in R_i^{-1} Q)$. Then, by property (i) in Definition 3.6 of stochastic bisimulation relation, we have

$$\begin{aligned} P(R_1 v_1 \in Q) &= P(v_1 \in R_1^{-1} Q) \\ &= P(v_2 \in \mathcal{R}((R_1^{-1} Q) \cap \text{supp}(v_1))). \end{aligned} \quad (56)$$

Obviously, $\mathcal{R}((R_1^{-1} Q) \cap \text{supp}(v_1)) \subseteq \mathcal{R}(R_1^{-1} Q)$, and by (13), we have $\mathcal{R}(R_1^{-1} Q) = R_2^{-1}(R_1(R_1^{-1} Q)) = R_2^{-1} Q$; then

$$P(v_2 \in \mathcal{R}((R_1^{-1} Q) \cap \text{supp}(v_1))) \leq P(v_2 \in R_2^{-1} Q). \quad (57)$$

Form (56), (57), and $P(v_2 \in R_2^{-1} Q) = P(R_2 v_2 \in Q)$, we get

$$P(R_1 v_1 \in Q) \leq P(R_2 v_2 \in Q) \quad (58)$$

contradicting the assumption $P(R_1 v_1 \in Q) > P(R_2 v_2 \in Q)$. This proves the Lemma. ■

Proof of Lemma 4.4: Pick any set $\bar{X}_i \subset \mathbb{R}^{n_i}$ satisfying (21), and define the shifted set $H_i = \bar{X}_i - x_i(t, x_i^0, u_i, \mu_i)$. By (19), $H_i \subseteq \text{Im}(\text{Reach}_t(A_i, G_i)) \subseteq \text{Im}(\text{Reach}(A_i, G_i))$, so that $x_i|_{x_i^0, u_i}(t) \in \bar{X}_i$ if and only if $\xi_i(t) \in H_i$ and $\mathbf{P}(x_i|_{x_i^0, u_i}(t) \in \bar{X}_i | x_i^0) = \mathbf{P}(\xi_i(t) \in H_i)$. The thesis is proven if we show that $\mathbf{P}(R_i \xi_i(t) \in R_i H_i) = \mathbf{P}(\xi_i(t) \in H_i)$. This is straightforward because $R_i \xi_i(t) \in R_i H_i$ implies $\xi_i(t) \in (H_i + \ker(R_i))$; by construction, $\xi_i(t) \in \text{Im}(\text{Reach}_t(A_i, G_i)) \subseteq \text{Im}(\text{Reach}(A_i, G_i))$, and therefore, by (20), $\xi_i(t) \notin \ker(R_i) \setminus \{0\}$. Thus, $\xi_i(t) \in (H_i + \ker(R_i))$ implies $\xi_i(t) \in H_i$, which concludes the proof. ■

REFERENCES

- [1] A. Abate, A. D'Innocenzo, and M. Di Benedetto, "Approximate abstractions of stochastic hybrid systems," *IEEE Trans. Automat. Control*, vol. 56, no. 11, pp. 2688–2694, Nov. 2011.
- [2] A. Abate, M. Prandini, J. Lygeros, and S. Sastry, "Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems," *Automatica*, vol. 44, no. 11, pp. 2724–2734, 2008.
- [3] R. Alur, T. Henzinger, G. Lafferriere, and G. Pappas, "Discrete abstractions of hybrid systems," *Proc. IEEE*, vol. 88, no. 7, pp. 971–984, Jul. 2000.
- [4] R. Blute, J. Desharnais, A. Edalat, and R. Panangaden, "Bisimulation for labelled Markov processes," in *Proc. 12th Annu. IEEE Symp. Logic Comput. Sci.*, 1997, pp. 149–158.
- [5] F. V. Breugel, M. Mislove, J. Ouaknine, and J. Worrell, "An intrinsic characterization of approximate probabilistic bisimilarity," in *Proc. Int. Conf. Found. Softw. Sci. Comput. Struct.*, 2003, pp. 200–215.
- [6] M. Bujorianu and J. Lygeros, "Reachability questions in piecewise deterministic Markov processes," in *Hybrid Systems: Computation and Control* (ser. Lecture Notes on Control and Information Sciences), vol. 2623, O. Maler and A. Pnueli, Eds. Berlin, Germany: Springer-Verlag, 2003, pp. 126–140.
- [7] M. Bujorianu, J. Lygeros, and M. Bujorianu, "Bisimulation for general stochastic hybrid systems," in *Hybrid Systems: Computation and Control* (ser. Lecture Notes in Computer Science), vol. 3414, M. Morari and L. Thiele, Eds. Berlin, Germany: Springer-Verlag, 2005, pp. 198–214.
- [8] P. Caspi and A. Benveniste, "Toward an approximation theory for computerized control," in *Proc. Int. Workshop Embedded Softw.*, 2002, pp. 294–304.
- [9] E. Clarke, O. Grumberg, and D. Peled, *Model Checking*. Cambridge, MA, USA: MIT Press, 1999.
- [10] L. de Alfaro, M. Faella, and M. Stoelinga, "Linear and branching metrics for quantitative transition systems," in *Proc. Int. Colloq. Automata Lang. Program.*, 2004, pp. 97–109.
- [11] J. Desharnais, V. Gupta, R. Jagadeesan, and P. Panangaden, "Approximating labeled Markov processes," in *Proc. 15th Annu. IEEE Symp. Logic Comput. Sci.*, Osaka, Japan, Jun. 2000, pp. 95–106.
- [12] J. Desharnais, V. Gupta, R. Jagadeesan, and P. Panangaden, "Metrics for labelled Markov processes," *Theor. Comput. Sci.*, vol. 318, no. 3, pp. 323–354, Jun. 2004.
- [13] M. Di Benedetto and G. Pola, "Networked embedded control systems: From modelling to implementation," in *Proc. 3rd Workshop Hybrid Auton. Syst.*, 2014, pp. 9–13.
- [14] A. Girard and G. Pappas, "Approximation metrics for discrete and continuous systems," *IEEE Trans. Automat. Control*, vol. 52, no. 5, pp. 782–798, May 2007.

- [15] A. Girard and G. Pappas, "Approximate bisimulation: A bridge between computer science and control theory," *Eur. J. Control*, vol. 17, nos. 5/6, pp. 568–578, 2011.
- [16] E. Haghverdi, P. Tabuada, and G. Pappas, "Bisimulation relations for dynamical and control systems," *Electron. Notes Theor. Comput. Sci.*, vol. 69, pp. 120–136, 2003.
- [17] H. Hermanns, *Interactive Markov Chains: And the Quest for Quantified Quality*. Berlin, Germany: Springer-Verlag, 2002.
- [18] A. Julius and G. Pappas, "Approximations of stochastic hybrid systems," *IEEE Trans. Automat. Control*, vol. 54, no. 6, pp. 1193–1203, Jun. 2009.
- [19] O. Junge, "A set oriented approach to global optimal control," *ESAIM: Control, Optim. Calculus Variations*, vol. 10, no. 2, pp. 259–270, 2004.
- [20] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1980.
- [21] K. G. Larsen and A. Skou, "Bisimulation through probabilistic testing," *Inf. Comput.*, vol. 94, no. 1, pp. 1–28, Sep. 1991.
- [22] J. Lunze and B. Nixdorf, "Representation of hybrid systems by means of stochastic automata," *Math. Comput. Model. Dyn. Syst.*, vol. 7, no. 4, pp. 383–422, 2001.
- [23] R. Milner, *Communication and Concurrency*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1989.
- [24] S. Mirzazad-Barijough and J. Lee, "Stability and transient performance of discrete-time piecewise affine systems," *IEEE Trans. Automat. Control*, vol. 57, no. 4, pp. 936–949, Apr. 2012.
- [25] D. Park, "Concurrency and automata on infinite sequences," in *Theoretical Computer Science* (ser. Lecture Notes in Computer Science), vol. 104. New York, NY, USA: Springer-Verlag, 1981, pp. 167–183.
- [26] A. D. Pierro, C. Hankin, and H. Wiklicky, "Quantitative relations and approximate process equivalences," in *Proc. Int. Conf. Concurrency Theory*, 2003, pp. 508–522.
- [27] G. Pola and M. Di Benedetto, "Symbolic models and control of discrete-time piecewise affine systems: An approximate simulation approach," *IEEE Trans. Automat. Control*, vol. 59, no. 1, pp. 175–180, Jan. 2014.
- [28] G. Pola, J. Lygeros, and M. Di Benedetto, "Invariance in Stochastic Dynamical Systems," in *Proc. 17th Int. Symp. Math. Theory Netw. Syst.*, Kyoto, Japan, Jul. 2006.
- [29] G. Pola, C. Manes, and M. D. Benedetto, "On external behavior equivalence of continuous-time stochastic linear control systems," in *Proc. 55th IEEE Conf. Decision Control*, Las Vegas, NV, USA, Dec. 2016, pp. 6583–6588.
- [30] G. Pola, C. Manes, A. van der Schaft, and M. D. Benedetto, "Equivalence notions for discrete-time stochastic linear control systems," *arXiv:1611.08352 [math.OC]*, 2016.
- [31] G. Pola, C. Manes, A. van der Schaft, and M. D. Benedetto, "Model reduction of continuous-time stochastic linear control systems via bisimulation equivalence," in *Proc. 55th IEEE Conf. Decision Control*, Las Vegas, NV, USA, Dec. 2016, pp. 6577–6582.
- [32] G. Pola, C. Manes, A. van der Schaft, and M. Di Benedetto, "On equivalence notions for discrete-time stochastic control systems," in *Proc. 54th IEEE Conf. Decision Control*, Osaka, Japan, Dec. 2015, pp. 1180–1185.
- [33] G. Pola and G. Pola, "A stochastic reachability approach to portfolio construction in finance industry," *IEEE Trans. Control Syst. Technol.*, vol. 20, no. 1, pp. 189–195, Jan. 2012.
- [34] G. Pola, A. van der Schaft, and M. Di Benedetto, "Equivalence of switching linear systems by bisimulation," *Int. J. Control*, vol. 79, pp. 74–92, Jan. 2006.
- [35] G. Reißig, "Computation of discrete abstractions of arbitrary memory span for nonlinear sampled systems," in *Proc. 12th Int. Conf. Hybrid Syst.: Comput. Control*, Apr. 2009, pp. 306–320.
- [36] J. Schroder, *Lecturing, State Observation and Diagnosis of Quantised Systems* (ser. Lecture Notes in Control and Information Sciences). New York, NY, USA: Springer, 2003.
- [37] S. Strubbe and A. van der Schaft, "Bisimulation for communicating piecewise deterministic Markov processes (CPDPs)," in *Hybrid Systems: Computation and Control* (ser. Lecture Notes in Computer Science), vol. 3414, M. Morari and L. Thiele, Eds. Berlin, Germany: Springer-Verlag, 2005, pp. 623–639.
- [38] S. Strubbe and A. van der Schaft, "Communicating piecewise deterministic Markov processes," in *Stochastic Hybrid Systems: Theory and Safety Critical Applications* (ser. Lecture Notes on Control and Information Sciences), vol. 337, H. Blom and J. Lygeros, Eds. Berlin, Germany: Springer-Verlag, 2006, pp. 65–104.
- [39] P. Tabuada, *Verification and Control of Hybrid Systems: A Symbolic Approach*. New York, NY, USA: Springer, 2009.
- [40] A. van der Schaft, "Equivalence of dynamical systems by bisimulation," *IEEE Trans. Automat. Control*, vol. 49, no. 12, pp. 2160–2172, Dec. 2004.
- [41] R. van Glabbeek, "The linear time-branching time spectrum," in *Proc. Int. Conf. Concurrency Theory*, 1990, pp. 278–297.
- [42] J. Willems, "Open stochastic systems," *IEEE Trans. Automat. Control*, vol. 58, no. 2, pp. 406–421, Feb. 2013.
- [43] B. Yordanov and C. Belta, "Formal analysis of discrete-time piecewise affine systems," *IEEE Trans. Automat. Control*, vol. 55, no. 12, pp. 2834–2840, Dec. 2010.
- [44] B. Yordanov, J. Tumova, I. Cerna, J. Barnat, and C. Belta, "Temporal logic control of discrete-time piecewise affine systems," *IEEE Trans. Automat. Control*, vol. 57, no. 6, pp. 1491–1504, Jun. 2012.
- [45] M. Zamani and A. Abate, "Approximately bisimilar symbolic models for randomly switched stochastic systems," *Syst. Control Lett.*, vol. 59, no. 12, pp. 3135–3150, 2014.
- [46] M. Zamani, P. Esfahani, R. Majumdar, A. Abate, and J. Lygeros, "Symbolic control of stochastic systems via approximately bisimilar finite abstractions," *IEEE Trans. Automat. Control*, vol. 59, no. 12, pp. 3135–3150, Nov. 2014 (Special Issue on Control of Cyber-Physical Systems).
- [47] M. Zamani and M. Rungger, "Compositional construction of approximate abstractions of interconnected control systems," *IEEE Trans. Control Netw. Syst.*, 2017, to be published.
- [48] M. Zamani, M. Rungger, and P. M. Esfahani, "Construction of approximations of stochastic control systems: A compositional approach," in *Proc. 54th IEEE Conf. Decision Control*, Osaka, Japan, Dec. 2015, pp. 525–530.
- [49] M. Zamani, M. Rungger, and P. M. Esfahani, "Approximations of stochastic hybrid systems: A compositional approach," *IEEE Trans. Automat. Control*, vol. 62, no. 6, pp. 2838–2853, Jun. 2017.

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