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Input–output finite-region stability and stabilization for discrete 2-D Fornasini–Marchesini models

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ABSTRACT

This paper establishes input–output finite-region stability (I–O FRS) and stabilization analysis methods to investigate the transient behavior of output signals for discrete two-dimensional (2-D) Fornasini–Marchesini (FM) models. For 2-D systems with two classes of input signals, by building special recursive formulas, sufficient I–O FRS conditions are formulated in terms of solvability of linear matrix inequalities (LMIs). Furthermore, aiming at designing static state feedback controllers, I–O finite-region stabilization issues are analyzed for the corresponding 2-D systems and sufficient conditions are reported to guarantee I–O FRS of the closed-loop systems. To conclude the paper, we provide some numerical examples to confirm the validity of the proposed methods.

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1. Introduction

In recent decades, discrete two-dimensional (2-D) models have been widely applied in iterative learning, image processing, satellite cloud picture analysis, seismological and geographical data processing, X-ray image enhancement, and other fields [1–3]. As is well known, Lyapunov asymptotic stability (LAS) theory [4–7] is a fundamental and important control issue for 2-D models. After decades of development, research on this topic has become quite sophisticated and has witnessed several notable achievements. To name a few, stability analysis for 2-D systems has been successfully investigated in [4,8] as well as stabilization and H_∞ control problems for 2-D systems with Markovian jump parameters [9]. Chen and Lam [10] explored stability analysis and control synthesis for 2-D fuzzy systems via basis dependent Lyapunov functions. Recent scholars have focused on stability issues of 2-D discrete switched systems, for example, the stability analysis of discrete-time 2-D switched systems in the Roesser form [11]. Xiang and Huang [12] studied stability analysis and stabilization problems for 2-D discrete switched systems also described by a Roesser model, and other researchers have focused on stability analysis and stabilization problems for Fornasini–Marchesini type 2-D switched systems in [13].

In addition to LAS, bounded-input bounded-output (BIBO) stability is also a basic concept in 2-D systems. BIBO stability means that the output signal $y(h, k)$ of a 2-D system is norm-bounded, if the input signal is norm-bounded. Note that BIBO stability is

related to the output $y(h, k)$ over the whole first quadrant, (i.e., $0 \leq h, k < \infty$). However, in certain practical applications, systems only operate over a finite region, $\{(h, k) | h + k \leq N, N \in \mathbb{N}_+\}$, and the systems' input, output and state signals are required to be limited within a specific bound. In this case, the transient performance of systems makes sense, however, it is not reasonable to consider the transient performance of the output $y(h, k)$ via BIBO stability theory. In this paper, we will focus on this issue for classical discrete 2-D Fornasini–Marchesini (FM) models.

In the case of dynamical systems [14] governed by ordinary differential equations, input–output stability (IOS) and input-to-state stability (ISS) are other effective tools for studying the property of output and state signals respectively. In addition, it is implied in [14] that IOS is related to ISS, and the corresponding conclusions are provided to describe the relationship between IOS and ISS. Moreover, consider a linear dynamical system [15,16]. It is concluded that if the control input converges to zero as time goes to infinity, then the state of the controlled system converges to zero. Together with LAS and BIBO, these issues consider the performance of output or state signals over a sufficiently long time interval (in principle $t \rightarrow \infty$), and then, it fails to describe the transient performance by these stability analyses. In 1-D state-space models, finite-time stability (FTS) and finite-time boundedness (FTB) are used to determine the transient performance of systems [17–28]. It describes that systems' state $x(k)$ satisfies a given bound during a specified time interval, so the concepts are very practical. For 2-D systems, Zhang and Wang [29,30] considered finite-region stability (FRS) and boundedness (FRB) issues, which analyzed transient performance of state variables in given finite regions for discrete Roesser models and FMII models respectively.

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In recent years, the concept of input–output finite-time stability (I–O FTS) has been introduced by Amato, Ambrosino. Generally speaking, a system is said to be I–O FTS, if for a class of norm-bounded input signals defined over a time interval, the output $y(k)$ does not exceed an assigned bound during the same time interval [31]. In other words, I–O FTS shows the transient performance of output $y(k)$ while FTS describes that of state $x(k)$. Up to now, there have been many valuable contributions on I–O FTS. For example, the I–O FTS issues of discrete-time linear systems and hybrid systems were studied in [32] and [33] respectively. In addition, Amato and Ambrosino [34] provided necessary and sufficient conditions for I–O FTS of linear systems. I–O FTS and stabilization of stochastic Markovian jump systems were studied in detail by Ma and Jia [35], as well as I–O FTS conditions in singular linear systems in [36]. Recently, in [37], the I–O FTS problem of discrete-time impulsive switched linear systems with state delays was considered.

Different from 1-D systems, most 2-D systems are characterized by two independent variables propagating information in two directions. In this case, there is no explicit time concept and it is impossible to formulate the concept of I–O FTS directly for such 2-D systems. For classical discrete 2-D FM models, it is expected that for the given input signals class $\mathcal{W}$, the transient performance of output signals is reflected by the behavior of $y(h, k)$, which is required to be bounded over a finite-region $\{(h, k) | h + k \leq N, N \in \mathbb{N}_+\}$. Of course, the input signal classes in 2-D systems should be introduced at first and such transient performance of output $y(h, k)$ cannot be obtained by BIBO stability. Motivated by the above discussions, in this paper, we are not only to address I–O FRS stability problem for discrete 2-D FM models, but also to formulate methods for transient performance analysis of output signals.

The most notable contributions of this paper can be summarized as follows: (1) The concept of I–O FRS is formulated for discrete 2-D FM models. (2) For two different classes of input signals, the I–O FRS problems are addressed accordingly by establishing special recursive formulas, and the corresponding sufficient conclusions and linear matrix inequalities (LMIs) conditions are also derived. (3) For designing proper feedback controllers, the I–O finite-region stabilization problems for the corresponding 2-D systems are tackled and the desired conclusions are obtained.

The remainder of this paper is organized as follows. In Section 2, background information, such as signal classes, the definition of I–O FRS, is extended to discrete 2-D FM models. In Section 3, I–O FRS issues are taken into consideration for two different class of input signals respectively, and the corresponding criteria and LMI conditions are derived. Section 4 addresses the I–O finite-region stabilization problems also under the above two classes of exogenous disturbances and obtains criteria and LMI conditions. Section 5 presents numerical examples to describe the use of the proposed results.

Remarks on notation: In this paper, we assume that vectors and matrices are always real and have appropriate dimensions. A matrix $P > 0$, means that it is a symmetric, positive definite real matrix; we denote $\mathbb{N}_+$ and $\mathbb{N}$ as the positive integer set and nonnegative integer set respectively. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and minimum eigenvalues of a symmetric matrix P . Superscript T denotes the transpose of a real vector or matrix.

2. Background information

In this paper, the following discrete 2-D FM model,

$$\begin{cases} x(h+1, k+1) = A_1x(h+1, k) + A_2x(h, k+1) \\ \quad + B_1u(h+1, k) + B_2u(h, k+1) \\ \quad + G_1w(h+1, k) + G_2w(h, k+1) \\ y(h, k) = Cx(h, k) + Dw(h, k), \end{cases} \quad (1)$$

which is defined on $\mathbb{N} \times \mathbb{N}$, will be considered, where $x(\cdot, \cdot) : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^n$ is the state vector, $u(\cdot, \cdot) : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^m$ is the control input, $w(\cdot, \cdot) \in \mathcal{W} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^q$ is the exogenous disturbance. A_1, A_2, B_1, B_2, G_1 and G_2 are matrices with appropriate dimensions.

In this paper, we denote a finite-region by the following:

$$\mathbf{S}(N) = \{(h, k) \in \mathbb{N} \times \mathbb{N} | h + k \leq N\}$$

with pre-given $N \in \mathbb{N}_+$.

Definition 2.1. For given scalars $N > 0, c > 0$ a matrix $Q > 0$ and a class of input signal $\mathcal{W}$ defined over $\mathbf{S}(N)$, the 2-D system (4) with boundary condition satisfying $x(h, 0) = 0$ and $x(0, k) = 0$, that is,

$$\begin{cases} x(h+1, k+1) = A_1x(h+1, k) + A_2x(h, k+1) \\ \quad + G_1w(h+1, k) + G_2w(h, k+1) \\ y(h, k) = Cx(h, k) + Dw(h, k), \\ x(h, 0) = 0 \text{ and } x(0, k) = 0, \end{cases} \quad (2)$$

is said to be I–O FRS with respect to $(\mathcal{W}, Q, N, c)$, if

$$w(\cdot, \cdot) \in \mathcal{W} \implies y^T(h, k)Qy(h, k) < c, \quad (h, k) \in \mathbf{S}(N). \quad (3)$$

Remark 1. It is worth noting that a 2-D system can be I–O FRS but not BIBO stable and vice versa. This fact is checked via Example 1 below.

In [30], authors presented the concept of finite-region boundedness (FRB) for the discrete FM model, and there is some certain relationship between FRB and I–O FRS. To clarify this point, the following definition needs to be introduced firstly.

Definition 2.2 ([30]). For given scalars c, c_1, c_2 with $0 < c_1, c_2 < c, N \in \mathbb{N}_+$ and matrices $R_1 > 0, R_2 > 0$ and signal class $\mathcal{W}$, the system (1) with zero control input, that is,

$$\begin{cases} x(h+1, k+1) = A_1x(h+1, k) + A_2x(h, k+1) \\ \quad + G_1w(h+1, k) + G_2w(h, k+1) \\ y(h, k) = Cx(h, k) + Dw(h, k), \end{cases} \quad (4)$$

is said to be FRB with respect to $(c, c_1, c_2, N, R_1, R_2, \mathcal{W})$, if

$$\begin{cases} x^T(h, 0)R_1x(h, 0) \leq c_1; \\ x^T(0, k)R_2x(0, k) \leq c_2 \end{cases} \implies \begin{cases} x^T(h, k)R_1x(h, k) < c; \\ x^T(h, k)R_2x(h, k) < c, \end{cases}$$

for any $(h, k) \in \mathbf{S}(N)$ and $w(\cdot, \cdot) \in \mathcal{W}$.

Remark 2. Learning from the definition of I–O FRS, the boundary conditions are always supposed to satisfy $x(h, 0) = 0$ and $x(0, k) = 0$, while these conditions do not necessarily be zero in the concept of FRB.

Remark 3. By Definition 2.2, if the 2-D FMII model (4) is FRB with respect to given parameter set $(c, c_1, c_2, N, R_1, R_2, \mathcal{W})$, then for

$$\begin{cases} x^T(h, 0)R_1x(h, 0) \leq c_1; \\ x^T(0, k)R_2x(0, k) \leq c_2 \end{cases} \implies \begin{cases} x^T(h, k)R_1x(h, k) < c; \\ x^T(h, k)R_2x(h, k) < c, \end{cases}$$

for any $(h, k) \in \mathbf{S}(N)$ and $w(\cdot, \cdot) \in \mathcal{W}$. Together with the output system:

$$y(h, k) = Cx(h, k) + Dw(h, k),$$

then a strict upper bound $\hat{c}$ of weighted-output $y^T(h, k)Qy^T(h, k)$ can be obtained. Obviously, the upper bound $\hat{c}$ depends on the parameter set $(c, c_1, c_2, N, R_1, R_2, \mathcal{W})$. However, when considering I–O FRS issue, we need to introduce another parameter index set $(\mathcal{W}, Q, N, \tilde{c})$. Then, bound $\tilde{c}$ has different meaning with bound $\hat{c}$. To be exact, the bound $\tilde{c}$ of weighted-output $y^T(h, k)Qy^T(h, k)$ is pre-given but not calculated by using weighted-state values and the output system, even though bounds $\tilde{c}$ and $\hat{c}$ are accidental equal.

On the other hand, as mentioned in Remark 2, for I–O FRS problem, we require boundary conditions $x(h, 0) = x(0, k) = 0$ for any $(h, k) \in \mathbf{S}(N)$. Under this zero-boundary conditions assumption, and as considered later in Theorem 3.1, we claim that the bound c for weighted-state $x^T(h, k)Px(h, k)$ over the same finite region $\mathbf{S}(N)$. Obviously, this bound c is also not pre-given, which is different from FRB issue.

Thus it can be seen for 2-D FMII model (4), I–O FRS and FRB in our earlier works [30] involve two different issues.

In the rest of this paper, input signal class $\mathcal{W}$ corresponds to either of the following two different classes of signals. For the given scalars $c_1 > 0$, $c_2 > 0$ and a matrix $R > 0$, signal classes $\mathcal{W}_2(N, c_1, R)$ and $\mathcal{W}_\infty(N, c_2, R)$ are defined as follows:

$$(i) \quad \mathcal{W}_2(N, c_1, R) = \{w(\cdot, \cdot) : \sum_{(h,k) \in \mathbf{S}(N)} w^T(h, k)Rw(h, k) \leq c_1\},$$

$$(ii) \quad \mathcal{W}_\infty(N, c_2, R) = \{w(\cdot, \cdot) : w^T(h, k)Rw(h, k) \leq c_2, (h, k) \in \mathbf{S}(N)\}.$$

In order to simplify the notation, the simplified notations $\mathcal{W}_2$ and $\mathcal{W}_\infty$ will be adopted in the rest of the paper.

At the end of this section, we provide a lemma which will be used frequently in the following discussion.

Lemma 2.1 ([38]). For arbitrary matrices X, Y and $P > 0$ with suitable dimensions, we have:

$$X^T Y + Y^T X \leq X^T P X + Y^T P^{-1} Y. \quad (5)$$

3. Input–output finite-region stability

In this section, we first focus on the I–O FRS problem for the system (2) under exogenous disturbances from the class $\mathcal{W}_2$. Subsequently, the corresponding analysis is discussed for the system (2) under exogenous disturbances from $\mathcal{W}_\infty$.

3.1. The case of exogenous disturbances in $\mathcal{W}_2$

Consider the system (2) with exogenous disturbance signals in $\mathcal{W}_2$. Then its I–O FRS criterion is formulated in the following theorem.

Theorem 3.1. Given scalars $N > 0$, $c \geq c_1 > 0$ and matrices $Q > 0$, $R > 0$, the 2-D system (2) is I–O FRS with respect to $(\mathcal{W}_2, Q, N, c)$, if there exist scalars $\alpha > 0$, $0 < \mu < 1$, $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$ and matrices $P > 0$, $H > 0$, such that

$$A^T P A - \Pi_1 + A^T P G H^{-1} G^T P A < 0, \quad (6)$$

$$G^T P G + H < \Lambda \tilde{R}, \quad (7)$$

$$C^T Q C \leq \frac{\mu}{2} P, \quad (8)$$

$$D^T Q D \leq \frac{(1 - \mu)}{2} R \quad (9)$$

$$(ii) \quad \alpha N c_1 + (\lambda_1 + \lambda_2) c_1 < c, \quad (10)$$

$$\text{where } A = [A_1 \ A_2], \ G = [G_1 \ G_2], \ \Pi_1 = \begin{bmatrix} P & 0 \\ 0 & \frac{c_1}{c} \alpha P \end{bmatrix}, \ \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}.$$

Proof. Consider the system (2) with $w(\cdot, \cdot) \in \mathcal{W}_2$. Obviously, it can be rewritten as,

$$\begin{aligned} x(h+1, k+1) &= [A_1 \ A_2] \begin{bmatrix} x(h+1, k) \\ x(h, k+1) \end{bmatrix} + [G_1 \ G_2] \begin{bmatrix} w(h+1, k) \\ w(h, k+1) \end{bmatrix} \\ &= A \tilde{x}(h, k) + G \tilde{w}(h, k), \end{aligned} \quad (11)$$

where $A = [A_1 \ A_2]$, $G = [G_1 \ G_2]$, $\tilde{x}(h, k) = \begin{bmatrix} x(h+1, k) \\ x(h, k+1) \end{bmatrix}$ and $\tilde{w}(h, k) = \begin{bmatrix} w(h+1, k) \\ w(h, k+1) \end{bmatrix}$.

For $P > 0$, define the following Lyapunov function,

$$V(x) = x^T P x, \quad (12)$$

then for any $(h, k) \in \mathbf{S}(N)$,

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) - x^T(h, k+1) \tilde{P}_1 x(h, k+1) \\ &= [A \tilde{x}(h, k) + G \tilde{w}(h, k)]^T P [A \tilde{x}(h, k) \\ &\quad + G \tilde{w}(h, k)] - x^T(h+1, k) P x(h+1, k) \\ &\quad - x^T(h, k+1) \tilde{P}_1 x(h, k+1) \\ &= \tilde{x}^T(h, k) A^T P A \tilde{x}(h, k) - \tilde{x}^T(h, k) \Pi_1 \tilde{x}(h, k) \\ &\quad + \tilde{w}^T(h, k) G^T P G \tilde{w}(h, k) \\ &\quad + \tilde{x}^T(h, k) A^T P G \tilde{w}(h, k) + \tilde{w}^T(h, k) G^T P A \tilde{x}(h, k), \end{aligned} \quad (13)$$

where $\Pi_1 = \begin{bmatrix} P & 0 \\ 0 & \tilde{P}_1 \end{bmatrix}$ and $\tilde{P}_1 = \frac{c_1}{c} \alpha P$. According to condition (6), then

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) - x^T(h, k+1) \tilde{P}_1 x(h, k+1) \\ &< \tilde{x}^T(h, k) A^T P G \tilde{w}(h, k) + \tilde{w}^T(h, k) G^T P A \tilde{x}(h, k) \\ &\quad - \tilde{x}^T(h, k) A^T P G H^{-1} G^T P A \tilde{x}(h, k) + \tilde{w}^T(h, k) G^T P G \tilde{w}(h, k). \end{aligned} \quad (14)$$

Let $v = H^{\frac{1}{2}} \tilde{w} - H^{-\frac{1}{2}} G^T P A \tilde{x}$, then

$$\begin{aligned} v^T(h, k) v(h, k) &= [H^{\frac{1}{2}} \tilde{w}(h, k) - H^{-\frac{1}{2}} G^T P A \tilde{x}(h, k)]^T [H^{\frac{1}{2}} \tilde{w}(h, k) \\ &\quad - H^{-\frac{1}{2}} G^T P A \tilde{x}(h, k)] \\ &= \tilde{w}^T(h, k) H \tilde{w}(h, k) + \tilde{x}^T(h, k) A^T P G H^{-1} G^T P A \tilde{x}(h, k) \\ &\quad - \tilde{x}^T(h, k) A^T P G \tilde{w}(h, k) - \tilde{w}^T(h, k) G^T P A \tilde{x}(h, k), \end{aligned} \quad (15)$$

which results in

$$\begin{aligned} \tilde{x}^T(h, k) A^T P G \tilde{w}(h, k) + \tilde{w}^T(h, k) G^T P A \tilde{x}(h, k) \\ &\quad - \tilde{x}^T(h, k) A^T P G H^{-1} G^T P A \tilde{x}(h, k) = \tilde{w}^T(h, k) H \tilde{w}(h, k) \\ &\quad - v^T(h, k) v(h, k). \end{aligned} \quad (16)$$

Together with conditions (7) and (16), the following holds,

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) - x^T(h, k+1) \tilde{P}_1 x(h, k+1) \\ &< \tilde{w}^T(h, k) H \tilde{w}(h, k) + \tilde{w}^T(h, k) G^T P G \tilde{w}(h, k) - v^T(h, k) v(h, k) \\ &< \tilde{w}^T(h, k) \Lambda \tilde{R} \tilde{w}(h, k), \end{aligned} \quad (17)$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$. Then, we conclude that:

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) \\ &< x^T(h, k+1) \tilde{P}_1 x(h, k+1) + \lambda_1 w^T(h+1, k) R w(h+1, k) \\ &\quad + \lambda_2 w^T(h, k+1) R w(h, k+1). \end{aligned} \quad (18)$$

Similarly, we have:

$$\begin{aligned} V(x(h+1, k)) - V(x(h+1, k-1)) \\ < x^T(h, k)\tilde{P}_1x(h, k) + \lambda_1 w^T(h+1, k-1)Rw(h+1, k-1) \\ + \lambda_2 w^T(h, k)Rw(h, k). \end{aligned} \quad (19)$$

$\vdots$

$$\begin{aligned} V(x(h+1, 2)) - V(x(h+1, 1)) \\ < x^T(h, 2)\tilde{P}_1x(h, 2) + \lambda_1 w^T(h+1, 1)Rw(h+1, 1) \\ + \lambda_2 w^T(h, 2)Rw(h, 2), \end{aligned} \quad (20)$$

$$\begin{aligned} V(x(h+1, 1)) - V(x(h+1, 0)) \\ < x^T(h, 1)\tilde{P}_1x(h, 1) + \lambda_1 w^T(h+1, 0)Rw(h+1, 0) \\ + \lambda_2 w^T(h, 1)Rw(h, 1). \end{aligned} \quad (21)$$

By summing these inequalities, we find that

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, 0)) \\ < \sum_{j=1}^{k+1} x^T(h, j)\tilde{P}_1x(h, j) + \lambda_1 \sum_{j=0}^k w^T(h+1, j)Rw(h+1, j) \\ + \sum_{j=1}^{k+1} \lambda_2 w^T(h, j)Rw(h, j). \end{aligned} \quad (22)$$

Noting that $x(h, 0) = 0$ and $w(\cdot, \cdot) \in \mathcal{W}_2$, that is $\sum_{(h,k) \in \mathbf{S}(N)} w^T(h, k)Rw(h, k) \leq c_1$, the following can be checked, for any $(h, k) \in \mathbf{S}(N)$,

$$\begin{aligned} x^T(h, k)Px(h, k) = V(x(h, k)) < \sum_{j=1}^k x^T(h-1, j)\tilde{P}_1x(h-1, j) \\ + (\lambda_1 + \lambda_2)c_1. \end{aligned} \quad (23)$$

On the other hand, for any $(h, k) \in \mathbf{S}(N)$, by using Lemma 2.1, we have:

$$\begin{aligned} y^T(h, k)Qy(h, k) &= (Cx(h, k) + Dw(h, k))^TQ(Cx(h, k) + Dw(h, k)) \\ &= x^T(h, k)C^TQCx(h, k) + x^T(h, k)C^TQDw(h, k) \\ &\quad + w^T(h, k)D^TQCx(h, k) + w^T(h, k)D^TQDw(h, k) \\ &\leq 2x^T(h, k)C^TQCx(h, k) + 2w^T(h, k)D^TQDw(h, k). \end{aligned} \quad (24)$$

By using conditions (8) and (9), we obtain:

$$\begin{aligned} y^T(h, k)Qy(h, k) &\leq 2x^T(h, k)C^TQCx(h, k) + 2w^T(h, k)D^TQDw(h, k) \\ &\leq \mu x^T(h, k)Px(h, k) + (1 - \mu)w^T(h, k)Rw(h, k). \end{aligned} \quad (25)$$

In the sequel, by mathematical induction to $l = h + k \leq N$ and condition (23), we will verify that, for any $(h, k) \in \mathbf{S}(N)$,

$$x^T(h, k)Px(h, k) < c. \quad (26)$$

For $h + k = 1$, that is $(h, k) \in \{(0, 1), (1, 0)\}$, according to the known boundary conditions, $x(1, 0) = x(0, 1) = 0 < c$, which implies inequality (26).

For $h + k = 2$, that is $(h, k) \in \{(2, 0); (0, 2); (1, 1)\}$, note that $\lambda_1 + \lambda_2 < 1$ and $0 < c_1 \leq c$, then by using condition (23) and boundary condition, $x(0, 1) = 0$,

$$\begin{aligned} x^T(1, 1)Px(1, 1) &= x^T(0, 1)\tilde{P}_1x(0, 1) + (\lambda_1 + \lambda_2)c_1 \\ &= (\lambda_1 + \lambda_2)c_1 \leq c, \end{aligned}$$

which implies inequality (26), where $\tilde{P}_1 = \frac{c_1}{c}\alpha P$.

Suppose that (26) holds for any $(h, k) \in \mathbf{S}(N)$ satisfying $h + k \leq N - 1$, we will show that if $h + k = N$, (26) is also satisfied. Firstly, note that the known boundary conditions $x(h, 0) = x(0, k) = 0$ for

any $(h, k) \in \mathbb{N} \times \mathbb{N}$, so it is also satisfied for $h + k = N$. According to the induction hypothesis, (26) holds for any $(h, k) \in \mathbf{S}(N)$ with $h + k \leq N - 1$, and by taking into account conditions (10) and (23),

$$\begin{aligned} x^T(N-1, 1)Px(N-1, 1) \\ &\leq x^T(N-2, 1)\tilde{P}_1x(N-2, 1) + (\lambda_1 + \lambda_2)c_1 \\ &< \alpha c_1 + (\lambda_1 + \lambda_2)c_1 \leq c. \end{aligned} \quad (27)$$

Continuing to use the induction assumptions and inequalities (10) and (23), we have,

$$\begin{aligned} x^T(N-2, 2)Px(N-2, 2) \\ &\leq [x^T(N-3, 2)\tilde{P}_1x(N-3, 2) \\ &\quad + x^T(N-3, 1)\tilde{P}_1x(N-3, 1)] + (\lambda_1 + \lambda_2)c_1 \\ &< 2\alpha c_1 + (\lambda_1 + \lambda_2)c_1 \leq c. \end{aligned} \quad (28)$$

Therefore, along lines as above and with the help of conditions (10) and (23), for any $0 \leq k \leq T$, we derive,

$$\begin{aligned} x^T(N-k, k)Px(N-k, k) \\ &\leq \sum_{l=1}^k x^T(N-k-1, j)\tilde{P}_1x(N-k-1, j) + (\lambda_1 + \lambda_2)c_1 \\ &< \alpha Nc_1 + (\lambda_1 + \lambda_2)c_1 \leq c. \end{aligned} \quad (29)$$

Noting that (29) holds $0 \leq k \leq N$, it can be rewritten equivalently as follows,

$$x^T(h, k)Px(h, k) < c, \quad (30)$$

for any $(h, k) \in \mathbf{S}(N)$.

Finally, by inequality (25), it can be seen that for any $(h, k) \in \mathbf{S}(N)$,

$$\begin{aligned} y^T(h, k)Qy(h, k) &\leq \mu x^T(h, k)Px(h, k) + (1 - \mu)w^T(h, k)Rw(h, k) \\ &\leq \mu c + (1 - \mu)c = c, \end{aligned} \quad (31)$$

which implies that the system (2) is I-O FRS with respect to $(\mathcal{W}_2, Q, N, c)$, and this completes the proof. $\square$

Obviously, the term $A^T P G H^{-1} G^T P A$ in condition (6) is nonlinear and so this condition does not have the form of an LMI. By using Schur complements however, the following result can immediately be obtained:

Theorem 3.2. Given scalars $N > 0, c \geq c_1 > 0$ and matrices $Q > 0, R > 0$, the 2-D system (2) is I-O FRS with respect to $(\mathcal{W}_2, Q, N, c)$, if there exist scalars $\alpha > 0, 0 < \mu < 1, \lambda_1 > 0, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$ and matrices $P > 0, H > 0$, such that

$$\begin{bmatrix} A^T P A - \Pi_1 & A^T P G \\ G^T P A & -H \end{bmatrix} < 0, \quad (32)$$

and

(ii) conditions (7)–(10) in Theorem 3.1 are satisfied,

$$\begin{aligned} \text{where } A &= [A_1 \ A_2], \ G = [G_1 \ G_2], \ \Pi_1 = \begin{bmatrix} P & 0 \\ 0 & \frac{c_1}{c}\alpha P \end{bmatrix}, \\ \Lambda &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}. \end{aligned}$$

Remark 4. It is worth pointing that inequality (5) also can be used to obtain another form of I-O FTS criterion and it is formulated as follows.

Proposition 3.1. For given scalars $N > 0, c \geq c_1 > 0$ and matrices $Q > 0, R > 0$, the 2-D system (2) is I-O FRS with respect to $(\mathcal{W}_2, Q, N, c)$, if there exist scalars $\alpha > 0, \theta > 0, 0 < \mu < 1, \lambda_1 > 0, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$ and matrices $P > 0, H > 0$, such that

(i)

$$(1 + \theta)A^T PA - \Pi_1 < 0, \quad (33)$$

$$(1 + \frac{1}{\theta})G^T PG < \Lambda \tilde{R}, \quad (34)$$

(ii) conditions (8)–(10) in Theorem 3.1 are satisfied, where A , G , Π_1 and $\tilde{R}$ are described as in Theorem 3.1.

Proof. Using a similar analysis as Eq.(13), we have

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) - x^T(h, k+1)\tilde{P}_1 x(h, k+1) \\ = \tilde{x}^T(h, k)A^T PA \tilde{x}(h, k) + \tilde{x}^T(h, k)A^T PG \tilde{w}(h, k) \\ + \tilde{w}^T(h, k)G^T PA \tilde{x}(h, k) + \tilde{w}^T(h, k)G^T PG \tilde{w}(h, k) \\ - \tilde{x}^T(h, k)\Pi_1 \tilde{x}(h, k), \end{aligned} \quad (35)$$

where $\Pi_1 = \begin{bmatrix} P & 0 \\ 0 & \tilde{P}_1 \end{bmatrix}$ with $\tilde{P}_1 = \frac{c_1}{c} \alpha P$. By using inequality (5), there exists some scalar $\theta > 0$, such that the following holds,

$$\begin{aligned} \tilde{x}^T(h, k)A^T PG \tilde{w}(h, k) + \tilde{w}^T(h, k)G^T PA \tilde{x}(h, k) \\ \leq \theta \tilde{x}^T(h, k)A^T PA \tilde{x}(h, k) + \frac{1}{\theta} \tilde{w}^T(h, k)G^T PG \tilde{w}(h, k), \end{aligned} \quad (36)$$

which implies that

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) - x^T(h, k+1)\tilde{P}_1 x(h, k+1) \\ \leq \tilde{x}^T(h, k)[(1 + \theta)A^T PA - \Pi_1] \tilde{x}(h, k) \\ + (1 + \frac{1}{\theta}) \tilde{w}^T(h, k)G^T PG \tilde{w}(h, k). \end{aligned} \quad (37)$$

Based on conditions (33)–(34), we then conclude that

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) - x^T(h, k+1)\tilde{P}_1 x(h, k+1) \\ \leq \Lambda \tilde{w}^T(h, k)\tilde{R} \tilde{w}(h, k), \end{aligned} \quad (38)$$

$$\text{where } \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ and } \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}.$$

Note that the inequality conditions (38) and (15) in Theorem 3.1 are consistent. Therefore, the remainder of the proof can be finished along the same lines as that of Theorem 3.1 and therefore we omit the details here.

Remark 5. Proposition 3.1 is provided here to show an alternative formulation of an I–O FRS criterion. This method is not difficult to realize and the interested readers can derive the corresponding results for the other conclusions in our paper via the similar analysis in Proposition 3.1. The conservative study about Proposition 3.1 and Theorem 3.2 will be discussed in the future.

3.2. The case of exogenous disturbances in $\mathcal{W}_\infty$

Theorem 3.3. Given scalars $N > 0$, $c \geq c_2 > 0$ and matrices $Q > 0$, $R > 0$, the 2-D system (2) is I–O FRS with respect to $(\mathcal{W}_\infty, Q, N, c)$, if there exist scalars $\alpha > 0$, $0 < \mu < 1$, $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$ and matrices $P > 0$, $H > 0$, such that conditions (7), (9), and

$$\begin{bmatrix} A^T PA - \Pi_2 & A^T PG \\ G^T PA & -H \end{bmatrix} < 0, \quad (39)$$

$$C^T QC \leq \mu P \quad (40)$$

$$\alpha c_2 + (\lambda_1 + \lambda_2)c_2 < \frac{c}{N}, \quad (41)$$

are satisfied, where $\Pi_2 = \begin{bmatrix} P & 0 \\ 0 & \tilde{P}_2 \end{bmatrix}$ with $\tilde{P}_2 = \frac{c_2}{c} \alpha P$.

Proof. By using the same arguments exploited in Theorem 3.1, it turns out that the following inequality holds,

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) \\ < x^T(h, k+1)\tilde{P}_2 x(h, k+1) + \lambda_1 w^T(h+1, k)Rw(h+1, k) \\ &+ \lambda_2 w^T(h, k+1)Rw(h, k+1), \end{aligned} \quad (42)$$

where $\tilde{P}_2 = \frac{c_2}{c} \alpha P$.

Since $w(h, k) \in \mathcal{W}_\infty$, that is for any $(h, k) \in \mathbf{S}(N)$, $w^T(h, k)Rw(h, k) \leq c_2$, it follows from this,

$$\begin{aligned} V(x(h+1, k+1)) - V(x(h+1, k)) \\ < x^T(h, k+1)\tilde{P}_2 x(h, k+1) + \lambda_1 w^T(h+1, k)Rw(h+1, k) \\ &+ \lambda_2 w^T(h, k+1)Rw(h, k+1) \\ &\leq x^T(h, k+1)\tilde{P}_2 x(h, k+1) + (\lambda_1 + \lambda_2)c_2. \end{aligned} \quad (43)$$

The following inequality can be derived via summing (43) over $j \in [0, k]$ and taking into account that $x(h, 0) = 0$:

$$\begin{aligned} x^T(h, k)Px(h, k) = V(x(h, k)) < \sum_{j=1}^k x^T(h-1, j)\tilde{P}_2 x(h-1, j) \\ &+ (\lambda_1 + \lambda_2)Nc_2. \end{aligned} \quad (44)$$

By using condition (41) and a similar analysis as used in deriving condition (30) in Theorem 3.1, we obtain

$$x^T(h, k)Px(h, k) \leq c. \quad (45)$$

Moreover, by inequality (40), it can be shown that for any $(h, k) \in \mathbf{S}(N)$,

$$\begin{aligned} y^T(h, k)Qy(h, k) &\leq \mu x^T(h, k)Px(h, k) + (1 - \mu)w^T(h, k)Rw(h, k) \\ &\leq \mu c + (1 - \mu)c = c, \end{aligned} \quad (46)$$

which implies that the system (2) is I–O FRS with respect to $(\mathcal{W}_\infty, Q, N, c)$. $\square$

4. Input–output finite-region stabilization

In the following, we will concentrate on studying the I–O finite-region stabilization issue for the 2-D system (1). Firstly, recall that the system is given by

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h+1, k) + A_2 x(h, k+1) \\ &+ B_1 u(h+1, k) + B_2 u(h, k+1) \\ &+ G_1 w(h+1, k) + G_2 w(h, k+1). \end{aligned} \quad (47)$$

The aim of I–O finite-region stabilization problem is to design a controller of the form

$$u(h, k) = Kx(h, k), \quad (48)$$

such that the following closed loop system

$$\begin{aligned} x(h+1, k+1) &= (A_1 + B_1 K)x(h+1, k) + (A_2 + B_2 K)x(h, k+1) \\ &+ G_1 w(h+1, k) + G_2 w(h, k+1), \end{aligned} \quad (49)$$

is I–O FRS.

Based on Theorems 3.1 and 3.2, we provide the following conditions in terms of feasibility of LMI's to test I–O FRS of the system (49) under exogenous disturbances in $\mathcal{W}_2$.

Theorem 4.1. Consider the 2-D system (49). Given scalars $N > 0$, $c \geq c_1 > 0$ and matrices $Q > 0$, $R > 0$, the 2-D system (2) is I–O finite-region stabilizable with respect to $(\mathcal{W}_2, Q, N, c)$, if there exist scalars $\alpha > 0$, $0 < \mu < 1$, $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$ and matrices $P > 0$, $H > 0$ and M , such that

(i)

$$\begin{bmatrix} -P^{-1} & 0 & 0 & P^{-1}A_1^T + MB_1^T \\ 0 & -\frac{c_1}{c}\alpha P^{-1} & 0 & P^{-1}A_2^T + MB_2^T \\ 0 & 0 & -H - G^T PG & G^T \\ A_1 P^{-1} + B_1 M^T & A_2 P^{-1} + B_2 M^T & G & -P^{-1} \end{bmatrix} < 0, \quad (50)$$

(ii) conditions (7)–(10) in Theorem 3.1 are satisfied.

If conditions (i) and (ii) hold, then a suitable controller K can be obtained with $K = M^T P$, where $G = [G_1 \ G_2]$, $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ and $\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$.

Proof. If we replace A_1 and A_2 in the 2-D system (2) with $A_1 + B_1 K$ and $A_2 + B_2 K$, respectively and apply Theorem 3.1, conditions (32) can be rewritten as

$$\begin{bmatrix} A^T P A - \Pi_1 & A^T P G \\ G^T P A & -H \end{bmatrix} < 0, \quad (51)$$

where $A = [A_1 + B_1 K \ A_2 + B_2 K]$, $G = [G_1 \ G_2]$.

By virtue of the Schur-Complement Lemma, condition (51) is equivalent to the following,

$$\begin{bmatrix} -\Pi_1 & 0 & A^T P \\ 0 & -H - G^T P G & G^T P \\ P A & P G & -P \end{bmatrix} < 0, \quad (52)$$

where $\Pi_1 = \begin{bmatrix} P & 0 \\ 0 & \frac{c_1}{c}\alpha P \end{bmatrix}$. Furthermore, condition (52) can be rewritten as

$$\begin{bmatrix} -P & 0 & 0 & (A_1^T + K^T B_1^T)P \\ 0 & -\frac{c_1}{c}\alpha P & 0 & (A_2^T + K^T B_2^T)P \\ 0 & 0 & -H - G^T P G & G^T P \\ P(A_1 + B_1 K) & P(A_2 + B_2 K) & P G & -P \end{bmatrix} < 0. \quad (53)$$

By pre- and post-multiplying (53) by

$$\begin{bmatrix} P^{-1} & 0 & 0 & 0 \\ 0 & P^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & P^{-1} \end{bmatrix} > 0 \quad (54)$$

we find that (53) holds if and only if:

$$\begin{bmatrix} -P^{-1} & 0 & 0 & P^{-1}A_1^T + MB_1^T \\ 0 & -\frac{c_1}{c}\alpha P^{-1} & 0 & P^{-1}A_2^T + MB_2^T \\ 0 & 0 & -H - G^T PG & G^T \\ A_1 P^{-1} + B_1 M^T & A_2 P^{-1} + B_2 M^T & G & -P^{-1} \end{bmatrix} < 0, \quad (55)$$

where $M = P^{-1}K^T$, which suggests as a suitable controller $K = M^T P$.

Together with conditions (7)–(10) and using Theorem 3.1, the proof is completed. $\square$

By using similar arguments as for Theorems 3.2 and 4.1, it is possible to prove the following result for the exogenous disturbances class $\mathcal{W}_\infty$.

Theorem 4.2. For given scalars $N > 0$, $c \geq c_2 > 0$ and matrices $Q > 0$, $R > 0$, the 2-D system (49) is I-O finite-region stabilizable with respect to $(\mathcal{W}_\infty, Q, N, c)$, if there exist scalars $\alpha > 0$, $0 < \mu < 1$, $\lambda_1 > 0$, $\lambda_2 > 0$ with $\lambda_1 + \lambda_2 \leq 1$ and matrices $P > 0$, $H > 0$ and M , such that

(i)

$$\begin{bmatrix} -P^{-1} & 0 & 0 & P^{-1}A_1^T + MB_1^T \\ 0 & -\frac{c_2}{c}\alpha P^{-1} & 0 & P^{-1}A_2^T + MB_2^T \\ 0 & 0 & -H - G^T PG & G^T \\ A_1 P^{-1} + B_1 M^T & A_2 P^{-1} + B_2 M^T & G & -P^{-1} \end{bmatrix} < 0, \quad (56)$$

(ii) conditions (7), (9), (40) and (41) are satisfied.

If conditions (i) and (ii) hold, then a suitable controller K is given by $K = M^T P$.

5. Numerical examples

Example 1a. Consider the following second-order FM model,

$$\begin{aligned} \begin{bmatrix} x_1(h+1, k+1) \\ x_2(h+1, k+1) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(h+1, k) \\ x_2(h+1, k) \end{bmatrix} \\ &+ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(h+1, k) \\ w_2(h+1, k) \end{bmatrix} \\ \begin{bmatrix} y_1(h, k) \\ y_2(h, k) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(h, k) \\ x_2(h, k) \end{bmatrix}. \end{aligned} \quad (57)$$

Clearly, for all (h, k) , we have:

$$\begin{aligned} x_1(h, k) &= x_1(h, k-1) + w_1(h, k-1) \\ &= \dots = x_1(h, 0) + \sum_{j=0}^{k-1} w_1(h, j), \end{aligned} \quad (58)$$

and

$$\|y(h, k)\| = |x_1(h, k)|. \quad (59)$$

Suppose that $Q = I$, $c = 20$, $w(h, k) \in \mathcal{W}_\infty$ with $R = I$, $c_2 = 1$, and consider the finite-region $\mathbf{S}(N) = \{(h, k) \in \mathbb{N} \times \mathbb{N} | 0 \leq h+k \leq 4\}$.

By using inequality (58) and taking $\tilde{w}_1(h, k) = 1$ and $\tilde{w}_2(h, k) = 0$ for any $(h, k) \in \mathbb{N} \times \mathbb{N}$, we have that $\tilde{w}(h, k) = [1, 0]^T \in \mathcal{W}_\infty$ with $c_2 = 1$, and

$$\begin{aligned} \|y(h, k)\| &= |x(h, k)| = |x_1(h, 0) + \sum_{j=0}^{k-1} \tilde{w}_1(h, j)| \\ &\geq \left| \sum_{j=0}^{k-1} \tilde{w}_1(h, j) \right| - |x_1(h, 0)| = k - L, \end{aligned} \quad (60)$$

where $L = \max_{h \in \mathbb{N}} |x_1(h, 0)|$. If $k \rightarrow \infty$, then $\|y(h, k)\| \rightarrow \infty$, which implies the system (57) is not BIBO stable.

On the other hand, for any $(h, k) \in \mathbf{S}(N)$ and $w(\cdot, \cdot) \in \mathcal{W}_\infty$ with $R = I$, $c_2 = 1$, and taking into account conditions (60) and boundary conditions $x(h, 0) = x(0, k) = 0$,

$$\|y(h, k)\| = \left| x_1(h, 0) + \sum_{j=0}^{k-1} w_1(h, j) \right| \leq \sum_{j=0}^{k-1} |w_1(h, j)| = 4, \quad (61)$$

and then

$$\|y(h, k)\|^2 \leq 16 < 20 = c,$$

which shows that the system (57) is I-O FRS with respect to $(\mathcal{W}_\infty, I, 4, 20)$.

Example 1b. Consider the second-order 2-D system,

$$\begin{aligned} \begin{bmatrix} x_1(h+1, k+1) \\ x_2(h+1, k+1) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(h+1, k) \\ x_2(h+1, k) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(h+1, k) \\ w_2(h+1, k) \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} y_1(h, k) \\ y_2(h, k) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(h, k) \\ x_2(h, k) \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1(h+1, k) \\ w_2(h+1, k) \end{bmatrix}. \quad (62)$$

Thus we have for all (h, k) ,

$$x_1(h, k) = x_1(h, k-1) = \dots = x_1(h, 0), \quad (63)$$

$$y_1(h, k) = x_1(h, k) + 2w(h, k) = x_1(h, 0) + 2w_1(h, k) \quad (64)$$

and

$$\|y(h, k)\| = |y_1(h, k)| = |x_1(h, 0) + 2w_1(h, k)|. \quad (65)$$

Suppose that $Q = I$, $c = 1.5$, $w(h, k) \in \mathcal{W}_\infty$ with $R = I$, $c_2 = 1$, and consider the finite-region $\mathbf{S}(N) = \{(h, k) \in \mathbb{N} \times \mathbb{N} | 0 \leq h+k \leq 5\}$. Obviously, for any $(h, k) \in \mathbb{N} \times \mathbb{N}$ and $w(h, k) \in \mathcal{W}_\infty$ with $R = I$, $c_2 = 1$,

$$\|y(h, k)\| = |y_1(h, k)| \leq |x_1(h, 0)| + |2w_1(h, k)| \leq M, \quad (66)$$

where $M = 2 + \max_{h \in \mathbb{N}} |x_1(h, 0)|$. It follows that the system (62) is BIBO stable.

We also take $\tilde{w}_1(h, k) = 1$ and $\tilde{w}_2(h, k) = 0$ for any $(h, k) \in \mathbf{S}(N)$. By using boundary conditions $x(h, 0) = x(0, k) = 0$ and equality (65),

$$\|y(h, k)\|^2 = |2\tilde{w}_1(h, k)|^2 = 4 > 1.5 = c, \quad (67)$$

which shows that the system (62) is not I-O FRS with respect to $(\mathcal{W}_\infty, I, 5, 1.5)$.

Example 2. Consider the following 2-D FM system,

$$\begin{cases} x(h+1, k+1) = A_1 x(h+1, k) \\ \quad + A_2 x(h, k+1) + B_1 u(h+1, k) + B_2 u(h, k+1) \\ \quad + G_1 w(h+1, k) + G_2 w(h, k+1) \\ y(h, k) = Cx(h, k) + Dw(h, k), \\ x(h, 0) = 0 \text{ and } x(0, k) = 0, \end{cases} \quad (68)$$

with

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.05 & 0 \\ 0.6 & 0.75 \end{bmatrix}; A_2 = \begin{bmatrix} 1.2 & 0 \\ 1.5 & 3 \end{bmatrix}; \\ B_1 &= \begin{bmatrix} 1 & 0 \\ 0.1 & 0.1 \end{bmatrix}; B_2 = \begin{bmatrix} 3 & 0.2 \\ 0 & 1.5 \end{bmatrix}; \\ G_1 &= \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}; G_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}; \\ C &= \begin{bmatrix} 2.2 & 0.2 \\ 0.2 & 2.1 \end{bmatrix}; D = \begin{bmatrix} 0.8 & 0.2 \\ 0.1 & 0.5 \end{bmatrix}. \end{aligned} \quad (69)$$

Assume that $Q = \frac{1}{4}I$, $R = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$, $c_1 = 3.5$, $c = 6$, $\mu = \frac{1}{2}$ and $N = 14$. According to Theorem 4.1, the 2-D FM model in system (68) is I-O finite-region stabilizable with respect to $(\mathcal{W}_2, I, 15, 6)$ since the conditions of the theorem are satisfied for $\alpha = 0.01$, $\lambda_1 = 0.65$, $\lambda_2 = 0.3$, $\mu = 0.75$,

$$P = \begin{bmatrix} 3.2770 & 0.5493 \\ 0.5493 & 3.3326 \end{bmatrix}, M = \begin{bmatrix} -1.0608 & -4.3116 \\ 0.2316 & -7.1692 \end{bmatrix}$$

and

$$H = \begin{bmatrix} H_1 & H_3 \\ H_3^T & H_2 \end{bmatrix}$$

$$\text{with } H_1 = \begin{bmatrix} 0.5005 & -0.0012 \\ -0.0012 & 0.3832 \end{bmatrix}, H_2 = \begin{bmatrix} 0.0894 & -0.0234 \\ -0.0234 & 0.0616 \end{bmatrix}, H_3 = \begin{bmatrix} -0.2549 & -0.0769 \\ -0.0769 & -0.3482 \end{bmatrix}.$$

Then, a suitable controller is given by $K = M^T P = \begin{bmatrix} -0.3449 & 0.1264 \\ -0.9822 & -1.9893 \end{bmatrix}$ and the corresponding closed-loop system is

Fig. 1. $y^T(h, k)Qy(h, k)$ before stabilization.

Fig. 2. $y^T(h, k)Qy(h, k)$ after stabilization.

described as follows:

$$\begin{cases} x(h+1, k+1) = \bar{A}_1 x(h+1, k) + \bar{A}_2 x(h, k+1) \\ \quad + G_1 w(h+1, k) + G_2 w(h, k+1) \\ y(h, k) = Cx(h, k) + Dw(h, k), \\ x(h, 0) = 0 \text{ and } x(0, k) = 0, \end{cases} \quad (70)$$

$$\text{where } \bar{A}_1 = A_1 + B_1 K = \begin{bmatrix} 0.7051 & 0.1264 \\ 0.4673 & 0.5637 \end{bmatrix} \text{ and } \bar{A}_2 = A_2 + B_2 K = \begin{bmatrix} -0.0311 & -0.0188 \\ 0.0266 & 0.0160 \end{bmatrix}.$$

For $0 \leq h+k \leq 14$, choose

$$w_0(h, k) = \begin{cases} [0.1, 0.1]^T, & 0 \leq k \leq 7; \\ [0.5, 0.2]^T, & k = 8; \\ [0.1, 0.1]^T, & 9 \leq k \leq 14, \end{cases}$$

then $w_0 \in \mathcal{W}_2$ with $c_1 = 3.5$ and the values of weighted-output $y^T(h, k)Qy(h, k)$ for the system (68) with control input $u(i, j) = 0$ are depicted in Fig. 1. It is indeed straightforward to verify that the system (68) is not I-O FRS with respect to $(\mathcal{W}_2, I, 15, 6)$. However, the weighted-output values $y^T(h, k)Qy(h, k)$ are limited by the given bound $c = 6$ for closed loop system (70) obtained after stabilization (see Fig. 2).

6. Conclusion

In this paper, we have investigated the I-O FRS and I-O finite-region stabilization problems for FM models. Based on the discussion of I-O FTS, the concept of I-O FRS was proposed for the FM models with input signals class $\mathcal{W}$. If input signal class $\mathcal{W}$ is taken to be $\mathcal{W}_2$, by building special recursive formulas, we have formulated a sufficient condition to guarantee I-O FRS. Subsequently, also, the input signal class $\mathcal{W}_\infty$ is taken into consideration, and the

corresponding I–O FRS criteria and LMIs conditions have been reported. Finally, aiming at designing state feedback controllers, the I–O finite-region stabilization issue has been analyzed for the corresponding 2-D closed-loop systems. To conclude the paper, some numerical examples have been provided to confirm the validity of the proposed methods.

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