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Nonperturbative confinement in quantum chromodynamics.

IV. Improved treatment of Schoenmaker's equation

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An improved ansatz for the three-gluon vertex function is treated; and it is shown that the gluon propagator has a double pole at the origin of the p^2 plane, as well as a tachyon on the spacelike real axis, at least in this approximation.

I. INTRODUCTION

The present paper constitutes the conclusion of a study of the infrared behavior of the gluon propagator in the Landau gauge. In previous papers,¹ to which we shall refer as I, II, and III, respectively, we investigated an approximation scheme for the gluon propagator Dyson-Schwinger equation that was initiated by Mandelstam.²

The basic idea of the approximations is to truncate the Dyson-Schwinger equation by introducing an ansatz for the three-gluon vertex that (a) involves only the propagator itself, and (b) is inspired by (Mandelstam's ansatz I and II), or is strictly consistent with (Schoenmaker's ansatz, III) the Slavnov-Taylor identity. It has been argued that the longitudinal part of the vertex function (i.e., the part that contributes to the Slavnov-Taylor identity) need not be relevant to the Dyson-Schwinger equation. Indeed Gardner³ constructed a model in which the vertex function consists of two parts, one of which contributes only to the Slavnov-Taylor identity, while the other contributes only to the Dyson-Schwinger equation. However, it can be shown that the two parts of Gardner's ansatz do not have the same scaling properties under renormalization: one part has the correct number of factors Z_3 , while the other does not. Accordingly, we may reject the Gardner ansatz as a valid criticism of the general method. On the other hand, a recent paper of Zhang⁴ is much more convincing. He shows that the contribution of the longitudinal part of the vertex function reduces essentially to a term that is indistinguishable from a tadpole, so that the nontrivial parts of the approximate Dyson-Schwinger equation arise from transverse contributions to the vertex function. Both Gardner's and Zhang's treatments apply specifically to an axial gauge propagator, which has special properties (in particular, orthogonality to the gauge vector); and hence they are not relevant to a study in the Landau gauge.

In III we studied an improved version of the Mandelstam ansatz, but with a simplification that allowed us to reduce the equation to a fourth-order nonlinear differential equation. In this paper, we complete this analysis by removing the simplification, which involves us in the treatment of a sixth-order nonlinear differential equation. This equation is subjected to numerical analysis as it stands: we confirm the p^{-4} behavior as $p \rightarrow 0$, p being the gluon momentum; and we also find a tachyon state, as in III. There are probably no first-sheet complex branch points—a deficiency of the model of I and II—although there is a neighborhood of the origin

that is inaccessible to the computer, because of large cancellations, so one cannot be completely sure. In order to remove these cancellations, the sixth-order differential equation is transformed into an integral equation by two successive implementations of the method of variation of parameters; and in this form the equation is suited to a rigorous demonstration of the existence of a solution. We show that the solution is analytic in a (cut) neighborhood of the origin, so that an accumulation of first-sheet complex branch points is excluded.

II. NUMERICAL ANALYSIS OF DIFFERENTIAL EQUATION

The form factor multiplying the bare gluon propagator (see III) can be written

$$F(x) = A/x + \gamma x + \phi(x)x^3, \quad (2.1)$$

where A is an unknown constant (which, however, can be scaled away), where

$$\gamma = \frac{348}{689} \approx 0.617, \quad (2.2)$$

and where $\phi(x)$ is a function that satisfies the nonlinear integral equation

$$\begin{aligned} x^6 G(x) = & -\frac{1}{4} \int_0^x dy (x-y)^3 (x^2 + 10xy + y^2) \phi(y) \\ & + \frac{1}{8} \int_0^{x/4} dy x^{3/2} (x-4y)^{3/2} \\ & \times (x^2 + 20xy + 12y^2) \phi(y), \end{aligned} \quad (2.3)$$

where

$$G(x) = [\gamma + x^2 \phi(x)] / [1 + \gamma x^2 + x^4 \phi(x)]. \quad (2.4)$$

The details can be found in III.

The second integral in (2.3), involving the surd, is rather awkward; but fortunately it has been shown numerically,⁵ by means of cubic splines, that the truncated equation

$$x^6 G(x) = -\frac{1}{4} \int_0^x dy (x-y)^3 (x^2 + 10xy + y^2) \phi(y) \quad (2.5)$$

has a solution that resembles that of the full equation (2.3), the difference being not qualitative, but merely quantitative and relatively minor. In order to study the qualitative properties of the solution, it suffices to look at (2.5). We shall

accordingly subject this equation to numerical analysis and to a rigorous proof of existence.

The integral equation can be converted into a differential equation, namely

$$\left[\frac{d}{dx} \right]^6 [x^6 G(x)] + 18x^2 \phi''(x) + 144x\phi'(x) + 138\phi(x) = 0. \quad (2.6)$$

This equation has been treated numerically (in double precision) by the Runge-Kutta method. As usual, an asymptotic series expansion must be made in a neighborhood of the infrared point $x = 0$.

The results of the numerical work⁶ can be summarized as follows.

(a) There are complex branch points on secondary Riemann sheets that are connected to the principal Riemann sheet through the timelike cut along $-\infty < x < 0$. There seem to be no complex branch points on the principal Riemann sheet, although this result is not completely conclusive, since a small region around the infrared point is inaccessible, due to large cancellations.

(b) There is a (ghost) pole on the spacelike axis at

$$x = x_p \approx 2.831, \quad (2.7)$$

much as in case III. This may be a signal that our vacuum is unstable, or it may be merely an artifact of our approximations.

III. REFORMULATION OF THE EQUATION

In this section, we reformulate Eq. (2.5) in such a way that there are no cancellations in the infrared region. Using such a form, we shall outline the proof that a solution exists (by means of the Banach Theorem). Moreover, we could also set up a computer analysis that is not plagued by cancellations (however, we have not done this).

The necessary manipulations are tedious, and we shall merely sketch the method here. Further details can be found in Ref. 6. Equation (2.6) is a nonlinear sixth-order differential equation. We transform it in two steps. First, observe that the linear, homogeneous equation

$$18x^2 \psi''(x) + 144x\psi'(x) + 138\psi(x) = 0 \quad (3.1)$$

has the two solutions

$$\psi_{\pm}(x) = x^{\pm\beta-7/2}, \quad (3.2)$$

where $\beta = (165)^{1/2}/6$. We resolve Eq. (2.6) by the method of variation of parameters, treating the first term, involving a sixth-order derivative of $x^6 G$, as an inhomogeneity. The result is an integral over this derivative, multiplied by a kernel. The six derivatives can be removed by six partial integrations, the result being

$$\begin{aligned} x^4 G''''''(x) + 20x^3 G''''(x) &+ \frac{355}{3} x^2 G''(x) + \frac{680}{3} x G'(x) + \frac{865}{9} G(x) \\ &= -18\phi(x) - \frac{455}{54\beta} x^{-7/2} \\ &\times \int_0^x dy y^{5/2} \left(\left(\frac{y}{x} \right)^{\beta} - \left(\frac{x}{y} \right)^{\beta} G(y) \right). \end{aligned} \quad (3.3)$$

The second step consists in resolving this fourth-order nonlinear integrodifferential equation by applying the method of variation of parameters again. In order to do this expeditiously, we add terms proportional to $x^2 G''$, $x G'$, and G to both sides of (3.3), as well as $18G/x^2$, the latter being the most singular part of $18\phi(x)$. The corresponding homogeneous equation is

$$\begin{aligned} x^6 H'''' + 20x^5 H''' + \frac{975}{8} x^4 H'' + 225x^3 H' \\ + \left[\frac{36465}{256} x^2 + 18 \right] H = 0, \end{aligned} \quad (3.4)$$

with the four solutions

$$H_i(x) = x^{-11/4} \exp[\beta_i x^{-1/2}], \quad (3.5)$$

$$\beta_1 = (72)^{1/4}(1+i), \quad \beta_2 = (72)^{1/4}(1-i), \quad (3.6)$$

$$\beta_3 = (72)^{1/4}(-1+i), \quad \beta_4 = (72)^{1/4}(-1-i).$$

These homogeneous solutions are used to resolve Eq. (3.3), the result being

$$G(x) = \sum_{i=1}^4 G_i(x), \quad (3.7)$$

where

$$\begin{aligned} G_i(x) &= \frac{1}{2} 72^{-1} \beta_i x^{-11/4} \exp(\beta_i x^{-1/2}) \\ &\times \int_0^x dy y^{5/4} \exp(-\beta_i y^{-1/2}) \Sigma(y), \end{aligned} \quad (3.8)$$

with

FIG. 1. Sketch of the domain of applicability of the Banach Theorem. (a) The full domain. (b) The region near the negative imaginary axis on a larger scale. Note that the boundary of the region of proved analyticity intercepts the negative real axis and penetrates the second Riemann sheet (dotted lines). Thus the origin cannot be an accumulation point of first-sheet singularities.

$$\begin{aligned}\Sigma(x) = & 18\gamma + \frac{106745}{2304}x^4G''(x) + \frac{85}{3}x^3G'(x) \\ & + \frac{85}{24}x^2G(x) + \frac{18x^2G^2(x)}{x^2G(x) - 1} - \frac{455}{54\beta}x^{-3/2} \\ & \times \int_0^x dy y^{5/2} \left(\left(\frac{y}{x} \right)^\beta - \left(\frac{x}{y} \right)^\beta G(y) \right). \quad (3.9)\end{aligned}$$

The first- and second-order derivatives in Eq. (3.9) can be removed by partially integrating Eq. (3.8); but there is a subtlety. One has to deform the contour $(0, x)$ in such a way that the boundary term at $y = 0$ vanishes; it turns out to be sufficient to ensure that the contour for G_2 and G_3 approaches the origin along the positive imaginary axis, while that for G_1 and G_4 approaches it along the negative imaginary axis. Such deformations are allowed if the G_i are analytic in the cut plane; and we can show that this is the case.

After this adjustment, Eqs. (3.7)–(3.9) constitute a nonlinear integral equation without derivatives. It is eminently suited to an existence proof by means of the contraction mapping (Banach) principle. As usual, the integration interval $(0, x)$ is transformed to $(0, \infty)$; and one finally proves

that a unique solution $G(x)$ exists, that is analytic in a half-circle in the right half-plane and in a curved region in the left half-plane (see Fig. 1). It is most significant that this curved region crosses the cut $(-\infty, 0)$ and penetrates the second Riemann sheet. Thus the infrared singularity is certainly *not* the accumulation point of first-sheet complex branch points (as it was in I and II). Further details can be found in Ref. 6.

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