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Published in:
Chaos

DOI:
[10.1063/1.4829631](https://doi.org/10.1063/1.4829631)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2013

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Yu, W., Chen, G., Cao, M., Lu, J., & Zhang, H-T. (2013). Swarming behaviors in multi-agent systems with nonlinear dynamics. *Chaos*, 23(4), 43118(1)-43118(12). [043118]. <https://doi.org/10.1063/1.4829631>

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Citation: *Chaos* **23**, 043118 (2013); doi: 10.1063/1.4829631

View online: <https://doi.org/10.1063/1.4829631>

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Swarming behaviors in multi-agent systems with nonlinear dynamics

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(Received 12 May 2013; accepted 28 October 2013; published online 11 November 2013)

The dynamic analysis of a continuous-time multi-agent swarm model with nonlinear profiles is investigated in this paper. It is shown that, under mild conditions, all agents in a swarm can reach cohesion within a finite time, where the upper bounds of the cohesion are derived in terms of the parameters of the swarm model. The results are then generalized by considering stochastic noise and switching between nonlinear profiles. Furthermore, swarm models with limited sensing range inducing changing communication topologies and unbounded repulsive interactions between agents are studied by switching system and nonsmooth analysis. Here, the sensing range of each agent is limited and the possibility of collision among nearby agents is high. Finally, simulation results are presented to demonstrate the validity of the theoretical analysis. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4829631>]

The dynamics of multi-agent systems are very complex. Each agent has its own nonlinear dynamics and there are connections among those agents. This paper investigates swarming behaviors in multi-agent systems with nonlinear dynamics. In particular, swarm model in multi-agent systems with stochastic noise, switching profiles, limited sensing range, and unbounded repulsive interactions are discussed. It is found that all agents can reach cohesion in a swarm within a finite time, where the upper bounds of the cohesion depend on the parameters of the swarm model.

I. INTRODUCTION

Swarming behaviors of groups of autonomous mobile agents have attracted increasing attention in recent years due to the extensive studies of biological systems and also because of the many applications in physics, engineering, and social science alike. Typical biological swarms include flocks of birds, schools of fish, herds of animals, and colonies of bacteria. The study of such swarms focuses on analyzing how coordinated collective behavior arises as a result of local interactions among individuals. In many applications of cooperative multi-agent systems, a group of agents only share and learn information locally and at the same time try to agree on certain global criteria of interest, such as cohesion of the whole group. As validated by biological field studies⁴⁷ and engineering robotic experiments,⁴⁶ swarm cohesion can be achieved in a distributed fashion despite the fact that each agent may only have local information regarding its nearest neighbors. An in-depth

understanding of the principles behind the swarming behaviors will help engineers to develop distributed cooperative control strategies and algorithms for networked dynamical systems, such as formations of unmanned air vehicles, teams of autonomous robots and networks of mobile sensors.

Recently, some progress has been made in analyzing collective behaviors in dynamical networks for which the closely related focal topics are consensus,^{1–11,44,45,50,51} swarming^{12–15,48,49,52,53} and synchronization.^{16–24} In Ref. 1, Vicsek *et al.* proposed a simple discrete-time model of autonomous agents moving in the plane with the same speed but different headings. Vicsek's model, often referred to as the consensus model in the literature, turns out to be a simplified version of the swarm model introduced earlier by Reynolds,² where the coordination is specified by nearest-neighbor rules. It has been proved that network connectivity is the key factor in reaching consensus.^{3,7,8,25,26} It has also been demonstrated that the consensus in a group with limited sensing range can be reached exponentially fast if and only if the union of the communication graphs contains a spanning tree with sufficient frequency as the networked system evolves. Synchronous distributed coordination rules for swarming groups in one or two-dimensional spaces were studied in Ref. 27 where convergence and stability analysis were given. In Refs. 12 and 13, stability properties of a continuous-time model for swarm aggregation in the n -dimensional space were discussed, and an asymptotic bound for the spatial size of the swarm was computed using the parameters of the swarm model. In Refs. 48 and 49, collective behavior of swarms with general nonlinear attraction and repulsion functions was investigated. More comprehensively, Reynolds' coordination rules were studied in detail in Ref. 28.

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Since the nonlinear profile of each agent and the communication topology of a swarming group may change from time to time, it may be convenient to describe the group dynamics using a switched system model. As a special case of hybrid systems, a switched system consists of a family of subsystems and a switching logic. Tools from switched systems have been successfully applied to complex systems^{29–31} which share some key features of swarming groups.

One main contribution of this paper is that a continuous-time swarm model with nonlinear profiles is proposed. Note that for the classical swarm model without the self-nonlinear profiles,¹³ the center of cohesion for agents is fixed, which is inconsistent with the common intuitive idea and practical applications, for example, fish can move together toward to another place under cohesion, robots maintain a group formation to capture a target, etc. From all these applications, the center of the agents is not a constant but rather time-varying. Therefore, it is of practical importance to study continuous-time swarm model with nonlinear profiles. Another contribution is that the swarm model with limited sensing range is also considered by nonsmooth analysis where the connections between agents can be disconnected at some time instants, which has rarely been investigated elsewhere.

In particular, this paper complements the existing swarm model by taking into account the intrinsic nonlinear dynamics of each individual agent. Stability analysis of the generalized swarm model is discussed by providing both spatial and temporal descriptions about how cohesion is achieved under various sufficient conditions. A stochastic swarm model is also formulated to incorporate the influence of a noisy environment. The swarming group is further modelled as a switched system where the switching signal describes how agents switch between different nonlinear profiles. Furthermore, the following challenging problems are dealt with: the communication topology changes with time, the sensing range of each agent is limited, and the repulsion forces between agents become gradually unbounded.

The rest of the paper is organized as follows. In Sec. II, some preliminaries are given. Stability analysis of the generalized swarm model is discussed in Sec. III. In Sec. IV, the cohesion of a swarm model with stochastic noise is considered. In Sec. V, cohesion of the model with switched profiles and stochastic noise is further investigated. Then, stability analysis of the swarm model with limited sensing range and unbounded repulsion is studied in Sec. VI. Simulation results are presented in Sec. VII.

II. PRELIMINARIES

The swarm model considered in Ref. 13 is first reviewed. In a swarm of N agents in the n -dimensional Euclidean space, the motion dynamics of the agent i , $1 \leq i \leq N$, are described by

$$\dot{x}_i(t) = \sum_{j=1, j \neq i}^N g(x_i(t) - x_j(t)), \quad (1)$$

where $x_i \in R^n$ is the position of agent i and $g(\cdot)$ represents the interaction force between the corresponding agents in the form of repulsion and attraction given by:

$$g(y) = -y \left(a - b e^{-\frac{\|y\|^2}{c}} \right), \quad (2)$$

where $y \in R^n$, a , b , and c are positive constants satisfying $b > a$. Here, the terms $-ay$ and $b y e^{-\frac{\|y\|^2}{c}}$ represent the attraction and repulsion between agents, respectively, and the correspondingly $g(\cdot)$ has two equilibria $y = 0$ and $\|y\| = \delta = \sqrt{c \ln(b/a)}$. Note that the attraction dominates when the two agents are far away from each other, and the repulsion dominates when they are close. Because of this particular property, $g(\cdot)$ is widely used to describe the interactions among agents in swarming biological systems. However, as it becomes apparent later in this paper, there are some drawbacks in using attraction/repulsion function (2), which motivates us to improve the swarm model (1) and (2) in this paper.

First, observe that the attraction/repulsion force between any chosen pair of agents i and j is anti-symmetric in i and j , namely, $g(x_i - x_j) = -g(x_j - x_i)$. As a result, if one examines the average position of the swarm, $\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$, it is easy to see that

$$\dot{\bar{x}} = -\frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N g(x_i - x_j) = 0, \quad (3)$$

which means that $\bar{x}$ is a constant and will not change with time. In real biological systems, however, each agent's motion dynamics are not just determined by inter-agent interactions, but also by each agent's intrinsic dynamics as well. For example, in a social foraging swarm, each agent tends to move towards a region with higher nutrient concentration. Consequently in a biological swarm, the average position of all agents is, in general, not a constant, but more likely a dynamical variable as the whole group of agents are in motion. In this paper, therefore, the following generalized swarm model with a nonlinear profile is considered:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1, j \neq i}^N g(x_i(t) - x_j(t)), \quad (4)$$

where $f(x_i) = (f_1(x_i), f_2(x_i), \dots, f_n(x_i))^T$ is a nonlinear function describing the intrinsic dynamics of each agent.

Assumption 1. For all $x, y \in R^n$, there exists a constant θ such that

$$\|f(x) - f(y)\| \leq \theta \|x - y\|. \quad (5)$$

$\sum_{j=1, j \neq i}^N g(x_i(t) - x_j(t))$ can be considered as a control input. Note that the Lipschitz condition (5) is very mild: if $\partial f_j / \partial x_{ij}$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, n$, are uniformly bounded, including in particular all linear time-invariant systems, then this condition is automatically satisfied.

Lemma 1. Let $A \in R^{N \times N}$. $A_{ij} = 1$ ($i \neq j$) and $A_{ii} = N - 1$, $i, j = 1, 2, \dots, N$, namely,

$$A = \begin{pmatrix} N-1 & 1 & \cdots & 1 \\ 1 & N-1 & \vdots & 1 \\ \vdots & \vdots & \ddots & 1 \\ 1 & 1 & \cdots & N-1 \end{pmatrix}. \quad (6)$$

Then, A has an eigenvalue $2(N-1)$ with multiplicity 1, and an eigenvalue $N-2$ with multiplicity $N-1$.

Proof. The proof can be completed by directly using the properties of the Laplace matrix of fully connected graphs.

Now, define the error vectors $e_i(t) = x_i - \bar{x}(t)$. Then one has the following error dynamical system:

$$\begin{aligned}\dot{e}_i(t) &= f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) + \sum_{j=1, j \neq i}^N g(x_i(t) - x_j(t)) \\ &= f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) \\ &\quad + \sum_{j=1, j \neq i}^N [-a(x_i(t) - x_j(t)) + b(x_i(t) \\ &\quad - x_j(t))e^{-\|x_i(t) - x_j(t)\|^2/c}] \\ &= f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) - aNe_i(t) \\ &\quad + \sum_{j=1, j \neq i}^N b(x_i(t) - x_j(t))e^{-\|x_i(t) - x_j(t)\|^2/c}.\end{aligned}\quad (7)$$

III. ANALYSIS OF SWARM COHESION

As a first step in the analysis of the generalized swarm model with a nonlinear profile, in this section stability analysis of swarm cohesion for reaching a hyperball at the center is investigated.

Theorem 1. Suppose that Assumption 1 holds. Consider the generalized swarm model (4) with an attraction/repulsion function (2). If

$$a > \frac{2(N-1)\theta}{N^2}, \quad (8)$$

then all the agents of the swarm will converge to a hyperball centered at $\bar{x}$,

$$B_\varepsilon = \left\{ (x_1, \dots, x_N) \left| \frac{1}{N} \sum_{i=1}^N \|x_i - \bar{x}\|^2 \leq \varepsilon \right. \right\}, \quad (9)$$

where $\varepsilon = \frac{b^2 c}{2e^{(a - \frac{2(N-1)\theta}{N^2})^2}}$. Furthermore, all agents will move into the hyperball B_ε in a finite time specified by

$$t = -\frac{1}{2\left(a - \frac{2(N-1)\theta}{N^2}\right)} \ln\left(\frac{N\varepsilon}{2V(0)}\right). \quad (10)$$

Proof. Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t). \quad (11)$$

Taking the derivative of $V(t)$ along the trajectories of Eq. (7) gives

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N e_i^T(t) \dot{e}_i(t) \\ &= \sum_{i=1}^N e_i^T(t) \left[f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) - aNe_i(t) \right. \\ &\quad \left. + \sum_{j=1, j \neq i}^N b(x_i(t) - x_j(t))e^{-\|x_i(t) - x_j(t)\|^2/c} \right].\end{aligned}\quad (12)$$

By assumption 1, one has

$$\begin{aligned}\sum_{i=1}^N e_i^T(t) \left[f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) \right] \\ \leq \frac{\theta}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \|e_i(t)\| (\|e_i(t)\| + \|e_j(t)\|).\end{aligned}\quad (13)$$

Note that the function $\|x_i(t) - x_j(t)\|e^{-\|x_i(t) - x_j(t)\|^2/c}$ is a bounded function with maximum value $\sqrt{\frac{c}{2}}e^{-\frac{1}{2}}$ attained when $\|x_i(t) - x_j(t)\| = \sqrt{\frac{c}{2}}$. Then, it follows that

$$\begin{aligned}\|e_i(t)\| \sum_{j=1, j \neq i}^N b\|x_i(t) - x_j(t)\|e^{-\|x_i(t) - x_j(t)\|^2/c} \\ \leq b(N-1)\|e_i(t)\|\sqrt{\frac{c}{2}}e^{-\frac{1}{2}}.\end{aligned}\quad (14)$$

Substituting Eqs. (13) and (14) into Eq. (12), one has

$$\begin{aligned}\dot{V} &\leq \left(-aN + \frac{N-1}{N}\theta\right) \sum_{i=1}^N \|e_i(t)\|^2 \\ &\quad + \frac{\theta}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \|e_i(t)\| \|e_j(t)\| \\ &\quad + b(N-1)\sqrt{\frac{c}{2}}e^{-\frac{1}{2}} \sum_{i=1}^N \|e_i(t)\| \\ &\leq -N\|e(t)\|\Omega\|e(t)\| + b(N-1)\sqrt{\frac{c}{2}}e^{-\frac{1}{2}} \sum_{i=1}^N \|e_i(t)\| \\ &\leq -N\lambda_{\min}(\Omega) \sum_{i=1}^N \|e_i(t)\|^2 + b(N-1)\sqrt{\frac{c}{2}}e^{-\frac{1}{2}} \\ &\quad \times \sum_{i=1}^N \|e_i(t)\|,\end{aligned}\quad (15)$$

where $\|e(t)\| = (\|e_1(t)\|, \|e_2(t)\|, \dots, \|e_N(t)\|)^T$, and

$$\Omega = \begin{pmatrix} a - \frac{N-1}{N^2}\theta & -\frac{\theta}{N^2} & \cdots & -\frac{\theta}{N^2} \\ -\frac{\theta}{N^2} & a - \frac{N-1}{N^2}\theta & \vdots & -\frac{\theta}{N^2} \\ \vdots & \vdots & \ddots & -\frac{\theta}{N^2} \\ -\frac{\theta}{N^2} & -\frac{\theta}{N^2} & \cdots & a - \frac{N-1}{N^2}\theta \end{pmatrix}.$$

By Lemma 1 and Eq. (8), one has $\lambda_{\min}(\Omega) = a - \frac{2(N-1)\theta}{N^2}$. Note that

$$\begin{aligned} \left(\sum_{i=1}^N \|e_i(t)\| \right)^2 &= \sum_{i=1}^N \sum_{j=1}^N \|e_i(t)\| \|e_j(t)\| \\ &\leq \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (\|e_i(t)\|^2 + \|e_j(t)\|^2) \\ &= N \sum_{i=1}^N \|e_i(t)\|^2. \end{aligned}$$

Then, it follows that

$$\begin{aligned} \dot{V}(t) &\leq -N \left(a - \frac{2(N-1)\theta}{N^2} \right) \sum_{i=1}^N \|e_i(t)\|^2 + b(N-1) \sqrt{\frac{Nc}{2}} \\ &\quad \times e^{-\frac{1}{2}} \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} \\ &= - \left(a - \frac{2(N-1)\theta}{N^2} \right) \sum_{i=1}^N \|e_i(t)\|^2 - \left(a - \frac{2(N-1)\theta}{N^2} \right) \\ &\quad \times (N-1) \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} \\ &\quad \times \left(\sqrt{\sum_{i=1}^N \|e_i(t)\|^2} - \frac{b\sqrt{\frac{Nc}{2}} e^{-\frac{1}{2}}}{a - \frac{2(N-1)\theta}{N^2}} \right). \end{aligned} \quad (16)$$

If $\frac{1}{N} \sum_{i=1}^N \|x_i(t) - \bar{x}\|^2 \geq \frac{b^2 c}{2e(a - \frac{2(N-1)\theta}{N^2})^2}$, then one has

$$\begin{aligned} \dot{V} &\leq - \left(a - \frac{2(N-1)\theta}{N^2} \right) \sum_{i=1}^N \|e_i(t)\|^2 \\ &= -2 \left(a - \frac{2(N-1)\theta}{N^2} \right) V(t). \end{aligned} \quad (17)$$

Therefore, the solutions of $V(t)$ satisfies

$$V(t) \leq V_0 e^{-2 \left(a - \frac{2(N-1)\theta}{N^2} \right) t}.$$

Now, it is easy to see that the trajectories enter the boundary $\frac{1}{N} \sum_{i=1}^N \|x_i(t) - \bar{x}\|^2 = \varepsilon$ in a finite time

$$t \leq - \frac{1}{2 \left(a - \frac{2(N-1)\theta}{N^2} \right)} \ln \left(\frac{N\varepsilon}{2V(0)} \right).$$

This completes the proof.

Remark 1. The bound $\varepsilon = \frac{b^2 c}{2e \left(a - \frac{2(N-1)\theta}{N^2} \right)^2}$ increases as the parameters b and c increase, while it decreases as the parameter a increases. This is consistent with the balance $\delta = \sqrt{c \ln(b/a)}$ between the attraction and repulsion. In Ref. 7, $\theta = 0$ is considered, and the bound is a constant for given a , b and c , and is independent of the size N . In this paper, the bound ε of cohesion increases as $\theta > 0$ increases, which is closer to biological reality.

Remark 2. Note that the bound ε of the swarm depends on the size N . If N increases, the bound ε decreases, which means that the density of the swarm increases. Furthermore,

for swarms with a very large number of agents, $\varepsilon \rightarrow \frac{b^2 c}{2ea^2}$. This, however, is inconsistent with the biological phenomena and is due to the fact that the attraction/repulsion function $g(\cdot)$ in (2) taken from Ref. 13 has an infinite long effective range for any chosen pair of agents. This function $g(\cdot)$ will be modified in Section VI so that it has only limited effective range. In other words, there will be no interaction between a pair of agents that are out of a pre-determined sensing range r .

Remark 3. In Ref. 6, a function f was investigated for stability analysis of social foraging swarms, where the assumptions on f are restrictive, e.g. requiring f to be bounded or linear. However, in this paper, assumption 1 is mild and applies to many well-known nonlinear systems.

IV. ANALYSIS OF SWARM COHESION IN A NOISY ENVIRONMENT

In this section, the cohesion of a swarm in a noisy environment is investigated. Consider the following stochastic swarm model:

$$\begin{aligned} dx_i(t) &= \left[f(x_i(t)) + \sum_{j=1, j \neq i}^N g(x_i(t) - x_j(t)) \right] dt \\ &\quad + v_i(t) d\nu_i, \end{aligned} \quad (18)$$

where $v_i(t) \in R^n$ is an external noise intensity function of agent i , and $\nu_i(t)$ is an independent one-dimensional Brownian motion with expectation $\mathbf{E}\{\nu_i(t)\} = 0$ and variance $\mathbf{D}\{\nu_i(t)\} = 1$. The model is defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by $\{\nu_i(s) : 0 \leq s \leq t\}$, where Ω is associated with the canonical space generated by $\nu_i(t)$ and $\mathcal{F}$ is the associated σ -algebra generated by $\{\nu_i(t)\}$ with probability measure $\mathbb{P}$.

Assumption 2. $v_i(t) \in R^n$ belongs to $L_\infty[0, \infty)$, i.e., $v_i(t)$ is a bounded vector function satisfying

$$v_i^T(t) v_i(t) \leq \alpha_i, \quad \forall t \in R, \quad (19)$$

where α_i is a positive constant, $i = 1, 2, \dots, N$.

Theorem 2. Suppose that assumptions 1 and 2 hold. Consider the swarm model (18) with the attraction/repulsion function $g(\cdot)$ defined by Eq. (2). If

$$a > \frac{2(N-1)\theta}{N^2}, \quad (20)$$

then the expectations of all the agents of the swarm centered at $\bar{x}$ satisfy

$$\frac{1}{N} \sum_{i=1}^N \mathbf{E}(\|x_i(t) - \bar{x}(t)\|^2) \leq \eta, \quad (21)$$

where

$$\begin{aligned} \eta &= \frac{\left(b\sqrt{\frac{Nc}{2e}} + \sqrt{\frac{b^2 Nc}{2e} + \frac{4\alpha(a - \frac{2(N-1)\theta}{N^2})}{N-1}} \right)^2}{4N \left(a - \frac{2(N-1)\theta}{N^2} \right)^2}, \quad \text{and} \\ \alpha &= \frac{1}{2} \sum_{i=1}^N \alpha_i. \end{aligned}$$

Furthermore, cohesion will be achieved within the bound η in a finite time

$$t = -\frac{1}{2\left(a - \frac{2(N-1)\theta}{N^2}\right)} \ln\left(\frac{N\eta}{2\mathbf{E}V(0)}\right). \quad (22)$$

Proof. Consider the same Lyapunov function candidate as in Eq. (11). From the Itô formula,^{32,33} one obtains the following stochastic differential:

$$dV(t) = \mathcal{L}V(t)dt + \sum_{i=1}^N e_i^T(t)[v_i(t)d\nu_i(t)]. \quad (23)$$

By assumption 2, the weak infinitesimal operator $\mathcal{L}$ of the stochastic process yields

$$\mathcal{L}V(t) = \sum_{i=1}^N \left[e_i^T(t)\dot{e}_i(t) + \frac{1}{2} v_i^T(t)v_i(t) \right] \leq \sum_{i=1}^N e_i^T(t)\dot{e}_i(t) + \alpha, \quad (24)$$

where $\alpha = \frac{1}{2} \sum_{i=1}^N \alpha_i$. Following the same steps as in the proof of Theorem 1, one obtains the following expression which is similar to Eq. (16):

$$\begin{aligned} \mathcal{L}V(t) &\leq -N\left(a - \frac{2(N-1)\theta}{N^2}\right) \sum_{i=1}^N \|e_i(t)\|^2 + b(N-1)\sqrt{\frac{Nc}{2}}e^{-\frac{1}{2}} \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} + \alpha \\ &= -\left(a - \frac{2(N-1)\theta}{N^2}\right) \sum_{i=1}^N \|e_i(t)\|^2 - \left(a - \frac{2(N-1)\theta}{N^2}\right) \\ &\quad \times (N-1) \left(\sum_{i=1}^N \|e_i(t)\|^2 - \frac{b\sqrt{\frac{Nc}{2}}e^{-\frac{1}{2}}}{a - \frac{2(N-1)\theta}{N^2}} \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} - \frac{\alpha}{\left(a - \frac{2(N-1)\theta}{N^2}\right)(N-1)} \right). \end{aligned} \quad (25)$$

Let $z = \sqrt{\sum_{i=1}^N \|e_i(t)\|^2}$ and $g(z) = z^2 - \frac{b\sqrt{\frac{Nc}{2}}e^{-\frac{1}{2}}}{a - \frac{2(N-1)\theta}{N^2}}z - \frac{\alpha}{\left(a - \frac{2(N-1)\theta}{N^2}\right)(N-1)}$. It is easy to see that $g(z)=0$ has two solutions:

$$z_{1,2} = \frac{b\sqrt{\frac{Nc}{2}} \pm \sqrt{\frac{b^2 Nc}{2e} + \frac{4\alpha\left(a - \frac{2(N-1)\theta}{N^2}\right)}{N-1}}}{2\left(a - \frac{2(N-1)\theta}{N^2}\right)}, \quad (26)$$

where $z_1 > 0$ and $z_2 < 0$. If $z(t) \geq z_1$, then one has $g(z) \geq 0$, and it follows that

$$\mathcal{L}V(t) \leq -\left(a - \frac{2(N-1)\theta}{N^2}\right) \sum_{i=1}^N \|e_i(t)\|^2. \quad (27)$$

From the Itô formula,³³ it follows that

$$\begin{aligned} \mathbf{E}V(t) - \mathbf{E}V(0) &= \mathbf{E} \int_0^t \mathcal{L}V(s)ds \\ &\leq -2\left(a - \frac{2(N-1)\theta}{N^2}\right) \int_0^t \mathbf{E}V(s)ds. \end{aligned}$$

Therefore, the solutions of $V(t)$ satisfy

$$\mathbf{E}V(t) \leq \mathbf{E}V(0)e^{-2\left(a - \frac{2(N-1)\theta}{N^2}\right)t}.$$

Thus, it is easy to show that the trajectories cross the boundary $\frac{1}{N}\mathbf{E}(\sum_{i=1}^N \|x_i(t) - \bar{x}\|^2) = \eta$ in a finite time

$$t \leq -\frac{1}{2\left(a - \frac{2(N-1)\theta}{N^2}\right)} \ln\left(\frac{N\eta}{2\mathbf{E}V(0)}\right).$$

This completes the proof.

V. ANALYSIS OF COHESION IN SWARMS WITH SWITCHED PROFILES

There are typically two types of switches for switching systems. One is time related as in Sec. V and the other is state related as in Sec. VI, which correspond to different scenarios. In Sec. V, the case that nonlinear functions and coupling topologies switch between different profiles at some time instants determined by a pre-designed switching signal is considered. However, in Sec. VI, network topologies can switch depending on the states of systems which may be non-smooth.

In a swarm system with a nonlinear profile, it is easy to check that the average position $\bar{x}$ evolves according to

$$\dot{\bar{x}} = \frac{1}{N} \sum_{i=1}^N f(x_i(t)). \quad (28)$$

Now let $f(y) = -\sigma(y - \rho)$, where $\sigma > 0$, $y, \rho \in \mathbb{R}^n$. Then $\dot{\bar{x}} = -\sigma(\bar{x} - \rho)$. Hence, one may check that $\bar{x}(t) \rightarrow \rho$ as $t \rightarrow \infty$. Here, ρ can be interpreted as the target average

position of the swarming group whose average velocity is determined by σ . Since in biological swarms, each agent's velocity may change from time to time and the target position may also move, this leads to switched systems. The phenomenon that the profiles of agents may change at particular times is also of special interest to applications since such behaviors appear naturally in automatic control systems, neural networks and communications.²⁹ In this section, therefore, the following switched swarming system is considered:

$$dx_i(t) = \left[f_\gamma(x_i(t)) + \sum_{j=1, j \neq i}^N g_\gamma(x_i(t) - x_j(t)) \right] dt + v_i(t) d\nu_i, \quad (29)$$

where γ is a switching signal which takes values from the finite set $\mathcal{I} = \{1, 2, \dots, \bar{N}\}$, and

$$g_\gamma(y) = -y \left(a_\gamma - b_\gamma e^{-\frac{\|y\|^2}{c_\gamma}} \right), \quad (30)$$

which means that the positive parameter values $(a_\gamma, b_\gamma, c_\gamma)$ are allowed to take values, at particular times, from the finite set $\{(a_1, b_1, c_1), \dots, (a_{\bar{N}}, b_{\bar{N}}, c_{\bar{N}})\}$.²⁹

Assumption 3. There exist constants θ_γ such that

$$\|f_\gamma(x) - f_\gamma(y)\| \leq \theta_\gamma \|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \quad \gamma = 1, 2, \dots, \bar{N}. \quad (31)$$

Theorem 3. Suppose that Assumptions 1 and 3 hold. In the swarm model (29) with an attraction/repulsion function (2), if

$$a_\gamma > \frac{2(N-1)\theta_\gamma}{N^2}, \quad \gamma = 1, 2, \dots, \bar{N}, \quad (32)$$

then the expectations of all the agents of the swarm centered at $\bar{x}$ satisfy

$$\frac{1}{N} \sum_{i=1}^N \mathbf{E}(\|x_i(t) - \bar{x}(t)\|^2) \leq \eta, \quad (33)$$

where $\eta_\gamma = \frac{\left(b_\gamma \sqrt{\frac{Nc_\gamma}{2e}} + \sqrt{\frac{b_\gamma^2 Nc_\gamma}{2e} + \frac{4a_\gamma \left(a_\gamma - \frac{2(N-1)\theta_\gamma}{N^2} \right)}{N-1}} \right)^2}{4N \left(a_\gamma - \frac{2(N-1)\theta_\gamma}{N^2} \right)^2}$, $\eta = \max_\gamma \eta_\gamma$,

and $\alpha = \frac{1}{2} \sum_{i=1}^N \alpha_i$. Furthermore, cohesion will be achieved within the bound η in a finite time

$$t = \max_\gamma \left(-\frac{1}{2 \left(a_\gamma - \frac{2(N-1)\theta_\gamma}{N^2} \right)} \ln \left(\frac{N\eta}{2\mathbf{E}V(0)} \right) \right). \quad (34)$$

Proof. Choose the same Lyapunov candidate as in Eq. (11) to be the common Lyapunov function. Then, the proof can be completed in the same way as in the proof of Theorem 2.

VI. ANALYSIS OF COHESION IN SWARMS WITH LIMITED SENSING RANGE

As discussed in Remark 2 above, the attraction function $-a_\gamma$ in Eq. (2) has an infinite sensing range, which is not realistic for biological systems.^{12,13} In this section, therefore, a swarm model with an attraction function having a limited sensing range r is considered:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1, j \neq i}^N h(x_i(t) - x_j(t)), \quad (35)$$

in which

$$h(x_i(t) - x_j(t)) = - \left(a a_{ij}(t) - b e^{-\frac{\|x_i(t) - x_j(t)\|^2}{c}} \right) (x_i(t) - x_j(t)), \quad (36)$$

where $a_{ij} = a_{ji} = 1$ if $\|x_i(t) - x_j(t)\| \leq r$, otherwise, $a_{ij} = a_{ji} = 0$ for $i \neq j$; $a_{ii} = 0$ for $i = 1, 2, \dots, N$. Let $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = \sum_{j=1, j \neq i}^N a_{ij}$. Then, system (35) can be written as

$$\begin{aligned} \dot{x}_i(t) = & f(x_i(t)) - \sum_{j=1}^N a l_{ij}(t) x_j(t) \\ & + \sum_{j=1, j \neq i}^N b e^{-\frac{\|x_i(t) - x_j(t)\|^2}{c}} (x_i(t) - x_j(t)). \end{aligned} \quad (37)$$

Note that $L = (l_{ij})_{N \times N}$ is the Laplacian matrix,³⁴ and has the following properties:

Lemma 2. Assume that an undirected graph is connected. Then, the Laplacian matrix L has an eigenvalue 0 with algebraic multiplicity one, and all the other eigenvalues are positive:¹⁷ $0 = \lambda_1(L) < \lambda_2(L) \leq \dots \leq \lambda_N(L)$.

Lemma 3. For an undirected graph with Laplacian matrix L , the algebraic connectivity of the network is described by^{35,36}

$$\lambda_2(L) = \min_{x^T \mathbf{1}_N = 0, x \neq 0} \frac{x^T L x}{x^T x}. \quad (38)$$

Again, let $e_i(t) = x_i - \bar{x}(t)$. Then, one obtains the following error dynamical system:

$$\begin{aligned} \dot{e}_i(t) = & f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) - a \sum_{j=1}^N l_{ij}(t) e_j(t) \\ & + \sum_{j=1, j \neq i}^N b (x_i(t) - x_j(t)) e^{-\|x_i(t) - x_j(t)\|^2/c}. \end{aligned} \quad (39)$$

Since the right-hand side of Eq. (39) is discontinuous, one can not study it by using ordinary differential equations with classical solutions (continuously differentiable). Here, the nonsmooth analysis is applied.³⁷

Definition 1. Suppose $E \subset \mathbb{R}^n$. Map $x \rightarrow F(x)$ is called a set-value map from $E \rightarrow \mathbb{R}^n$, if each point x of a set $E \subset \mathbb{R}^n$ corresponds to a non-empty set $F(x) \subset \mathbb{R}^n$.³⁸

Let $\dot{e}(t) = \varphi(x)$, where $e(t) = (e_1^T, e_2^T, \dots, e_N^T)^T$, and $x(t) = (x_1^T, x_2^T, \dots, x_N^T)^T$.

Definition 2. (Filippov solution) A set-valued map is defined as^{37,39}

$$\psi(x) = \bigcap_{\delta>0} \bigcap_{\mu(\bar{N})=0} \mathbb{K}[\varphi(B(x, \delta) - \bar{N})],$$

where $\mathbb{K}(E)$ is the closure of the convex hull of set E , $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$, and $\mu(\bar{N})$ is the Lebesgue measure of set $\bar{N}$. A solution in the sense of Filippov of Eq. (39) with initial condition $x(0) = x_0$ is an absolutely continuous function $x(t)$, which satisfies $x(0) = x_0$ and the differential inclusion:

$$\dot{x}(t) \in \psi(x), a.e. t.$$

The concept of an Filippov solution is very important in engineering applications. All sets of measure zero are disregarded, which allows solutions to be defined at points even where the vector $\psi(x)$ is discontinuous. In addition, an arbitrary set of measure zero in $B(x, \delta)$ is excluded when evaluating x such that the result is the same for any two vector fields that differ on a set of measure zero. By Ref. 40, the concept of Filippov solution is extended to the following:

$$\begin{aligned} \dot{x}_i(t) \in & f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) - a \sum_{j=1}^N \mathbb{K}[l_{ij}(t)] e_j(t) \\ & + \sum_{j=1, j \neq i}^N b(x_i(t) - x_j(t)) e^{-\|x_i(t) - x_j(t)\|^2/c}, \quad a.e. t, \end{aligned} \quad (40)$$

or equivalently,

$$\begin{aligned} \dot{x}_i(t) = & f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) - a \sum_{j=1}^N d_{ij}(t) e_j(t) \\ & + \sum_{j=1, j \neq i}^N b(x_i(t) - x_j(t)) e^{-\|x_i(t) - x_j(t)\|^2/c}, \quad a.e. t, \end{aligned} \quad (41)$$

where $d_{ij}(t) = \begin{cases} 1 & \|x_i(t) - x_j(t)\| < r \\ 0 & \|x_i(t) - x_j(t)\| > r \\ \xi_{ij} & \|x_i(t) - x_j(t)\| = r \end{cases}$ for $i \neq j$,
 $d_{ii} = -\sum_{j=1}^N d_{ij}$, and $\xi_{ij} \in [0, 1]$. Let $D = (d_{ij})_{N \times N}$.

Theorem 4. Suppose that Assumption 1 holds, and the network with the Laplacian matrix L is connected at all times. Consider the swarm model (35) with an attraction/repulsion function (36). If

$$a\phi_t > \frac{2(N-1)\theta}{N}, \forall t > 0, \quad (42)$$

then all the agents of the swarm will converge to a hyperball centered at $\bar{x}$,

$$B_{\varepsilon_t} = \left\{ (x_1, \dots, x_N) \left| \frac{1}{N} \sum_{i=1}^N \|x_i - \bar{x}\|^2 \leq \varepsilon_t \right. \right\}, \quad (43)$$

where

$$\phi_t = \min_{\tau \in [0, t]} \lambda_2(D(\tau)), \quad \varepsilon_t = \frac{b^2 c N^2}{2e \left(a\phi_t - \frac{2(N-1)\theta}{N} \right)^2}.$$

Furthermore, cohesion will be achieved within the bound ε_t in a finite time

$$t = - \frac{N}{2 \left(a\phi_t - \frac{2(N-1)\theta}{N} \right)} \ln \left(\frac{N\varepsilon_t}{2V(0)} \right). \quad (44)$$

Proof. Consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t) e_i(t). \quad (45)$$

Taking the derivative of $V(t)$ along the trajectories of Eq. (40), and using Eqs. (13)–(16) and Lemma 3, one has

$$\begin{aligned} \dot{V} = & \sum_{i=1}^N e_i^T(t) \dot{x}_i(t) = \sum_{i=1}^N e_i^T(t) \left[f(x_i(t)) - \frac{1}{N} \sum_{j=1}^N f(x_j(t)) - a \sum_{j=1}^N d_{ij}(t) e_j(t) + \sum_{j=1, j \neq i}^N b(x_i(t) - x_j(t)) e^{-\|x_i(t) - x_j(t)\|^2/c} \right] \\ \leq & -a \sum_{i=1}^N \sum_{j=1}^N d_{ij}(t) e_i^T(t) e_j(t) + \frac{2(N-1)}{N} \theta \sum_{i=1}^N \|e_i(t)\|^2 + b(N-1) \sqrt{\frac{Nc}{2}} e^{-\frac{1}{2}} \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} \\ = & -a e^T(t) (D(t) \otimes I_n) e(t) + \frac{2(N-1)}{N} \theta \sum_{i=1}^N \|e_i(t)\|^2 + b(N-1) \sqrt{\frac{Nc}{2}} e^{-\frac{1}{2}} \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} \\ \leq & \left[-a(\lambda_2(D(t)) + \frac{2(N-1)}{N} \theta) \sum_{i=1}^N \|e_i(t)\|^2 + b(N-1) \sqrt{\frac{Nc}{2}} e^{-\frac{1}{2}} \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} \right], \end{aligned} \quad (46)$$

By using a similar argument as in the proof Theorem 1, the claim can be proved.

Remark 4. Note that the bound of the swarm, $\varepsilon_t = \frac{b^2 c N^2}{2e \left(a\phi_t - \frac{2(N-1)\theta}{N} \right)^2}$, depends on the size N . If N increases, the

bound ε increases, which means that the boundary of cohesion increases due to the large number of agents. It is consistent with the biological phenomena, thanks to the limited sensing range r in the attraction/repulsion function $h(\cdot)$ in Eq. (36).

Generally, suppose that the network structure $L(t)$ changes very slowly:

Assumption 4. There is at most one connection that changes in each time, for example, the connection between agents i and j , $i \neq j$.

Lemma 4. For any given graph G of size N , its nonzero eigenvalues grow monotonically with the number of added edges, i.e., for any added edge $\tilde{e}$, $\lambda_2(G + \tilde{e}) \geq \lambda_2(G)$.⁴¹

Corollary 1. Suppose that assumptions 1 and 4 hold, and the network with the Laplacian matrix L is connected at all times. Consider the swarm model (35) with an attraction/repulsion function (36). If

$$a\phi_t > \frac{2(N-1)\theta}{N}, \forall t > 0,$$

then all the agents of the swarm will converge to a hyperball centered at $\bar{x}$,

$$B_{\varepsilon_t} = \left\{ (x_1, \dots, x_N) \left| \frac{1}{N} \sum_{i=1}^N \|x_i - \bar{x}\|^2 \leq \varepsilon_t \right. \right\},$$

where

$$\phi_t = \min_{\tau \in [0, t]} \lambda_2(L(\tau)), \quad \varepsilon_t = \frac{b^2 c N^2}{2e \left(a\phi_t - \frac{2(N-1)\theta}{N} \right)^2}.$$

Furthermore, cohesion will be achieved within the bound ε_t in a finite time

$$t = - \frac{N}{2 \left(a\phi_t - \frac{2(N-1)\theta}{N} \right)} \ln \left(\frac{N\varepsilon_t}{2V(0)} \right).$$

Proof. By assumption 4 and Lemma 4, one knows that

$$\min\{\lambda_2(L(\tau-)), \lambda_2(L(\tau+))\} \leq \lambda_2(D(\tau)) \leq \max\{\lambda_2(L(\tau-)), \lambda_2(L(\tau+))\}.$$

The proof is completed.

Remark 5. The difference between Theorem 4 and Corollary 1 is that under Assumption 4, the term $D(\tau)$ in Theorem 4 is replaced by $L(\tau)$ in Corollary 1. Therefore, when the network changes very slowly, the condition can be simplified by using L instead. In the following theorems, only D are used which can also be replaced by L under assumption 4. Detailed analysis is omitted.

It is still not easy to verify whether or not the condition in Eq. (42) is satisfied for all $t \in R$. If the passivity degree $\theta = 0$, then system (8) is more likely a linear model; if $\theta > 0$, then $\phi_t > 0$ must be satisfied. Since for chaotic nodes, $\theta > 0$, one may be interested in the condition under which $\phi_t > 0$. In the following, some conditions are given to ensure $\phi_t > 0$ for all $t \in R$.

Lemma 5. For a connected graph G of order N , its second Laplacian eigenvalue λ_2 imposes upper bounds on the diameter $\text{diam}(G)$ and the mean distance $\rho(G)$ of G as follows:^{42,43}

- (i) $\lambda_2 \geq \frac{4}{N \text{diam}(G)}$, and
- (ii) $\lambda_2 \geq \frac{2}{(N-1)\rho(G) - (N-2)/2}$.

Theorem 5. Suppose that assumption 1 holds, and the network with the Laplacian matrix L is connected at all times. Consider the swarm model (35) with an attraction/repulsion function (36). If

$$a\delta_t > \frac{2(N-1)\theta}{N}, \forall t > 0, \quad (47)$$

then all the agents of the swarm will converge to a hyperball centered at $\bar{x}$,

$$B_{\varepsilon_t} = \left\{ (x_1, \dots, x_N) \left| \frac{1}{N} \sum_{i=1}^N \|x_i - \bar{x}\|^2 \leq \varepsilon_t \right. \right\}, \quad (48)$$

where $\delta_t = \min_{\tau \in [0, t]} \max \left\{ \frac{4}{N \text{diam}(D(\tau))}, \frac{2}{(n-1)\rho(D(\tau)) - (n-2)/2} \right\}$, and $\varepsilon_t = \frac{b^2 c N^2}{2e \left(a\delta_t - \frac{2(N-1)\theta}{N} \right)^2}$. Furthermore, cohesion will be achieved within the bound ε_t in a finite time

$$t = - \frac{N}{2 \left(a\delta_t - \frac{2(N-1)\theta}{N} \right)} \ln \left(\frac{N\varepsilon_t}{2V(0)} \right). \quad (49)$$

Proof. By Lemma 5, it is easy to see that $\delta_t \geq \phi_t$. The proof can be completed by using the same method as in the proof of Theorem 4.

Next, consider the stochastic switched swarm model (29), where

$$g_\gamma(x_i(t) - x_j(t)) = - \left(a_\gamma a_{ij}(t) - b_\gamma e^{-\frac{\|x_i(t) - x_j(t)\|^2}{c_\gamma}} \right) (x_i(t) - x_j(t)). \quad (50)$$

Theorem 6. Suppose that Assumptions 1 and 3 hold, and the network with the Laplacian matrix L is connected at all times. Consider the swarm model (29) with an attraction/repulsion function (50). If

$$a_\gamma \phi_t > \frac{2(N-1)\theta_\gamma}{N}, \quad \forall t > 0, \quad \gamma = 1, 2, \dots, \bar{N}, \quad (51)$$

then the expectations of all the agents of the swarm centered at $\bar{x}$ satisfy

$$\frac{1}{N} \sum_{i=1}^N \mathbf{E}(\|x_i(t) - \bar{x}(t)\|^2) \leq \eta, \quad (52)$$

where

$$\phi_t = \min_{\tau \in [0, t]} \lambda_2(D(\tau)), \quad \eta_{t,\gamma} = \frac{\left(b_\gamma \sqrt{\frac{Nc_\gamma}{2e}} + \sqrt{\frac{b_\gamma^2 Nc_\gamma}{2e} + \frac{4a \left(a_\gamma \phi_t - \frac{2(N-1)\theta_\gamma}{N} \right)}{N(N-1)}} \right)^2}{4N \left(a_\gamma \phi_t - \frac{2(N-1)\theta_\gamma}{N} \right)^2}, \quad \eta = \max_{t,\gamma} \eta_{t,\gamma}, \quad \text{and}$$

$\alpha = \frac{1}{2} \sum_{i=1}^N \alpha_i$. Furthermore, cohesion will be achieved within the bound η in a finite time

$$t = \max_{\gamma} \left(-\frac{N}{2 \left(a_{\gamma} \phi_t - \frac{2(N-1)\theta_{\gamma}}{N} \right)} \ln \left(\frac{N\eta_t}{2EV(0)} \right) \right). \quad (53)$$

In order to avoid possible collision when two agents are moving close to each other, the repulsive function should be sufficiently large around the origin. In what follows, an unbounded repulsive function is adopted. Consider the following swarm model:

$$\begin{aligned} \dot{x}_i(t) = & f(x_i(t)) - \sum_{j=1}^N a_{ij}(t)x_j(t) \\ & + \sum_{j=1, j \neq i}^N g_r(\|x_i(t) - x_j(t)\|)(x_i(t) - x_j(t)), \end{aligned} \quad (54)$$

where $g_r(\cdot)$ is the repulsive function to be further described below.

Assumption 5. For all $x, y \in R^n$, there exists a constant b such that

$$|g_r(\|y\|)| \leq \frac{b}{\|y\|^2}. \quad (55)$$

Theorem 7. Suppose that Assumptions 1 and 5 hold. Consider the swarm model (54). If

$$a\phi_t > \frac{2(N-1)\theta}{N}, \forall t > 0, \quad (56)$$

then all the agents of the swarm will converge to a hyperball centered at $\bar{x}$,

$$B_{\varepsilon_t} = \left\{ (x_1, \dots, x_N) \left| \frac{1}{N} \sum_{i=1}^N \|x_i - \bar{x}\|^2 \leq \varepsilon_t \right. \right\}, \quad (57)$$

where

$$\phi_t = \min_{\tau \in [0, t]} \lambda_2(D(\tau)), \varepsilon_t = \frac{bN}{2 \left(a\phi_t - \frac{2(N-1)\theta}{N} \right)}.$$

Furthermore, cohesion will be achieved within the bound ε_t in a finite time

$$t = -\frac{N}{2 \left(a\phi_t - \frac{2(N-1)\theta}{N} \right)} \ln \left(\frac{N\varepsilon_t}{2V(0)} \right). \quad (58)$$

Proof. Consider the same Lyapunov function candidate as in Eq. (11). By Theorems 1 and 4, one obtains

$$\begin{aligned} \dot{V}(t) \leq & - \left(a\lambda_2(D(t)) - \frac{2(N-1)\theta}{N} \right) \sum_{i=1}^N \|e_i(t)\|^2 \\ & + \sum_{i=1}^N \sum_{j=1, j \neq i}^N g_r(\|x_i(t) - x_j(t)\|) \\ & \times e_i^T(t)(x_i(t) - x_j(t)). \end{aligned} \quad (59)$$

By assumption 5 and based on the fact that $e_i(t) - e_j(t) = x_i(t) - x_j(t)$, one has

$$\begin{aligned} & \sum_{i=1}^N \sum_{j=1, j \neq i}^N g_r(\|x_i(t) - x_j(t)\|) e_i^T(t)(x_i(t) - x_j(t)) \\ & = \sum_{i=1}^N \sum_{j=1, j > i}^N [g_r(\|x_i(t) - x_j(t)\|) e_i^T(t)(x_i(t) - x_j(t)) \\ & \quad + g_r(\|x_j(t) - x_i(t)\|) e_j^T(t)(x_j(t) - x_i(t))] \\ & = \frac{1}{2} \sum_{i=1}^N \sum_{j=1, j \neq i}^N g_r(\|x_i(t) - x_j(t)\|) \|x_i(t) - x_j(t)\|^2 \\ & \leq \frac{b}{2} N(N-1). \end{aligned} \quad (60)$$

Then, it follows that

$$\begin{aligned} \dot{V}(t) \leq & - \left(a\phi_t - \frac{2(N-1)\theta}{N} \right) \sum_{i=1}^N \|e_i(t)\|^2 + \frac{b}{2} N(N-1) \\ & = -\frac{1}{N} \left(a\phi_t - \frac{2(N-1)\theta}{N} \right) \sum_{i=1}^N \|e_i(t)\|^2 \\ & \quad - (N-1) \left[\left(a\phi_t - \frac{2(N-1)\theta}{N} \right) \frac{1}{N} \sum_{i=1}^N \|e_i(t)\|^2 - \frac{b}{2} N \right]. \end{aligned} \quad (61)$$

If $\frac{1}{N} \sum_{i=1}^N \|e_i(t)\|^2 \geq \varepsilon_t$, then one has

$$\dot{V}(t) \leq \frac{1}{N} \left(a\phi_t - \frac{2(N-1)\theta}{N} \right) \sum_{i=1}^N \|e_i(t)\|^2. \quad (62)$$

Consequently, it is easy to verify that the trajectories enter the boundary $\frac{1}{N} \mathbf{E}(\sum_{i=1}^N \|x_i(t) - \bar{x}\|^2) = \varepsilon_t$ in a finite time

$$t \leq -\frac{N}{2 \left(a - \frac{2(N-1)\theta}{N} \right)} \ln \left(\frac{N\varepsilon_t}{2V(0)} \right).$$

This completes the proof.

Remark 6. The bound of the swarm, $\varepsilon_t = \frac{bN}{2(a\phi_t - \frac{2(N-1)\theta}{N})}$, depends on the size N . If N increases, the bound ε increases, which means that the boundary of cohesion increases due to the large number of agents. This is consistent with the biological phenomena, thanks to the particular attraction/repulsion function $g_r(\cdot)$ in Eq. (55).

Remark 7. The above analysis can still be useful even if the network is not connected. Suppose that the graph has k components of orders $n_1, n_2, \dots, n_k$, where $n_1 + n_2 + \dots + n_k = N$. Let L_i be the Laplacian matrix of component i of order n_i , $i = 1, 2, \dots, k$. All the results can be used to study the component i with the Laplacian matrix L_i , for each $i = 1, 2, \dots, k$, and then all components achieve cohesion. Detailed analysis is omitted here.

VII. SIMULATION EXAMPLES

Example 1. Consider the following stochastic switched swarm model:

FIG. 1. Agents' trajectories on the x - y plan in the stochastic switched swarm model of example 1.

$$dx_i(t) = \left[f_\gamma(x_i(t)) + \sum_{j=1, j \neq i}^N g_\gamma(x_i(t) - x_j(t)) \right] dt + v_i(t) dv_i,$$

where $i = 1, 2, \dots, 20$, $\gamma = 1, 2$, g_γ is shown in Eq. (2) with $a = 1$, $b = 20$, and $c = 0.2$, $v_i(t) = 0.02$, $f_1(y) = 4(1, 1)^T$, and $f_2(y) = 6(1, -1)^T$.

When $\gamma = 1$, $\mathbf{E}d\bar{x} = 4(1, 1)^T dt$, and when $\gamma = 2$, $\mathbf{E}d\bar{x} = 6(1, -1)^T dt$. Suppose that there is a triangular obstacle between the starting point and the target point. Simulation shows that the 20 agents first move in the direction of the 45° line ($\gamma = 1$), and after reaching a middle point move in the direction of the -45° line ($\gamma = 2$). Finally, they reach the target as shown in Fig. 1.

Hence, the proposed switched swarm model (29) is effective as the agents can change their directions at the middle point. By Theorem 3, cohesion of the swarm can be reached, as verified in Fig. 1. Note that $\theta = 0$, and the estimated bound of cohesion is $\sqrt{\varepsilon} = 3.836$, which is much larger than the actual value.

Example 2. Consider the following stochastic swarm model with a limited sensing range:

FIG. 2. Agents' trajectories on the x - y plan in the stochastic swarm model of example 2 with a limited sensing range.

FIG. 3. Agents' trajectories on the x - y plan in the stochastic swarm model with a limited sensing range and the nonlinear profile.

$$dx_i(t) = \left[f(x_i(t)) + \sum_{j=1, j \neq i}^N h(x_i(t) - x_j(t)) \right] dt + v_i(t) dv_i, \quad (63)$$

where $i = 1, 2, \dots, 20$, h is shown in Eq. (36) with $a = 1$, $b = 20$, $c = 0.2$ and $r = 1$, $v_i(t) = 0.01$, and $f(y) = 0.3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[y - \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right]$.

Here, $\mathbf{E}d\bar{x} = 0.3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\bar{x} - \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right] dt$, which is a periodic orbit. Suppose that there is a semi-circle obstacle between the starting point and the target, and an agent can only move to the target along that orbit.

By Theorem 6, cohesion of the swarm can be reached, as verified by the simulation result shown in Fig. 2. Note that $\theta = 0.3$, and $\phi_t = 0.66135$, so condition (47) is satisfied.

Consider the same model as in Eq. (63) except that $f(y) = (\sin(y_1), \cos(y_2))^T$, where $y = (y_1, y_2)^T$. The trajectories of agents are shown in Fig. 3 from which one can see that all the agents can eventually form a swarm centered at the average states. Specifically, all the agents move from the right bottom with random initiations to the left top side to form a swarm as in Fig. 3. Then, by Theorem 6, cohesion can still be reached in such a swarm model (63) with the nonlinear profile.

VIII. CONCLUSIONS

As demonstrated by the fast growing literature on complex swarming behaviors, models and corresponding analysis are in urgent need to gain insight into biological collective behaviors and thus guide novel design of distributed coordination rules for engineering multi-agent systems. In this paper, the stability of a continuous-time swarm model was investigated. It was shown that, under some mild conditions, all agents of the swarm can reach cohesion in a finite time. In addition, by incorporating stochastic noise and switched profiles, more realistic swarm models were studied; i.e. the bounds of cohesion were derived based on the parameters of

the model. Furthermore, swarms with limited sensing range and unbounded repulsions were studied by nonsmooth analysis, where the sensing range of each agent is limited and the possibility of collision among nearby agents is high. In future work, different types of attraction/repulsion interactions and the effects of different communication topologies on the cohesion of swarms will be further investigated.

ACKNOWLEDGMENTS

The authors would like to thank Fan Chung Graham, Wenlian Lu, W. K. W. Thong, Zhixi Wu, and Yanli Zou for their helpful suggestions.

This work was supported by the National Natural Science Foundation of China under Grant Nos. 61104145, 61322302, 61025017, 11072254, and 91023034, the Natural Science Foundation of Jiangsu Province of China under Grant No. BK2011581, the Research Fund for the Doctoral Program of Higher Education of China under Grant No. 20110092120024, the Fundamental Research Funds for the Central Universities of China, the Australian Research Council (ARC) Discovery Scheme under Grant DP130104765, and the Hong Kong Research Grants Council under the GRF Grant CityU1114/11.

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