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The Coulomb unitarity relation and some series of products of three Legendre functions

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We obtain from the off-shell Coulomb unitarity relation a closed expression for \( \Sigma_{l=0}^{\infty}(2l + 1)P_l(x) \times Q_l^{(1)}(y) Q_l^{(-1)}(z) \), and we consider some related series of products of Legendre functions.

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In this paper we shall consider the Coulomb unitarity relation\(^1\) and derive from this relation a closed expression for an infinite series of products of three Legendre functions, \( P_l, Q_l^{(1)}, \) and \( Q_l^{(-1)} \) [see Eq. (12)]. By taking the limit \( \gamma \to 0 \) we obtain agreement with an expression\(^1\) for the corresponding series, which exists in the literature. However, our expression has a much simpler form, which means that we have obtained a substantial reduction of the expression given in.\(^5\) After the derivation of our main result, Eq. (12), we shall briefly consider some related series of products of Legendre functions [see Eqs. (14)-(25)].

The unitarity relation, or generalized optical theorem, or Low equation, in quantum-mechanical scattering theory establishes a simple relation between the imaginary part of the off-shell \( T \) matrix and its half-off-shell elements.\(^4,5\) Suppressing the energy, \( E = k^2 + i\eta, \eta > 0 \), we have

\[
\langle p | T - T^\dagger | p' \rangle = -i\pi \kappa \int \langle p | T | k \rangle \langle k | T^\dagger | p' \rangle dk, \tag{1}
\]

where the integration is over the unit sphere. Equation (1) is valid when the potential associated with \( T \) has a short range. However, for the Coulomb potential \( V_c \), Eq. (1) has to be modified because the half-shell limit of the off-shell Coulomb \( T \) matrix \( T_c \) does not exist. Instead we have\(^6\)

\[
\langle p | T_c - T_c^\dagger | p' \rangle = -i\pi \kappa \int \langle p | T_c | k \rangle \langle k | T_c^\dagger | p' \rangle dk, \tag{2}
\]

where \( |k\infty\rangle \) is the so-called Coulombian asymptotic state and \( |k + \rangle \) is the Coulomb scattering state with energy \( (k + i\epsilon)^2, \epsilon > 0 \). The left-hand side of Eq. (2) is known in closed form [Ref. 4]. We rewrite the right-hand side by inserting

\[
\langle p | V_c | k + \rangle = \sum_{l=0}^{\infty} (4\pi)^{-1}(2l + 1)P_l(\hat{p} \cdot \hat{k}) \langle p | V_c | k + \rangle, \tag{3}
\]

and using the orthogonality relation

\[
\int P_l(\hat{p} \cdot \hat{k}) P_l(\hat{p} \cdot \hat{k}) dk = 4\pi(2l + 1)^{-1} \delta_{l0}. \tag{4}
\]

In Eq. (3), \( |k + \rangle \) is the partial-wave Coulomb scattering state. Denoting \( (p^2 + k^2)/(2pk) \) by \( y \) and assuming \( p > k \), we have\(^6\)

\[
\langle p | V_c | k + \rangle = 2\pi(\pi p)^{-1} e^{(1/2)p y} Q_l^{(1)}(y), \tag{5}
\]

where \( y \) is Sommerfeld's parameter, which is real \( (k > 0) \). It is important to note that \( Q_l^{(1)}(y) \) is not real-analytic: For the complex conjugate of both members of Eq. (5) we obtain

\[
\langle p | V_c | k + \rangle^\dagger = 2\pi(\pi p)^{-1} e^{(1/2)p y} Q_l^{(-1)}(y). \tag{6}
\]

In the above indicated way we obtain from Eqs. (2)-(6),

\[
\sum_{l=0}^{\infty} (2l + 1)P_l(x) Q_l^{(1)}(y) Q_l^{(-1)}(z) = -\frac{1}{\pi} \sin(\gamma y) (-\alpha + \alpha_-)^{1/2} \sinh \gamma y. \tag{7}
\]

Here \( x = \hat{p} \cdot \hat{k}, z = (p^2 + k^2)/(2pk), p' > k, \alpha_+ = yz - x \pm (y^2 - 1)^1/2(z^2 - 1)^1/2, \alpha_- = (\alpha_- + \alpha_-)^1/2. \tag{8}
\]

For convenience we introduce the quantity \( W \),

\[
W = W(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1. \tag{10}
\]

Then we have \( \alpha_+ + \alpha_- = W > 0 \)

\[
Y = (y^2 + z^2)/(2xyz - 1). \tag{11}
\]

so that Eq. (7) can be rewritten as

\[
\sum_{l=0}^{\infty} (2l + 1)P_l(x) Q_l^{(1)}(y) Q_l^{(-1)}(z) = -\frac{1}{\pi} \sin(\gamma y ln|Y|/W^{1/2} \sinh \gamma y. \tag{12}
\]

By analytic continuation it follows that Eq. (12) is valid for complex \( x, y, z, \gamma \). The series in Eq. (12) is convergent if \( Re x > 0, Re y > 0, Re z > 0, \) and

\[
|x + (x^2 - 1)^{1/2} < |y + (y^2 - 1)^{1/2}|, |z + (z^2 - 1)^{1/2}|. \tag{13}
\]

When \( Re x < 0, \) one should replace \( x \) by \( -x \) in Eq. (13), and similarly for \( y \) and \( z \). It may be noted that

\[
P_l(-y) = (-1)^l P_l(y), \tag{14}
\]

\[
Q_l^{(1)}(-z) = (-1)^l + 1 Q_l^{(-1)}(z). \tag{15}
\]

Now we are going to consider the more general expression

\[
F_{mn}(x_1, \ldots, x_n, z_1, \ldots, z_n) = \sum_{l=0}^{\infty} (2l + 1)P_l(x_1) \cdots P_l(x_m) Q_l(z_1) \cdots Q_l(z_n) \tag{16}
\]
(cf. Ref. 5) for \( n, m = 0, 1, 2, 3, x_i \in \mathbb{C}, y_i \in \mathbb{C} \setminus [0,1,1]. \) When \( \Re x_i > 0, \Re y_i > 0, \) this series is convergent if
\[
\prod_{i=1}^{m} |x_i + (x_i^2 - 1)^{1/2}| < \prod_{j=1}^{n} |z_j + (y_j^2 - 1)^{1/2}|.
\]

(15)

Let us first consider \( F_{12}. \) By taking the limit for \( y \to 0 \) in Eq. (12) we obtain
\[
F_{12}(x,y,z) = \sum_{i=0}^{\infty} (2l + 1)P_l(x)\bar{Q}_l(y)Q_l(z)
\]
\[
= \frac{1}{2} \int_{-1}^{1} dx \sum_{l=0}^{\infty} (2l + 1)P_l(x)\bar{Q}_l(y)Q_l(z)
\]
\[
= \frac{1}{4} \int_{-1}^{1} W^{-1/2} \ln \frac{y - x + W^{1/2}}{y - x - W^{1/2}} dx.
\]

(16)

Putting \( a = (y^2 - 1)^{1/2}(x^2 - 1)^{1/2}, \) \( v = \cosh((yz - x)/a), \)
\( v_{\pm} = \cosh((yz + 1)/a) \) we get \( W^{1/2} = a \sinh v \) and
\[
F_{12}(x,y,z) = (1/2a) \int_{-\infty}^{+\infty} v dv \cosh v - (y - x)/a.
\]

(25)

According to formula 2.478.7 of Ref. 7 we have
\[
\int_{-1}^{1} x dx \cosh 2x - \cos 2t
\]
\[
= \frac{1}{2 \sin 2t} \left[ L(u + t) - L(u - t) - 2L(t) \right],
\]

(26)

where \( u = \arctan(\tanh x /t) \) and \( L \) is Lobachevski's function, defined by
\[
L(x) = - \int_{0}^{x} \ln(\cosh t) dt.
\]

(27)

This implies that \( F_{03} \) cannot be expressed in terms of elementary functions.

By using the series representation
\[
L(x) = - x \ln 2 + (1/2) \sum_{n=1}^{\infty} (-1)^{n-2} \sin 2nx,
\]

(28)

the right member of Eq. (26) can be rewritten as
\[
\frac{1}{4 \sin 2t} \sum_{n=1}^{\infty} (-1)^{n-2} \sin 2nt \cos 2nu.
\]

(29)

We point out that on p. 377 of Ref. 6, Eq. (56.8.1), a closed formula is given for the series
\[
\sum_{l=0}^{\infty} (2l + 1)P_l(x)P_l(y)P_l(-z),
\]

(30)

where \( m \in \mathbb{N} \) and \( x, y, z \in [-1,1]. \)

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