A note on applications of a trajectory-based small-gain theorem to decentralized stabilization of switching networks with generalized dead zones

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Abstract

We propose a method for uniform decentralized stabilization for a class of large-scale networks of nonlinear switched systems in lower-triangular form with unknown arbitrary switchings. In contrast to the class of networks with the same structure of interconnections that was considered in [1], the dynamics of each subsystem of our network satisfies more general conditions and its input–output maps are not necessarily right invertible, for instance, they can have dead zones. Another difference from [1] is that the current result is proved by a different approach. Instead of using a small-gain theorem in terms of ISS Lyapunov functions as in [1], we use the trajectory-based small-gain theorem for switched systems proved in [2].

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1. Introduction

Beginning with [3], small-gain theorems became very useful for solving nonlinear control problems in presence of dynamic uncertainties [4–6] and for studying stability properties of

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interconnected systems [7–9]. For the general case of \(N \geq 2\) interconnected nonlinear systems of ordinary differential equations (ODE), small-gain theorems were proved in [10–12]. For other classes of systems, small-gain theorems were proved in [13–15] and in other related papers, and their various applications are proposed, for instance, in [22,18,1], etc.

During the last decade, the problem of uniform stabilization with arbitrary or unknown switching signals has become popular [16–18]. This interest was motivated by the fact that, if each constant value of the switching parameter produces a globally asymptotically stable system of ODE, then it does not necessarily imply that every piecewise constant switching signal produces a stable system of ODE [19]. New conditions for ISS of switched systems and their applications were obtained in recent works [20] (the ISS and integral ISS of time-varying switched systems with time delays by means of the Lyapunov–Razumikhin method), [21] (stability conditions for randomly switched systems with applications to the problem of consensus for multi-agent systems with nonlinear dynamics).

In [2], a new small-gain theorem was proved for uniform input-to-state stability (UISS) of large-scale networks composed of nonlinear switched systems with arbitrary switching signals. In general, there are two kinds of small-gain theorems, namely, small-gain theorems expressed in terms of trajectories [3,10] and small-gain conditions formulated in terms of Lyapunov functions and Lyapunov gains [11]. Work [2] addresses trajectory-based small-gain conditions for switched nonlinear systems. In the current paper, we apply [2] to decentralized stabilization of networks of nonlinear systems with unknown switching signals. To provide some motivation for studying the problem of decentralized stabilization, let us consider the interconnection of one-dimensional control systems \(\dot{x}_{i,1} = u_i + \varrho_{i,1}(x_{1,1}, \ldots, x_{N,1}), \ i = 1, \ldots, N\) with scalar states \(x_i\), scalar controls \(u_i\), with some nonlinear and smooth \(\varrho_{i,1}(\cdot)\) such that \(\varrho_{i,1}(0) = 0\), and without any switchings (as the most trivial example). It can be immediately stabilized by some controller \(u_i = \hat{u}_i(x_{1,1}, \ldots, x_{N,1})\) with \(\hat{u}_i(0) = 0\) but design of a decentralized feedback \(u_i = \hat{u}_i(x_{i,1})\) requires some efforts. Of course, this goal can be achieved by the small-gain approach and the solution to this problem is just a corollary of our auxiliary Lemma 1 given below.

Large-scale networks appear in many applications such as teleoperations, power networks, multi-agent systems of autonomous vehicles, platooning, logistics, production and transport networks, etc. Very often, design of decentralized or distributed controllers has a certain advantage. For instance, suppose that we want to stabilize a large-scale network composed of \(N = 10^5\) or \(N = 10^7\) interconnected agents or nodes, and the output of each agent works as a destabilizing disturbance for its several “neighbors”. However, a feedback controller defined as a function of all the \(N\) states would be not convenient, of course. Then it is natural to look for a decentralized or distributed feedback due to the complexity of the network, so that each agent with its feedback needs to know only its own state and perhaps the states of its several “neighbors”. Then the challenge is to design a feedback with a suitable gain assignment for each agent to satisfy the corresponding small-gain condition.

We consider a hierarchically interconnected large-scale network of nonlinear switched systems each of which has a lower triangular form, whose feedback linearizable and classical version was proposed in [25]. Beginning with early 90th, the triangular form systems as well as backstepping and adding an integrator techniques became very popular and efficient in theory and applications, see, for instance, [26–35]. In our problem formulation, the structure of interconnections of triangular form systems is the same as in the recent paper [1] by the current authors. However, we replace the assumption of right invertibility of the input-output maps considered in [1] with more general practical right invertibility, which, in particular,
includes dead zones as a special case. Decentralized control of such interconnections in the special case of strict-feedback forms of ODE is important in engineering applications, e.g. in power networks [24,23]. In our case, switching signals can appear, for instance, when we deal with instant changes of some parameters of the dynamics or with changes of the topology of interconnections, in particular, caused by instant damage of some nodes, interconnecting channels, etc. We propose a solution to the problem of uniform, switching-independent and decentralized ISS stabilization for the above-mentioned class. While the result from [1] is obtained by a Lyapunov-based gain assignment, the background of our current method is the recent trajectory-based small-gain theorem proved in [2]. Since [2] also addresses the small-gain conditions for practical stabilization, we extend the class of networks under consideration. Furthermore, our main result can be extended to some cases when the network has dynamic uncertainties and only the trajectory-based gains of these uncertainties are known. This case is discussed in Section 7.

2. Notation, preliminaries, and main definitions

Our basic notation is summarized in the following table:

| $A \times B$ | the Cartesian product of sets $A$ and $B$, i.e., the set of all ordered pairs $(a, b)$, such that $a \in A$ and $b \in B$ |
| $[a, b]$ for any $a \in \mathbb{R}$, $b \in \mathbb{R}$ | the set defined by $\{ t \in \mathbb{R} \mid a \leq t \leq b \}$ |
| $[a, b]$ for any $a \in \mathbb{R}$, $b \in \mathbb{R}$ | the set defined by $\{ t \in \mathbb{R} \mid a \leq t < b \}$ |
| $[a, b]$ for any $a \in \mathbb{R}$, $b \in \mathbb{R}$ | the set defined by $\{ t \in \mathbb{R} \mid a < t \leq b \}$ |
| $[a, b]$ for any $a \in \mathbb{R}$, $b \in \mathbb{R}$ | the set defined by $\{ t \in \mathbb{R} \mid a < t < b \}$ |
| $\mathbb{R}$ | the set defined by $\{ t \in \mathbb{R} \mid b \leq t \leq a \}$ |
| $\mathbb{R}^+$ | the set defined by $\{ t \in \mathbb{R} \mid t > a \}$ |
| $\mathbb{Z}$ | the set defined by $\{ t \in \mathbb{R} \mid t \geq a \}$ |
| $\mathbb{Z}_{\geq k}$ for any $k \in \mathbb{Z}$ | the set defined by $\{ t \in \mathbb{R} \mid t \geq k \}$ |
| $\mathbb{Z}_{< k}$ for any $k \in \mathbb{Z}$ | as in [36] |
| $\mathbb{A}^T$ for any $A \in \mathbb{R}^{m \times n}$ | the transpose of this matrix $A$ |
| $\xi \in \mathbb{R}^N$ | $\xi$ is an $N$-dimensional column vector |
| $\langle \xi, \eta \rangle$ for any $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$ | the scalar product of $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^N$, the Euclidean norm of $\xi$, i.e., $|\xi| = \sqrt{\langle \xi, \xi \rangle}$ |
| $|\xi|$ for any $\xi \in \mathbb{R}^N$ | the standard norm of matrix $A$, i.e., $\|A\| = \sup_{\xi \in \mathbb{R}^n,|\xi|=1} |A\xi|$ |
| $\|A\|$ for any $A \in \mathbb{R}^{m \times n}$ | the normed vector space of continuous functions $x(\cdot) : [a, b] \to \mathbb{R}^n$ with the norm defined by $\max_{t \in [a, b]} |x(t)|$ |
| $C([a, b]; \mathbb{R}^k)$ for any $a \in \mathbb{R}$, $b \in [a, +\infty[$, $k \in \mathbb{N}$ | the normed vector space of functions $x(\cdot) : [a, b] \to \mathbb{R}^k$ whose $n$ derivatives are well-defined and continuous on $[a, b]$ with |
Remark 1. Accordingly, we may consider $C^∞(I; \mathbb{R}^k), L_∞(I; \mathbb{R}^k)$, etc. for any interval $I$, for instance, for $I = \mathbb{R}$. Sometimes we just say “a function of class $L_∞, C^∞,$ ” etc. if it is clear what are the corresponding domains, and, in such cases, we just write $\| \cdot \|_{L_∞}$. Also we can write $\| \cdot \|_{C^∞}$, instead of $\| \cdot \|_{L_∞(\mathbb{R}; \mathbb{R}^k)}$. Let us also note that, although the symbol “×” is reserved for the Cartesian product, as it is stated above, in several exceptional cases, we also use it to denote the usual product with slight abuse of notation (see, e.g., (23) below).

By Hadamard’s Lemma, we call the following simple statement: If $F \in C^{m+1}(\mathbb{R}^N; \mathbb{R})$ with $\mu \in \mathbb{Z}_{\ge 0}$ then $F(\xi) = \Phi(\xi, \eta)(\xi - \eta)$ for all $\xi \in \mathbb{R}^N$, $\eta \in \mathbb{R}^N$, where $\Phi(\xi, \eta) = \int_0^1 \nabla F(\eta + s(\xi - \eta))ds$ is of class $C^\mu$.

A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$, and $\mathcal{K}_∞$ is the set of all the unbounded $\mathcal{K}$-functions. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class $\mathcal{K}\mathcal{L}$ if for each fixed $t \ge 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed $s \ge 0$, we have $\beta(s, t) \rightarrow 0$ as $t \rightarrow +\infty$ and $\beta(s, t) \rightarrow 0$ for all $t \ge 0$.

Consider the following nonlinear switched system

$$x(t) = F_\sigma(t)(x(t), D(t)), \quad t \in \mathbb{R}, \quad (1)$$

with the piecewise constant switching signal $\mathbb{R} \ni t \mapsto \sigma(t) \in \{1, \ldots, M\}$, with the state $x \in \mathbb{R}^n$, and with the external disturbance input $D \in \mathbb{R}^N$. Each $F_\sigma$ is continuous in $(t, x, D)$ and locally Lipschitz continuous in $(x, D)$. Given any $D(\cdot) \in L_∞(\mathbb{R}; \mathbb{R}^N)$, let $\|D(\cdot)\|_{L_∞}$ denote its $L_∞$-norm as we proposed above.

Definition 1. System (1) is said to be uniformly input-to-state practically stable (UISpS) if there are $\beta \in \mathcal{K}\mathcal{L}, \gamma \in \mathcal{K}$, and $d \ge 0$ such that for each $(t_0, x^0) \in \mathbb{R} \times \mathbb{R}^n$, each $D(\cdot) \in L_∞(\mathbb{R}; \mathbb{R}^N)$ and each piecewise constant $\sigma(\cdot)$ the solution $x(t)$ of the Cauchy problem $x(t_0) = x^0$ for (1) with these $D(\cdot)$, $\sigma(\cdot)$ satisfies the following condition

$$\forall t \ge t_0 \quad |x(t)| \le \max\{\beta(|x(t_0)|), t - t_0\}; \gamma(\|D(\cdot)\|_{L_∞}); d). \quad (2)$$

Definition 2. System (1) is said to be uniformly input-to-state stable (UISS), if it is UISpS with $d = 0$ in Eq. (2).

For the ODE systems, the notion of ISS was introduced in [37], and it was studied later, for instance, in [38], whereas the current definition of UISS for switched systems was proposed in [39].
Remark 2. The difference between Definition 1 and Definition 2 is as follows. The UISS property means that, if the system is UISS (Definition 2), then it becomes globally asymptotically stable, whenever $D(\cdot) = 0$. Accordingly, all the trajectories converge to the equilibrium (which is the origin in our case), whenever the external disturbance $D(\cdot)$ vanishes, in the case of UISS (Definition 2). However, if the system is UISpS only (Definition 1) and the external disturbance $D(\cdot)$ vanishes, then the trajectories converge to the equilibrium with some “overshoot” $d > 0$ (for UISS from Definition 2 the “overshoot” $d$ equals zero, i.e., the convergence is exact). In the general case $D(\cdot) \neq 0$, there is an “overshoot” anyway, but, for UISpS (Definition 1), this overshoot equals $\max \{ \gamma(\|D(\cdot)\|_\infty); d \}$ (see Eq. (2)), and, for UISS, the overshoot is equal to $\gamma(\|D(\cdot)\|_\infty)$ as it follows from Definition 2.

3. Main purpose and results

As in the related paper [1], we consider a network of switched control systems in the following form

$$
\dot{x}_{i,j} = f_{i,j}(x_{i,j}, \ldots, x_{i,j+1}) + \pi_{i,j,\sigma(t)}(\bar{X}_j, D(t)), \quad 1 \leq j \leq v_i - 1, \\
\dot{x}_{i,v_i} = f_{i,v_i}(x_i, \ldots, x_{i,v_i}, u_i) + \pi_{i,v_i,\sigma(t)}(\bar{X}_v, D(t)), \\
\text{for} \quad i = 1, \ldots, N,
$$

(3)

with the state vector components $x_{i,j} \in \mathbb{R}^{m_{i,j}}$ (with $m_{i,j} \leq m_{i,j+1}$), the controls $u_i = x_{i,v_i+1} \in \mathbb{R}^{m_{v_i+1}}$, the external disturbances $D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^d)$, and with the unknown piecewise constant switching signals $\mathbb{R} \ni t \mapsto \sigma(t) \in \{1, \ldots, M\}$, where $X_{i,p} := [x_{i,1}^T, \ldots, x_{i,p}^T]^T$ for all $p = 1, \ldots, v_i$, $i = 1, \ldots, N$, and $\bar{X}_p := [X_{1,\min[p,v_i]}^T, \ldots, X_{N,\min[p,v_N]}^T]^T$ for all $p = 1, \ldots, \max v_i$.

In [1], the physical motivation of this structure (3) is discussed in detail (see e.g. Remark 3 in [1]). Let us just briefly note that such structure appears for instance in power networks [23]. In contrast to [1], we assume that system (3) satisfies the following conditions:

(AES) (Equilibrium and Smoothness) All $f_{i,j}$, $\pi_{i,j,\sigma}$ are of class $C^{v+1}$, where $v = \max_{1 \leq i \leq N}\{v_i\}$ and $f_{i,j}(0) = \pi_{i,j,\sigma}(0) = 0 \in \mathbb{R}^{m_{i,j}}$.

(ApRI) (Practical Right Invertibility) Every map $x_{i,j+1} \mapsto f_{i,j}(X_{i,j}, x_{i,j+1})$, $1 \leq i \leq N$, $1 \leq j \leq v_i$, is practically right invertible, i.e., there exists $R_{i,j}(\cdot)$ of class $C(\mathbb{R}^{m_{i,j} + \cdots + m_{i,j+1}}; [0, +\infty[)$ and there is a map $(X_{i,j}, w) \mapsto \alpha_{i,j}(X_{i,j}, w)$ of class $C^v(\mathbb{R}^{m_{i,j} + \cdots + m_{i,j+1}} \times \mathbb{R}^{m_{i,j}})$ such that $\alpha_{i,j}(0, 0) = 0$ and such that $f_{i,j}(X_{i,j}, \alpha_{i,j}(X_{i,j}, w)) = w$, whenever $|w| \geq R_{i,j}(X_{i,j})$.

Remark 3. Note that, if some $R_{i,j}(\cdot)$ from (ApRI) satisfies $R_{i,j}(0) = 0$, then without loss of generality we may assume that there is $r_{i,j}^* > 0$ s.t. $|X_{i,j}| \leq r_{i,j}^* \Rightarrow R_{i,j}(X_{i,j}) = 0$. Indeed, if $R_{i,j}(0) = 0$, i.e., $\forall w \quad f_{i,j}(0, \alpha_{i,j}(0, w)) = w$, then $\exists R_{i,j} > 0$ such that, if $|X_{i,j}| \leq \bar{R}_{i,j}$ and $|w| \leq \bar{R}_{i,j}$, then, $\frac{\partial}{\partial w} f_{i,j}(X_{i,j}, \alpha_{i,j}(X_{i,j}, w))$ is positive definite. Take any $r_{i,j}^* \in [0, \frac{1}{2}\bar{R}_{i,j}]$ such that $R_{i,j}(X_{i,j}) < \frac{1}{2}\bar{R}_{i,j}$ whenever $|X_{i,j}| \leq 2r_{i,j}^*$. Then for each $|X_{i,j}| \leq 2r_{i,j}^*$ the map $\bar{w} \mapsto \mathcal{F}(X_{i,j}, \bar{w}) := f_{i,j}(X_{i,j}, \alpha_{i,j}(X_{i,j}, \bar{w}))$ is a diffeomorphism of $\mathbb{R}^{m_{i,j}}$ onto $\mathbb{R}^{m_{i,j}}$. Let $\Psi(\cdot, \cdot)$ be its inverse, i.e., $\forall w \quad \mathcal{F}(X_{i,j}, \Psi(X_{i,j}, w)) = w$, whenever $|X_{i,j}| \leq 2r_{i,j}^*$. Take any $\lambda(\cdot)$ in $C^\infty(\mathbb{R}; [0, 1])$ such that $(|r| \leq \frac{1}{2}r_{i,j}^* ) \Rightarrow (\lambda(r) = 1)$ and $(|r| \geq 2r_{i,j}^* ) \Rightarrow (\lambda(r) = 0)$ and such that $0 \leq \lambda(r) \leq 1$ for all $r \in \mathbb{R}$. Then the new feedback $\hat{\alpha}_{i,j}(X_{i,j}, w) := \alpha_{i,j}(X_{i,j}, 1 - \hat{\lambda}(|X_{i,j}|))w + \hat{\lambda}(|X_{i,j}|)\Psi(X_{i,j}, w)$ also satisfies (ApRI) with some new continuous “radius of noninvertibility” $\hat{R}_{i,j}(\cdot)$ such that $|X_{i,j}| \leq r_{i,j}^* \Rightarrow \hat{R}_{i,j}(X_{i,j}) = 0$. 


If we have $R_{i,j}(X_{i,j}) = 0$ for all $X_{i,j} \in \mathbb{R}^{m_{i,1}+\ldots+m_{i,j}}$, in Assumption (ApRI), then (ApRI) becomes the same property of right invertibility as Assumption (II) from [1]:

(ARI) (Right Invertibility) Each map $x_{i,j+1} \mapsto f_{i,j}(X_{i,j},x_{i,j+1})$, is right invertible, i.e., there is a map $(X_{i,j}, w) \mapsto \alpha_{i,j}(X_{i,j}, w)$ of class $C^1(\mathbb{R}^{m_{i,1}+\ldots+m_{i,j}} \times \mathbb{R}^{m_{i,j}}; \mathbb{R}^{m_{i,j}})$ with $\alpha_{i,j}(0,0) = 0$ such that $f_{i,j}(X_{i,j}, \alpha_{i,j}(X_{i,j}, w)) = w$ for all $w \in \mathbb{R}^{m_{i,j}}, \ X_{i,j} \in \mathbb{R}^{m_{i,1}+\ldots+m_{i,j}}, \ 1 \leq i \leq N, \ 1 \leq j \leq v_i$.

Also note that, if (ApRI) holds, then $f_{i,j}(X_{i,j}, \mathbb{R}^{m_{i,j+1}}) = \mathbb{R}^{m_{i,j}}$ for each $X_{i,j}$. This follows from the Brouwer fixed-point theorem and can be proved in the same way as in [40]. Hence, according to [40], if system (3) satisfies (AES), (ApRI), then system (3) is globally controllable for every fixed $(\sigma(\cdot), D(\cdot))$. Our main goal is to prove the following theorem.

**Theorem 1.** 1. Under Assumptions (AES),(ApRI), there is a decentralized feedback in the form $u_i = u_i(X_{i,v_i}), \ i = 1, \ldots, N$ of class $C^1$ with $u_i(0) = 0$ such that the closed-loop system (3) with $u_i = u_i(X_{i,v_i})$ is UISpS at $X^* = 0$.

2. If $R_{i,j}(0) = 0$, for all $i = 1, \ldots, N, \ j = 1, \ldots, v_i$, then there is a decentralized feedback $u_i = u_i(X_{i,v_i}), \ i = 1, \ldots, N$ of class $C^1$ with $u_i(0) = 0$ such that the closed-loop system (3) with $u_i = u_i(X_{i,v_i})$ is UIS at $X^* = 0$. In particular, this is true under Assumptions (AES),(ARI).

**Remark 4.** To show the difference between (ApRI) and (ARI), take any $\delta \in [0, 1]$ and consider the following system

$$
\dot{x}_1 = h_\delta(x_2), \quad \dot{x}_2 = u,
$$

with the state $[x_1, x_2]^T \in \mathbb{R}^2$, and the control $u \in \mathbb{R}$, where $h_\delta(\cdot)$ is any function of class $C^\infty(\mathbb{R}; \mathbb{R})$ such that

$$
\begin{align*}
\left| h_\delta(x_2) \right| &= \delta + \delta^3, & \text{whenever } |x_2| \geq \delta + \delta^3, \\
\left| h_\delta(x_2) \right| &= 0, & \text{whenever } |x_2| \leq \delta, \\
\left| h_\delta(x_2) \right| &\leq \delta + \delta^3, & \text{whenever } |x_2| \leq \delta + \delta^3.
\end{align*}
$$

It is clear that, if $0 < \delta \leq 1$, then system (4) has the dead zone $|x_2| \leq \delta$, and cannot be asymptotically stabilized by any $C^1$-feedback $u = u(x_1, x_2)$ with $u(0) = 0$ at the origin. It is also clear that system (4) satisfies (ApRI) but does not satisfy (ARI), whenever $\delta > 0$, therefore, Item 1 of Theorem 1 is applicable to system (4), but Item 2 is not. However, for $\delta = 0$, we have (ARI), and system (4) becomes globally asymptotically stabilizable. Finally, let us note that systems with dead zones like system (4) have not only academic physical motivation like in Example 3.4 from [40], but they also appear in many engineering problems, like those considered in [41] or [42].

**Remark 5.** The special case of Theorem 1 summarized in the last sentence of Item 2 was proved in [1] by the current authors. Our goal is not only to extend [1] to the case of practical right invertibility (ApRI) but also to obtain such a generalization by another method. While [1] uses Lyapunov-based small-gain theorems, we want to show that the recent trajectory-based small-gain theorem from [2] can be also efficiently applied for even more general class of problems (see also our Section 7 given below).

Thus, our main tool is the following theorem proved in [2].

**Theorem 2.** [2] Consider the following nonlinear switched system

$$
\dot{x}_i(t) = f_{i,\sigma(t)}(x_1, x_2, \ldots, x_N, D(t)), \quad i = 1, \ldots, N, \quad t \in \mathbb{R}
$$


with the state $x = [x_1^T, ..., x_N^T]^T \in \mathbb{R}^n$, $x_i \in \mathbb{R}^n$, $n = n_1 + \ldots + n_N$, and the disturbance input $D(\cdot) \in L_\infty([0, +\infty[; \mathbb{R}^m)$ and assume that each $x_i$-subsystem of system (6) is UISpS with $x_j$ ($j \neq i$), $D$ treated as disturbance inputs, i.e., there are $\beta_i(\cdot) \in \mathcal{K}_\infty \mathcal{L}$, $\gamma_i(\cdot) \in \mathcal{K}_\infty \mathcal{L}$, $d_i \geq 0$ and $\gamma_{i,j}(\cdot) \in \mathcal{K}$, $j \in \{1, \ldots, N\} \setminus \{i\}$, $i = 1, \ldots, N$ such that for each piecewise constant $\mathbb{R} \ni t \mapsto \sigma(t) \in \{1, \ldots, M\}$, each $i = 1, \ldots, N$, each $D(\cdot) \in L_\infty([0, +\infty[; \mathbb{R}^m)$, each $x_j(\cdot)$ in $L_\infty([0, +\infty[; \mathbb{R}^n)$, $j \in \{1, \ldots, N\} \setminus \{i\}$, every trajectory $x_i(\cdot)$ of $i$-th subsystem of system (6) with $x_j = x_j(t)$, $j \neq i$, $\sigma = \sigma(t)$, $D = D(t)$ satisfies the following inequality

$$\forall t \geq 0 \quad |x_i(t)| \leq \max \left\{ \beta_i(|x_i(0)|, t); \max_{j \in \{1, \ldots, N\} \setminus \{i\}} \left\{ \gamma_{i,j}(\|x_j(\cdot)\|_{[0,t]}); \gamma_i(\|D(\cdot)\|_{[0,t]}); d_i \right\} \right\}. \quad (7)$$

Then the following two statements hold.

1) If the small-gain condition

$$\exists \mathcal{R}_0 \geq 0 \quad \forall \mathcal{R} > \mathcal{R}_0 \quad (\gamma_{i_1,i_2}, \gamma_{i_2,i_3}, \ldots, \gamma_{i_{n_i-1},i_{n_i}})(r) < r \quad (8)$$

holds for all $1 \leq i_1 \leq N$ s.t. $i_i \neq i_{i'}$ if $l \neq l'$ with $1 \leq l \leq k$, and $1 \leq k \leq N$, then (6) is UISpS in sense of Definition 1.

2) If $d_i = 0$ for all $i = 1, \ldots, N$ in Eqs. (7), and (8) holds with $R_0 = 0$ for all $1 \leq i_1 \leq N$ s.t. $i_i \neq i_{i'}$ if $l \neq l'$ with $1 \leq l \leq k$, and $1 \leq k \leq N$, then (6) is UIS in sense of Definition 2.

(Of course, each $f_{i,\sigma}$ is locally Lipschitz continuous, since this assumption is included into the definitions of the UIS and the UISpS).

We will prove Theorem 1 by a constructive design of stabilizers which satisfy the small-gain conditions of Theorem 2.

4. Adding an integrator

The proof of Theorem 1 is based on our next Theorem 3 given below. Consider a switched nonlinear system

$$\dot{z} = g(z, w) + \varphi_{\sigma(t)}(z, \xi(t)), \quad t \in \mathbb{R}, \quad (9)$$

with the state $z \in \mathbb{R}^k$, the control $w \in \mathbb{R}^n$, the external disturbance $\xi(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^N)$, and the switching signal $\sigma(\cdot)$. Along with system (9) consider its dynamic extension

$$\begin{align*}
\dot{z} &= g(z, w) + \varphi_{\sigma(t)}(z, \xi(t)) \\
\dot{w} &= h(z, w; \nu) + \phi_{\sigma(t)}(z, w; \xi(t), \eta(t)), \quad t \in \mathbb{R} \quad (10)
\end{align*}$$

with the state $y = [z^T, w^T]^T \in \mathbb{R}^k \times \mathbb{R}^n$, the control $\nu \in \mathbb{R}^m$, $n \leq m$, with the external disturbance $\zeta(\cdot) = [\xi^T(\cdot), \eta^T(\cdot)]^T \in L_\infty(\mathbb{R}; \mathbb{R}^{N + m})$, and with the switching signal $\sigma(\cdot)$. We assume that:

(CES) (Equilibrium and Smoothness) $g \in C^{p+1}(\mathbb{R}^k \times \mathbb{R}^n; \mathbb{R}^k)$; $h \in C^{p+1}(\mathbb{R}^{k+n} \times \mathbb{R}^m; \mathbb{R}^n)$; $\varphi_{\sigma} \in C^{p+1}(\mathbb{R}^k \times \mathbb{R}^N; \mathbb{R}^k)$ and $\phi_{\sigma} \in C^{p+1}(\mathbb{R}^{k+n} \times \mathbb{R}^{N+1}; \mathbb{R}^n)$ for all $\sigma = 1, \ldots, M$ with some $p \in \mathbb{N}$; and $g(0, 0) = \varphi_{\sigma}(0, 0) = 0 \in \mathbb{R}^k$ and $h(0, 0, 0) = \phi_{\sigma}(0, 0, 0) = 0 \in \mathbb{R}^n$ for all $\sigma = 1, \ldots, M$;

(CpRl) (Practical Right Invertibility) Each map $\nu \mapsto h(y, \nu)$ is practically right invertible, i.e., there are a nonnegative $\varrho(\cdot) \in C(\mathbb{R}^{k+n}, [0, +\infty[)$ and $\psi(\cdot) \in C^{p+1}(\mathbb{R}^{k+n} \times \mathbb{R}^n; \mathbb{R}^m)$ with $\psi(0, 0) = 0$ such that $h(y, \psi(\omega)) = \omega$, whenever $[y^T, \omega^T]^T \in \mathbb{R}^{k+n} \times \mathbb{R}^n$ satisfies $|\omega| \geq \varrho(y)$.
Remark 6. As above in Remark 3, if \(q(\cdot)\) from (CpRI) satisfies \(q(0)=0\), then without loss of
generality we assume throughout the paper that there is \(\bar{r}^*>0\) s.t. \(|y|\leq\bar{r}^*\Rightarrow q(y)=0\).

As above for (ApRI), if we have \(q(y)=0\) for all \(y\in\mathbb{R}^{k+n}\) in Condition (CpRI), then
(CpRI) becomes the same property of right invertibility as Condition (C2) assumed in [1):

(CRI) (Right Invertibility) The map \(v\mapsto h(y,v)\) is right invertible, i.e., there is \(\mathcal{V}(\cdot,\cdot)\in \mathbb{C}^{p+1}(\mathbb{R}^{k+n}\times\mathbb{R}^n;\mathbb{R}^m)\) s.t. \(h(y,\mathcal{V}(y,\omega))=\omega\) for all \([y^T,\omega^T]^T\in\mathbb{R}^{k+n}\times\mathbb{R}^n\).

For systems (9),(10), given any feedbacks \(w(\cdot):\mathbb{R}^k\rightarrow\mathbb{R}^n\) and \(v(\cdot,\cdot):\mathbb{R}^{k+n}\rightarrow\mathbb{R}^m\), define their Lyapunov functions \(W(z)\) and \(V(y)\), \(y=[z^T,w^T]^T\in\mathbb{R}^k\times\mathbb{R}^n\), and the derivatives of \(W(z)\) and \(V(y)\) w.r.t. (9) and (10) with and without these feedbacks by

\[
W(z) := \langle z, z \rangle \quad \text{and} \quad V(y) := \langle y, y \rangle = W(z) + \langle w, w \rangle, \quad y = [z^T, w^T]^T,
\]

\[
\frac{dW(v,y,\sigma)}{dt}|_{(9)} := (2z, g(y) + \phi_\sigma(z, \xi)),
\]

\[
\frac{dW(v,y,\sigma)}{dt}|_{(10)} := (2z, g(y) + \phi_\sigma(z, \xi)) + (2w, h(y, v) + \phi_\sigma(y, \zeta));
\]

\[
\frac{dV(v,y,\sigma,0)}{dt}|_{(9),w=0}|_{v(y)} := (2z, g(y) + \phi_\sigma(z, \xi)) \quad \text{and} \quad \frac{dV(v,y,\sigma,0)}{dt}|_{(10),v=0}|_{v(y)} := (2z, g(y) + \phi_\sigma(z, \xi)) + (2w, h(y, v(y)) + \phi_\sigma(y, \xi, \eta))
\]

for all \([y^T, v^T]^T\in\mathbb{R}^{k+n+m}\), \(\zeta=[\xi^T,\eta^T]^T\in\mathbb{R}^{N+h}\), \(\sigma\in\{1,\ldots,M\}\).

Fix any \(T>0\), \(\lambda^*>0\), and sequences of positive numbers \(\{r_q\}_{q=-\infty}^{+\infty}\subset\mathbb{R^+}\) and \(\{\rho_q\}_{q=-\infty}^{+\infty}\subset\mathbb{R^+}\) such that

\[
\forall q \in \mathbb{Z}_+ \quad r_{q+1} = r_q e^{\lambda^* T}, \quad \rho_{q+1} = \rho_q e^{\lambda^* T}. \tag{11}
\]

Our main statement on adding an integrator is as follows.

Theorem 3. (a) Assume that (CES),(CpRI) hold and there are \(q_0\in\mathbb{Z}, \kappa\in\mathbb{N}\), and \(\gamma(\cdot)\in\mathcal{H}_\infty\), \(\gamma_0>0\), \(r'>0\) such that \(\forall \varepsilon\in[0, \frac{\sqrt{2}}{2}]\), and for \(\hat{q}:=q_0\), there is \(\mathcal{V}(\cdot)\) of class \(\mathbb{C}^p(\mathbb{R}^k\times\mathbb{R}^n;\mathbb{R}^m)\) with \(v(0)=0\in\mathbb{R}^m\) such that

\[
\forall q \in \mathbb{Z}_+ \quad \forall \varepsilon \in [0, \frac{\sqrt{2}}{2}], \quad \forall y \in \mathbb{R}^{k+n} \forall \xi \in \mathbb{R}^N
\]

\[
\left(r_q \leq |z| \leq r_{q+\kappa} \quad \text{and} \quad |\xi| \leq \gamma^{-1}(\rho_q)\right) \Rightarrow \frac{dW(v,y,\sigma)}{dt}|_{(9),w=0} \leq -\lambda^* W(z). \tag{12}
\]

Then for every \(\varepsilon\in[0, \frac{\sqrt{2}}{2}]\), and for \(\hat{q}:=q_0\), there is \(\mathcal{V}(\cdot)\) of class \(\mathbb{C}^p(\mathbb{R}^k\times\mathbb{R}^n;\mathbb{R}^m)\) with \(v(0)=0\in\mathbb{R}^m\) such that

\[
\forall q \in \mathbb{Z}_+ \quad \forall y \in \mathbb{R}^{k+n} \forall \xi \in \mathbb{R}^{N+h}\quad \forall \sigma \in \{1,\ldots,M\}
\]

\[
\left(r_q \leq |y| \leq r_{q+\kappa} \quad \text{and} \quad |\xi| \leq \gamma^{-1}(\rho_q)\right) \Rightarrow \frac{dV(v,y,\sigma,0)}{dt}|_{(10),v=0}|_{v(y)} \leq -\left(\lambda^* - \frac{\varepsilon}{2}\right) V(y). \tag{13}
\]

(b) If (CES),(CpRI) hold and Eq. (12) holds with \(q_0=-\infty\), i.e., for all \(q\in\mathbb{Z}\), then for each fixed \(\hat{q}\in\mathbb{Z}\) there exists \(v(\cdot)\) of class \(\mathbb{C}^p(\mathbb{R}^k\times\mathbb{R}^n;\mathbb{R}^m)\) with \(v(0)=0\in\mathbb{R}^m\) such that (13) holds true with this \(\hat{q}\in\mathbb{Z}\).

(c) If (CES),(CpRI) hold with \(q(0)=0\) and (12) holds with \(q_0=-\infty\), i.e., (12) holds for all \(q\in\mathbb{Z}\), then we obtain \(\hat{q}=-\infty\) in Eq. (13), i.e., there exists \(v(\cdot)\) of class \(\mathbb{C}^p(\mathbb{R}^k\times\mathbb{R}^n;\mathbb{R}^m)\) with \(v(0)=0\in\mathbb{R}^m\) such that Eq. (13) holds for all \(q\in\mathbb{Z}\). In particular, this is true for (CES),(CRI) (Next, in this Case (c), we always assume without loss of generality that \(\exists r^*>0\) \(|y|\leq r^* \Rightarrow q(y)=0\) following Remark 6).

Remark 7. If we compare Theorem 3 from the current paper with its analog from [1] also entitled “Theorem 3”, then we see the following two differences: (I) extension to the case of practically right invertible input-output maps (CpRI) characterized by adding \(q_0\) into Eq.
(12) and $\hat{q}$ into Eq. (13), and (II) trajectory-based gain $\gamma(\cdot)$ in both Eqs. (12) and (13). Note that the conditions of Theorem 3 from [1] hold only in Case (c) of the above Theorem 3 with the additional requirement $\hat{q}(y) = 0$ for all $y \in \mathbb{R}^{k+n}$, i.e. (CRI). Therefore, the new proof will be redesigned as follows.

**Proof of Theorem 3** As in [1], we begin with the identity

$$
\frac{dV}{dt}\bigg|_{(10)} = \frac{dW}{dt}\bigg|_{(9) w = 0} + \{2w, h(y, \nu) + J^T(y)z + \phi_\sigma(y, \zeta)\},
$$

(14)

where $J(\cdot) \in C^p(\mathbb{R}^k \times \mathbb{R}^n; \mathbb{R}^{k \times m})$ is defined by the above-mentioned Hadamard's lemma: $g(y) - g(z, 0) = J(y)w$ for all $y = [z^T, w^T]^T$. First, we replace Eq. (14) with some switching-independent estimate. In contrast to [1], $\nu \mapsto h(y, \nu)$ is not necessarily right invertible and our feedback should provide the gain assignment (13), which is different from [1], to satisfy the trajectory-based small-gain conditions of Theorem 2.

**Step 1.** Let us eliminate $\sigma$ in (14). By Hadamard's lemma,

$$
g(\zeta'), 0 - g(\zeta''), 0 = A(\zeta', \zeta'') (\zeta' - \zeta''), \varphi_\sigma(\zeta', \xi') - \varphi_\sigma(\zeta'', \xi'')
$$

(15)

for all $[z^T, w^T, \xi^T, \eta^T]^T \in \mathbb{R}^{k+n} + (\mathcal{N} + k)$, $[z^T, \xi^T]^T \in \mathbb{R}^{k+N}$, $[z''^T, \xi''^T]^T \in \mathbb{R}^{k+N}$, $\sigma \in \{1, \ldots, M\}$ with some $\Phi_\sigma^z(\cdot), \Phi_\sigma^w(\cdot), \Phi_\sigma^x(\cdot), \Phi_\sigma^y(\cdot), A(\cdot), C_\sigma(\cdot), \chi_\sigma(\cdot)$ of class $C^p$. Then, we define the following switching-independent continuous function of the upper boundaries of the states and the disturbances:

$$
M(r, D):= \max_{|\nu| \leq r, |\xi| \leq D} \{ \| J^T(y)J(y) \| + \sum_{\sigma = 1}^M \{ \| \Phi_\sigma^z(y, \zeta) \|^T \Phi_\sigma^z(y, \zeta) \| \\
+ \| \Phi_\sigma^w(y, \zeta) \|^T \Phi_\sigma^w(y, \zeta) \| + \| \Phi_\sigma^x(y, \zeta) \|^T \Phi_\sigma^x(y, \zeta) \| \\
+ \| \Phi_\sigma^y(y, \zeta) \|^T \Phi_\sigma^y(y, \zeta) \| \} \} + \max_{|\nu| \leq r, |\xi| \leq D} \sum_{\sigma = 1}^M \{ 2[\| A(z_1, z_2) \|^T A(z_1, z_2) \| \\
+ 2 \| C_\sigma(z_1, z_2, \eta_1, \eta_2) \|^T C_\sigma(z_1, z_2, \eta_1, \eta_2) \| \\
+ 2 \| \chi_\sigma(z_1, z_2, \eta_1, \eta_2) \|^T \chi_\sigma(z_1, z_2, \eta_1, \eta_2) \| ] + 1 \}.
$$

(16)

To characterize our trajectory-based gain assignment, define

$$
\varepsilon_q^*: = \min_{\gamma_0, \gamma_0^2} \frac{1}{16(\gamma_0 + 1)^2} \min_{\varepsilon, 1} \frac{r_\varepsilon^2 - \kappa - 1}{\gamma^{-1}(\rho_\varepsilon + 4 + k)}^2, \quad q \in \mathbb{Z}.
$$

(17)

**Remark 8.** By the assumptions of Theorem 3, $\gamma(\cdot) \in \mathcal{K}_\infty$, $\gamma^{-1}(\cdot) \in \mathcal{K}_\infty$ are locally linear around the origin. Hence, the infimum in Eq. (17) and $\varepsilon_q^*$ defined by Eq. (17) asymptotically tend to some strictly positive and finite constants as $q \to -\infty$.

Take any $\varepsilon \in [0, \frac{\varepsilon_0}{q}]$ and any $q \in \mathbb{Z}$. Using the Cauchy-Bunyakovsky inequality and the Cauchy inequality $a^2 + b^2 \geq 2ab$, we get the following estimate for all $y = [z^T, w^T]^T \in \mathbb{R}^{k+n}$ and $\zeta \in \mathbb{R}^{N+k}$:

$$
\forall \sigma \in \{1, \ldots, M\} \quad \{ |y| \leq R \text{ and } |\zeta| \leq d \} \Rightarrow \langle 2w, J^T(y)z \rangle
\leq \frac{\varepsilon}{8} (z, z) + \frac{M(R, d)}{\varepsilon} \langle w, w \rangle \quad \text{and} \quad \langle 2w, \phi_\sigma(y, \zeta) \rangle
\leq \frac{\varepsilon}{8} (z, z) + \frac{1}{\varepsilon} (1 + M(R, d) \left( 1 + \frac{8}{\varepsilon^2} + \frac{8}{\varepsilon^2} \right) \langle w, w \rangle + \varepsilon_q^* |\zeta|^2)
$$

(18)
Then, from Eqs. (15),(18), we obtain for every $y = [z^T, w^T]^T$ in $\mathbb{R}^{k+n}$, every $\zeta = [\xi^T, \eta^T]^T$ in $\mathbb{R}^{N+\theta_0}$, and for all $\sigma \in \{1, \ldots, M\}$, $q \in \mathbb{Z},$

$$\frac{dV}{dt}(10) \leq \frac{dW}{dt}(9)_{y=\eta} + \frac{1}{4}(2w, h(y, v) + (1 + M(r, D))(1 + \frac{16}{\epsilon})$$

$$+ \frac{8}{\epsilon^2}w) + \frac{1}{4}|\xi|^2 + \frac{\epsilon^2}{4}|\eta|^2, \quad \text{whenever} \ |y| \leq r, \ |\zeta| \leq D. \quad (19)$$

Thus, we obtained Eq. (19) uniformly in $\sigma \in \{1, \ldots, M\}$. The goal of the next Step 2 is to replace $M(r, D)$ in Eq. (19) with some coefficients depending on $|y|$ and $|\zeta|$.

**Step 2.** Fix any nondecreasing function $\Lambda(\cdot) \in C^\infty([0, +\infty[; 0, +\infty])$, which is locally constant around $r=0$, and satisfies the following condition

$$\forall q \in \mathbb{Z} \quad \forall r \in [r_q, r_{q+1}] \quad \Lambda(r) \geq 2\lambda^* + \frac{1}{\gamma_0}(\lambda^* + 2)$$

$$+ \frac{2}{2\gamma_0^2} + \frac{8}{\gamma_0 \gamma^2} \left( 1 + \frac{\epsilon^2}{4}\right) + \frac{\epsilon^2}{4} \left( 1 + \frac{\epsilon^2}{4}\right) |\xi|^2.$$ 

$$\Lambda(r_q, \frac{1}{\gamma_0}(\lambda^* + 2)) = \frac{1}{\gamma_0}(\lambda^* + 2)$$

$$+ \frac{2}{2\gamma_0^2} + \frac{8}{\gamma_0 \gamma^2} \left( 1 + \frac{\epsilon^2}{4}\right) + \frac{\epsilon^2}{4} \left( 1 + \frac{\epsilon^2}{4}\right) |\xi|^2.$$ 

$$\Lambda(r_q, (1 + \frac{\epsilon^2}{4})(1 + \frac{\epsilon^2}{4})) = \frac{1}{\gamma_0}(\lambda^* + 2)$$

$$+ \frac{2}{2\gamma_0^2} + \frac{8}{\gamma_0 \gamma^2} \left( 1 + \frac{\epsilon^2}{4}\right) + \frac{\epsilon^2}{4} \left( 1 + \frac{\epsilon^2}{4}\right) |\xi|^2.$$ 

Thus, we obtained Eq. (19) uniformly in $\sigma \in \{1, \ldots, M\}$. The goal of the next Step 2 is to replace $M(r, D)$ in Eq. (19) with some coefficients depending on $|y|$ and $|\zeta|$.

**Remark 9.** By Remark 8, the limit of $\epsilon_q^* \gamma_0$ and the limit of the right-hand side of (20) as $q \to -\infty$ are well-defined and they are both strictly positive, finite constant numbers.

Then from Eq. (19) we obtain for all $y = [z^T, w^T]^T$ in $\mathbb{R}^{k+n}$ and $\zeta = [\xi^T, \eta^T]^T$ in $\mathbb{R}^{N+\theta_0}$ uniformly in $\sigma \in \{1, \ldots, M\}$:

$$\forall q \in \mathbb{Z} \quad \forall r \in [r_q, r_{q+1}] \quad \Lambda(r) \geq 2\lambda^* + \frac{1}{\gamma_0}(\lambda^* + 2)$$

$$+ \frac{2}{2\gamma_0^2} + \frac{8}{\gamma_0 \gamma^2} \left( 1 + \frac{\epsilon^2}{4}\right) + \frac{\epsilon^2}{4} \left( 1 + \frac{\epsilon^2}{4}\right) |\xi|^2.$$ 

$$\Lambda(r_q, \frac{1}{\gamma_0}(\lambda^* + 2)) = \frac{1}{\gamma_0}(\lambda^* + 2)$$

$$+ \frac{2}{2\gamma_0^2} + \frac{8}{\gamma_0 \gamma^2} \left( 1 + \frac{\epsilon^2}{4}\right) + \frac{\epsilon^2}{4} \left( 1 + \frac{\epsilon^2}{4}\right) |\xi|^2.$$ 

Fix any $y = [z^T, w^T]^T \neq 0 \in \mathbb{R}^{k+n}$, and fix any corresponding $q \in \mathbb{Z}$ such that $r_q \leq |y| \leq r_{q+\theta_0}$. Let $\zeta = [\xi^T, \eta^T]^T \in \mathbb{R}^{N+\theta_0}$ satisfy $|\zeta| \leq \gamma^{-1}(\rho_q)$. Next, $y$, $\zeta$ and $q$ are assumed to be fixed. Consider the following two possible cases.

**Case 1. Assume that** $0 < |w|^2 \leq \gamma_0|z|^2$. **Define** $\mu := |y||z| = (1 + (|w||z|)^2)^{\frac{1}{2}}$, and, for $w = 0$, we denote: $\lambda_0 := 1$. To design a suitable feedback for Eq. (21), our main goal is to estimate $\Delta(z, \xi, \gamma) \leq \Delta(\mu z, \xi, \gamma) + |\Delta(z, \xi, \gamma) - \Delta(\mu z, \xi, \gamma)|$ in Eqs. (21)–(22) and then refer to Eq. (12). Using Eqs. (15),(16), and the inequality $0 \leq \mu - \mu_0 \leq \frac{|w|^2}{2\gamma_0^2}$, we obtain

$$\forall \sigma \in \{1, \ldots, M\}:$$

$$\forall q \in \mathbb{Z} \quad (r_q \leq |y| \leq r_{q+\theta_0} \quad \text{and} \quad |\zeta| \leq \gamma^{-1}(\rho_q) \Rightarrow (\mu z, g(\mu z, 0)$$

$$+ \phi_0(\mu z, \xi)) - (\mu z, g(\mu z, 0) + \phi_0(\mu z, \xi)) \leq \frac{|w|^2}{2\gamma_0^2} \times$$

$$((8M(r_{q+\theta_0}, \gamma^{-1}(\rho_q)) + \frac{1}{\gamma^2}(M(r_{q+\theta_0}, \gamma^{-1}(\rho_q)) |\xi|^2 + \epsilon^2 |\xi|^2)$$

$$\Lambda(r_q, \frac{1}{\gamma_0}(\lambda^* + 2)) = \frac{1}{\gamma_0}(\lambda^* + 2)$$

$$+ \frac{2}{2\gamma_0^2} + \frac{8}{\gamma_0 \gamma^2} \left( 1 + \frac{\epsilon^2}{4}\right) + \frac{\epsilon^2}{4} \left( 1 + \frac{\epsilon^2}{4}\right) |\xi|^2.$$ 

**Case 2. Assume that** $0 < |z|^2 \leq \frac{1}{\gamma_0}|w|^2$. **Then, using** Eqs. (15),(16), **we get:** if $r_q \leq |y| \leq r_{q+\theta_0}$ and $|\zeta| \leq \gamma^{-1}(\rho_q)$, then

$$|2z, g(z, 0) + \phi_0(\mu z, \xi))| \leq \frac{1}{\gamma_0}(2w, (M(r_{q+\theta_0}, \gamma^{-1}(\rho_q)) |1 + \frac{1}{\gamma^2}| + 1)w) + \frac{\epsilon^2}{8} |\xi|^2.$$ 

Combining Eqs. (20)–(22) with (23) in Case 1, and with Eq. (24) in Case 2, we finally obtain for each $y = [z^T, w^T]^T \neq 0 \in \mathbb{R}^{k+n}$:

$$|w|^2 < \gamma_0 |z|^2 \Rightarrow \delta^*(y, \xi, \gamma) \leq \Delta(\mu z, \xi, \gamma) - \frac{\lambda(|y| - 2\lambda^* + \epsilon)}{2} |w|^2$$

$$|w|^2 \geq \gamma_0 |z|^2 \Rightarrow \delta^*(y, \xi, \gamma) \leq 0$$

(25)
The goal of Step 2 is now achieved, since Eqs. (21), (22), (25) hold for every \( y = [z^T, w^T]^T \neq 0 \in \mathbb{R}^{k+n} \) regardless of \( M(R, d) \) and \( \sigma \). (Note that switching-independent (12) is applicable to \( \Delta(mz, \xi, \sigma) \).

**Step 3.** The goal of Step 3 is to define the stabilizing controller and to prove that it resolves our main problem.

**Remark 10.** Note that Theorem 3 in the special Case (c) with (CRI) can already be proved by using (CRI) and resolving the equation \( h(y, v) = -\Lambda(|y|)w \) globally w.r.t. \( v \). However, in general, we now have (CpRI) in contrast to [1], and we need to handle Cases (a)-(b) as well. Our general solution incorporating all the Cases (a)-(c) is as follows.

Take any nondecreasing \( \tilde{\varrho}(\cdot) \in \mathcal{C}^\infty(0, +\infty] \) such that \( \tilde{\varrho}(|y|) \geq \varrho(y) \) for all \( y \in \mathbb{R}^{k+n} \) with \( \hat{\varrho}(r) = \varrho_{\text{const}} > 0 \) in some neighborhood of \( r=0 \). To characterize the ball in the \( w \)-subspace, in which \( h(y, v) = -\Lambda(|y|)w \) is not resolvable by (CpRI), we fix any \( r^* > 0 \) and define for all \( r \geq 0 \):

\[
\mathcal{M}^*(r) := \max_{|y| \leq r, \omega \leq \tilde{\varrho}(|y|)} |h(y, V(y, \omega))|;
\]

\[
\rho^*(r) := \min \left\{ 1, \frac{\sqrt{\gamma_0}}{\sqrt{y_0+1}}, r^*, \frac{r^2 \min[1, \epsilon/2]}{16|y_0+1|/\varrho^2(r) + \Lambda(r) + 1} \right\}
\]

(26)

Let us prove that our final feedback \( v(\cdot) \) defined by

\[
v(y) := V(y, -\frac{\Lambda(|y|) + \tilde{\varrho}(|y|)}{\rho^*(|y|)} w) \text{ for all } y = [z^T, w^T]^T \in \mathbb{R}^{k+n}
\]

satisfies all the Statements (a)-(c) of our Theorem 3.

Let the assumptions of Case (a) hold. Define \( r^* := r_{q_0} \) in Eq. (26) and take any \( y = [z^T, w^T]^T \neq 0 \in \mathbb{R}^{k+n} \). By (12), in Case 1 with \( 0 < |w|^2 \leq \gamma_0 |z|^2 \), we have for all \( \sigma \):

\[
\forall q \in \mathbb{Z}_{\geq q_0} (r_{q} \leq |y| \leq r_{q+k}, |\xi| \leq \gamma^{-1}(\rho_q) \Rightarrow \Delta(mz, \xi, \sigma) \leq 0
\]

(28)

Then, we combine Eqs. (21),(25), with Eqs. (26)–(27) and with \( \tilde{\varrho}(|y|) \geq \varrho(y) \), and we obtain from (CpRI) uniformly in \( \sigma \in \{1, \ldots, M\} \):

\[
\forall q \in \mathbb{Z}_{\geq q_0} (r_{q} \leq |y| \leq r_{q+k} \text{ and } |\xi| \leq \gamma^{-1}(\rho_q) \Rightarrow \frac{dV(y, \xi, \sigma)}{dt} \big|_{(y, \xi, \sigma) = (y)} \leq -\left(k^* - \frac{\epsilon}{2}\right)V(y).
\]

(29)

Similarly, by the second part of Eq. (25), we obtain Eq. (29) for all \( q \in \mathbb{Z} \) without the subscript “\( \geq q_0 \)” and \( \forall \sigma \in \{1, \ldots, M\} \), whenever Case 2 with \( 0 < |w|^2 \leq \frac{1}{\gamma_0} |z|^2 \) holds. This proves Eq. (29) for all \( y = [z^T, w^T]^T \neq 0 \in \mathbb{R}^{k+n} \), with Item (a) of Theorem 3.

In Case (b), given any \( \hat{q} \in \mathbb{Z} \), define \( r^* := r_{\hat{q}} \) in Eq. (26). For every \( y = [z^T, w^T]^T \neq 0 \in \mathbb{R}^{k+n} \), we repeat all the previous paragraph replacing \( q_0 \) with \( \hat{q} \) everywhere and taking into account that Eq. (28) holds true whenever \( 0 < |w|^2 \leq \gamma_0 |z|^2 \) and regardless of \( q \in \mathbb{Z}, \sigma \in \{1, \ldots, M\} \). Then the Proof of Statement (b) of our Theorem 3 is complete as well.

Finally, in Case (c), we define \( r^* := \bar{r}^* \) in Eq. (26), where \( \bar{r}^*>0 \) is from Remark 6. Fix any \( q_0 \in \mathbb{Z} \) s.t. \( 0 < r_{q_0+k} < r^* := \bar{r}^* \). Take any \( y = [z^T, w^T]^T \neq 0 \in \mathbb{R}^{k+n} \). Then Eq. (28) again holds true for all \( q \in \mathbb{Z} \), i.e., without the subscript “\( \geq q_0 \)” and regardless of whether \( 0 < |w|^2 \leq \gamma_0 |z|^2 \) or \( 0 < \gamma_0 |z|^2 \leq |w|^2 \). Thus, all the Statements (a)-(c) of Theorem 3 are proved. ■
5. Proof of Theorem 1

Proof of Theorem 1 is immediately obtained as a corollary of the above Theorem 3 by induction on \( \nu \). Due to space limits it is only sketched. In the Base Case, system (3) has the form

\[
\dot{x}_{i,1} = f_{i,1}(x_{i,1}, x_{i,2}) + \pi_{i,1}(\sigma(t)) x_{i,1}, \quad i = 1, \ldots, N, \tag{30}
\]

with the state \( x_{i,1} \), with the control \( x_{i,2} \), with the disturbance input \( \xi_i := [x_{i,1}^T, \ldots, x_{i-1,1}^T, x_{i,1}^T, \ldots, x_{N,1}^T, D_1^T]^T \), and with the unknown piecewise constant switching signals \( \sigma(\cdot) \). As in [1], let \( V_{i,1}(x_{i,1}) := |x_{i,1}|^2 \) be the Lyapunov function for each \( x_{i,1} \)-subsystem of system (30). Fix any \( \lambda^* > 0 \), any \( T > 0 \), and any sequence \( \{r_q\}_{q=-\infty}^{+\infty} \in ]0, +\infty[ \) as in Eq. (11) and define \( \{\rho_q\}_{q=-\infty}^{+\infty} \) by \( \rho_q := r_{q+5}, \ q \in \mathbb{Z} \).

Lemma 1. (i) Under Assumptions (AES), (ApRI), for every \( i = 1, \ldots, N \), and every locally linear \( \hat{\gamma}_i(\cdot) \in \mathcal{X}_\infty \), s.t. \( \forall r \in [0, \bar{r}_i^0] \) \( \hat{\gamma}_i(r) = \bar{y}_i^0 r \) with some \( \bar{r}_i^0 > 0, \bar{y}_i^0 > 0 \), there are \( q_0, i \in \mathbb{Z} \) and \( \hat{\nu}_i(\cdot) \in C^0(\mathbb{R} \times \mathbb{R}^{m_{i,1}}; \mathbb{R}^{m_{i,2}}) \) such that for all \( (x_{i,1}, \xi_i) \) and \( \forall \sigma \in \{1, \ldots, M\} \) we have

\[
\forall q \in \mathbb{Z}_{\geq q_0}, r_q \leq |x_{i,1}| \leq r_{q+3} \text{ and } |\xi_i| \leq \hat{\gamma}_i^{-1}(\rho_q) \Rightarrow \frac{dV_{i,1}}{dt}_{(30), x_{i,2}=\hat{v}_i(x_{i,1})} \leq -2\lambda^* V_{i,1}(x_{i,1}). \tag{31}
\]

(ii) If \( R_i(0)=0 \) in (ApRI), then \( q_0, i = -\infty \) in Eq. (31), i.e., Eq. (31) holds for all \( q \in \mathbb{Z} \).

Proof of Lemma 1 is a significantly simplified version of the above Proof of Theorem 3; we just follow the above Steps 1-3 of Proof of Theorem 3 simplifying Eqs. (15)-(21) by omitting \( z \) everywhere and putting \( y = w = x_{i,1}, \nu = x_{i,2} \) (see also Proofs of Lemmas 2,5 in [35]).

Finally, we note that Theorem 3 and Lemma 1 imply Theorem 1. For \( \nu = 1 \), take any locally linear \( \hat{\gamma}_1(\cdot) \in \mathcal{X}_\infty \) such that Eq. (8) holds with \( R_0=0 \) and find the corresponding \( \hat{\nu}_1(\cdot) \) by applying Lemma 1. Arguing as in [17] (Sect. 4, pp 148–149), we obtain the existence of \( \beta(\cdot, \cdot) \in \mathcal{X}_\mathcal{L} \) such that every trajectory \( n \to x_{i,1}(t) \) of each closed-loop \( x_{i,1} \)-subsystem of system (30) with \( x_{i,2} = \hat{v}_1(x_{i,1}) \), \( i = 1, \ldots, N \), satisfies the following inequality:

\[
|x_{i,1}(t)| \leq \max\{\beta(|x_{i,1}(0)|, t - t_0); \hat{\gamma}_1(\|\xi_i(\cdot)\|_L, r_{q_{0, i}})\} + 3
\]

uniformly in \( t_0 \in \mathbb{R}, \ t \geq t_0, \ x_{i,1}(t_0) \in \mathbb{R}^{m_{i,1}}, \ xi_i(\cdot) \in L_\infty, \sigma(\cdot) \). Then taking any small enough \( \hat{\gamma}_1(\cdot) \in \mathcal{X}_\infty \) to satisfy (8) with \( R_0=0 \), one makes (30) with \( x_{i,2} = \hat{v}_1(x_{i,1}) \) satisfying Theorem 2. Taking the backstepping transformation \( z = x_{i,1}, w = x_{i,2} - \hat{v}_1(x_{i,1}), \) we update gains keeping (8) with \( R_0=0 \), apply Theorem 3, and prove Theorem 1 for \( \nu = 2 \). Arguing by induction on \( \nu \), we complete the Proof of Theorem 1.

6. Examples and simulation

Let \( \hat{H}_1(\cdot) \in C^2([-2, 2]; \mathbb{R}) \) be any odd and nondecreasing function such that \( \hat{H}_1(\eta_2) = 0 \), whenever \( |\eta_2| \leq 1, \hat{H}_1(2) = 2, \hat{H}_1(-2) = -2, \hat{H}_1''(2) = 1, \hat{H}_1''(-2) = \hat{H}_1''(2) = 0 \). Define the odd function \( H_1(\cdot) \in C^2(\mathbb{R}; \mathbb{R}) \) by \( H_1(\eta_2) := \hat{H}_1(\eta_2) \), whenever \( |\eta_2| \leq 2 \), and \( H_1(\eta_2) := \eta_2 \), whenever \( |\eta_2| \geq 2 \).

Define \( H_0(\eta_2) := \eta_2 \) for all \( \eta_2 \in \mathbb{R} \). Given any \( \delta \in ]0, 1[ \), let function \( H_3(\cdot) \in C^2(\delta; \mathbb{R}) \) be defined by \( H_3(\eta_2) := \delta H_1(\frac{\eta_2}{\delta}) \) for all \( \eta_2 \in \mathbb{R} \). Thus, we have defined the family of func-
tions \( H_\delta(\cdot) \in C^2(\mathbb{R}; \mathbb{R}) \), where \( \delta \in [0, 1] \), such that \( \sup_{\eta_2 \in \mathbb{R}} |H_\delta(\eta_2) - H_0(\eta_2)| = \max_{\eta_2 \in \mathbb{R}} |H_\delta(\eta_2) - H_0(\eta_2)| \to 0 \) as \( \delta \to +0 \).

Define \( \rho_\sigma = -p_\sigma = q_\sigma = -\Delta_\sigma = \frac{(-1)^\sigma \gamma}{2} \), \( \sigma \in \{1, 2\} \). Consider the network

\[
\begin{aligned}
\dot{y}_1 &= u + \frac{\sigma(t)}{16} y_1 \eta_1 + \frac{y_1 \xi_1}{8}, \\
\dot{\eta}_1 &= H_\delta(\eta_2) + \frac{\eta_1}{2} + \eta_1 + q_\sigma(t) \xi_1 \eta_1, \\
\dot{\eta}_2 &= v + \Delta_\sigma(t) \xi_2 \eta_1, \\
\dot{\xi}_1 &= \xi_2 - 3\xi_1 - 3\xi_1^3 + \rho_\sigma(t) \xi_1 \eta_1 + p_\sigma(t) \xi_1 + D, \\
\dot{\xi}_2 &= \zeta,
\end{aligned}
\]

composed of \( N = 3 \) subsystems with states \( y_1 \in \mathbb{R}^1 \), \( \eta = [\eta_1, \eta_2]^T \in \mathbb{R}^2 \), \( \xi = [\xi_1, \xi_2]^T \in \mathbb{R}^2 \), and controls \( u \in \mathbb{R} \), \( v \in \mathbb{R} \), \( \zeta \in \mathbb{R} \) respectively, with the switching signals \( \mathbb{R} \ni t \mapsto \sigma(t) \in \{1, 2\} \) and with the disturbances \( D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}) \). Our goal is to design a decentralized feedback \( u = u(y_1), \zeta = \zeta(\xi), v = v(\eta) \) which renders (32) UISpS for \( \delta > 0 \) and UISS for \( \delta = 0 \) as in Theorem 1.

Let us first stabilize the \( y_1 \)-subsystem. Take the Lyapunov function \( V(y_1) = y_1^2 \). Then

\[
\dot{V} = 2y_1(u + \frac{\sigma(t)}{16} y_1 \eta_1 + \frac{y_1 \xi_1}{8}) \leq y_1(2u + \frac{y_1^3}{4}) + \frac{1}{8} \eta_1^2 + \frac{1}{8} \xi_1^2
\]
and the feedback $u(y_1) = -\frac{y_1}{2} - \frac{y_3}{8}$ provides the following inequality:

$$|y_1(t)| \leq \max\{|y_1(t_0)|\sqrt{2e^{-\frac{t-t_0}{2}}}, \frac{\sqrt{2}}{2} \|\eta_1(\cdot)\|_{\infty}, \frac{\sqrt{2}}{2} \|\xi_1(\cdot)\|_{\infty}\}$$

for each trajectory of the $y_1$-subsystem with this feedback and with $t \rightarrow \eta_1(t), \ t \rightarrow \xi_1(t)$ treated as the external disturbance inputs. Inequality (33) means that the gains for the $y_1$-subsystem with the feedback $u(y_1) = -\frac{y_1}{2} - \frac{y_3}{8}$ are strictly less than $1$. Let us stabilize the $\eta$-subsystem and the $\xi$-subsystem by our version of backstepping approach with a suitable gain assignment.

In the Base Case for the $\eta$-subsystem, we take $Q(\eta_1) := \eta_1^2$ and consider the system $\dot{\eta}_1 = H_5(\eta_2) + \frac{\eta_1}{2} + \eta_1 + \sigma(\eta_1)\xi_1\eta_1$ with states $\eta_1 \in \mathbb{R}$, controls $\eta_2 \in \mathbb{R}$, and disturbances $x_1 \in \mathbb{R}$.

Define the virtual feedback $\alpha(\eta_1) := -\frac{3}{2}\eta_1 - \frac{1}{8}\eta_1^3$. Then, by the Cauchy inequality $a^2 + b^2 \geq 2ab$,

$$\frac{dQ}{dt}|_{(32),\eta_2=\alpha(\eta_1)} \leq -Q(\eta_1) + \frac{\epsilon}{2}(\xi_1^2 + y_1^2), \text{ whenever } |\eta_1| \geq \frac{4\delta}{\epsilon} > 0, \epsilon > 0 \ (34)$$

Define the Lyapunov function $V(\eta_1, \chi) := Q(\eta_1) + \chi^2$, where $\eta_1 = \eta_1, \ \chi = \eta_2 - \alpha(\eta_1)$. Then $H_5(\chi + \alpha(\eta_1)) - H_5(\alpha(\eta_1)) = h(\eta_1, \chi)\chi$, where $h(\eta_1, \chi) = \int_0^1 \frac{\partial H_5(\alpha(\eta_1) + \theta\chi)}{\partial \eta_2} d\theta$. Eliminating $\sigma$...
by the Cauchy inequality $a^2 + b^2 \geq 2ab$, as above in Eq. (34), we obtain:

$$\frac{dW}{dt}|_{(32)} \leq 2|\eta_1[H_\delta(\alpha(\eta_1)) + \eta_1 + \varrho \xi_1 \eta_1]| + 2\chi[v + \eta_1 \bar{h}(\eta_1, \chi) + \Delta_\varrho \xi_2 \eta_1 - \frac{\partial \alpha(\eta_1)}{\partial \eta_1}[H_\delta(\chi + \alpha(\eta_1)) + \eta_1 + \varrho \xi_1 \eta_1]]$$

$$\leq 2|\eta_1[H_\delta(\alpha(\eta_1)) + \eta_1 + \varrho \xi_1 \eta_1]| + 2\chi[\eta_1 \bar{h}(\eta_1, \chi) + v + \frac{\chi \eta_1^2}{2\epsilon} + 1 + \frac{\partial \alpha(\eta_1)}{\partial \eta_1}[H_\delta(\chi + \alpha(\eta_1)) + \eta_1]| + \epsilon(\xi_1^2 + \xi_2^2 + \gamma_1^2)$$

Define the feedback

$$v(\eta_1, \chi) := -\chi - \eta_1 \bar{h}(\eta_1, \chi) - \frac{\chi \eta_1^2}{2\epsilon} + 1 + \frac{\partial \alpha}{\partial \eta_1}[H_\delta(\chi + \alpha(\eta_1)) + \eta_1].$$

From Eqs. (35), (36), (34), it follows that, if $0 < \epsilon \leq 1$ then

$$\eta_1^2 + \chi^2 \geq \frac{4}{\epsilon}\left(\left(\frac{4\delta}{3}\right)^2 + \left(\frac{4\delta}{3}\right)^4\right) \Rightarrow \frac{dW}{dt}|_{(32), v = v(\eta_1, \chi)} \leq -W + \epsilon(\xi_1^2 + \xi_2^2 + \gamma_1^2).$$

Define $\vartheta^* = \frac{2}{\sqrt{\epsilon}}\left(\frac{4\delta}{3}\right)^2 + \left(\frac{4\delta}{3}\right)^4\frac{1}{\epsilon}$. Then each trajectory $\vartheta(t) = (\eta_1(t), \chi(t))$ of the closed-loop $(\eta_1, \eta_2) = (\eta_1, \chi)$-subsystem of system (32) with $v = v(\eta_1, \chi)$ defined by Eq. (36), satisfies the following inequality:

$$\forall t \geq t_0 \quad |\vartheta(t)| \leq \sqrt{2} \max\left\{e^{-\frac{\epsilon}{2\delta}}|\vartheta(t_0)|; 2\sqrt{\epsilon}\|\xi(\cdot)\|_\infty; 2\sqrt{\epsilon}\|\eta(\cdot)\|_\infty; |\vartheta^*|\right\}.$$

For the $\xi$-subsystem of system (32), we define $\Lambda(\xi) := \xi_1^2$. Applying the Cauchy inequality $a^2 + b^2 \geq 2ab$, we obtain $\frac{d\Lambda}{dt}|_{(32), \xi_1 = 0} \leq -3\Lambda + \tau_1^2 + D^2$, and then for Lyapunov func-
tion $\Omega(\xi):=\Delta(\xi_1)+\xi_2^2$, and for the feedback $\zeta^{}(\xi)=-\xi_1-\xi_2$, we obtain $\frac{d\Omega}{dt}|_{(32),\zeta^{}(\xi)} \leq -\Omega+\eta_2^2+D^2$. Hence each trajectory $\xi(t)=(\xi_1(t),\xi_2(t))$ of the closed-loop $(\xi_1,\xi_2)$-subsystem of system (32) with $\zeta^{}(\xi)=-\xi_1-\xi_2$ satisfies

$$\|\xi(t)\| \leq \sqrt{3} \max \left\{ e^{\frac{-t}{2\beta}} \|\xi(t_0)\|; \|\eta_1(\cdot)\|_{\infty}; \|D(\cdot)\|_{\infty} \right\},$$

for all $t \geq t_0$. Combining Eqs. (33),(38),(37) with Theorem 2 we obtain that, if $0<\varepsilon<\frac{1}{35}$, then the decentralized feedback $u=u(y_1), v=v(\eta_1, \chi), \zeta=\zeta(\xi_1, \xi_2)$ renders the closed-loop system (32) UISS at the origin. If $\delta = 0$, then this feedback renders the closed-loop (32) UISS at the origin.

For simulation, let our initial condition be $[y_1(0), \eta_1(0), \eta_2(0), \xi_1(0), \xi_2(0)]^T = [1.5, 0.5, -1.5, 1, -1]^T$, and we take $\sigma_1(t)=1$ if $t \in \left[\frac{2k+1}{10}, \frac{2k+1}{10}\right]$ and $\sigma_2(t)=2$ if $t \in \left[\frac{2k+2}{10}, \frac{2k+2}{10}\right]$ for each $k \in \mathbb{Z}$. We consider all possible cases, namely, systems without dead zones and without disturbances ($\delta = 0, D(t) = 0$, see Fig. 3); systems with dead zones, but without disturbances (see Fig. 7, Fig. 8); systems with disturbances, but without dead zones (Fig. 4, Fig. 5, Fig 6); systems with both disturbances and dead zones ($\delta \neq 0, D(t) \neq 0$, see Fig. 1, Fig. 2). According to our theory, the closed-loop system (32) with our decentralized feedback should be globally asymptotically stable in the case when $\delta = 0, D(t) = 0$, and, indeed, this is true as we conclude from Fig. 3. If $\delta \neq 0$, or $D(t) \neq 0$, or both these conditions hold, then we have some overshoot as we can see in the other seven plots. If, at least, either $D(\cdot) \neq 0$ or $\delta \neq 0$ (or both the conditions are satisfied) and it is “relatively large” (i.e., it is “comparable” with the initial conditions), then the overshoot is also “relatively large”.

Fig. 4. No deadzones, large disturbances.
even in the case when the other parameter is zero (see Fig. 1, Fig. 4, Fig. 7). If both the parameters are different from zero, but are “rather small”, or one of them is “rather small” and the other is zero, then the overshoot is “rather small” as well (Fig. 2, Fig. 6, Fig. 8), i.e., roughly speaking, the trajectories tend to the asymptotically stable case from Fig. 3, as $(\delta, D(\cdot)) \to (0, 0)$. This can be concluded from the following chains of Figures: Fig. 1 \→ Fig. 2 \→ Fig. 3; Fig. 4 \→ Fig. 5 \→ Fig. 6 \→ Fig. 3; and Fig. 7 \→ Fig. 8 \→ Fig. 3.

7. Some generalizations. Stabilization in presence of dynamic uncertainties

In this section, we sketch a certain extension of our main results, which becomes possible, since we are applying the trajectory-based result from [2], instead of the Lyapunov-based approach applied in [1]. Motivated by [3,6], and [35], we assume that our network (3) has some dynamic uncertainty, whose dynamics is also UISS (or maybe UISpS), but it is assumed that we know only its trajectory-based gain. Then, the applicability of the small-gain approach in terms of Lyapunov functions becomes questionable, in general.

The point is that it is not always possible to find constructively an ISS Lyapunov function for a subsystem with the corresponding Lyapunov gain, even if we know its trajectory-based gain. This is not surprising, since even the proof of the classical Massera’s theorem is not constructive, and, of course, the proofs of the converse ISS Lyapunov theorems are not constructive as well, see [43]. That is one of the main reasons why, for instance, [3] and [35] are based on the corresponding trajectory-based versions of small-gain theorems (for two interconnected subsystems only).
Motivated by [3] and [35], let us assume that our network (3) has some dynamic uncertainties. More specifically, let us consider a network of switched control systems in the following form

\[
\begin{align*}
\dot{\xi} &= F_{\sigma(t)}(\bar{X}_1, \xi, D), \\
\dot{x}_{i,j} &= f_{i,j}(x_{i,1}, \ldots, x_{i,j+1}) + \pi_{i,j,\sigma(t)}(\bar{x}_j, \xi, D(t)), \quad 1 \leq j \leq v_i - 1, \\
\dot{x}_{i,v_i} &= f_{i,v_i}(x_{i,1}, \ldots, x_{i,v_i}, u_i) + \pi_{i,v_i,\sigma(t)}(\bar{x}_{v_i}, \xi, D(t)); \\
i &= 1, \ldots, N,
\end{align*}
\]  

(39)

with the dynamic uncertainty \( \xi \in \mathbb{R}^\kappa \), with the state vector components \( x_{i,j} \in \mathbb{R}^{m_{i,j}} \) (and with \( m_{i,j} \leq m_{i,j+1} \)), with the controls \( u_i = x_{i,v_i+1} \in \mathbb{R}^{m_{i,v_i+1}} \), with the external disturbances \( D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^b) \), and with the unknown piecewise constant switching signals \( \sigma(\cdot) \) as for system (3), where \( \bar{X}_1 := [x_{1,1}^\top, \ldots, x_{N,1}^\top]^\top \), and the other \( \bar{X}_p, p \geq 2 \) are defined in the same way as for system (3).

For any \( (t_0, \xi^0) \in \mathbb{R} \times \mathbb{R}^\kappa \), any \( \bar{X}_1(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^{m_{1,1}+\ldots+m_{1,N}}), D(\cdot) \in L_\infty(\mathbb{R}; \mathbb{R}^b) \), and any piecewise constant \( \sigma(\cdot) \), let \( t \mapsto \xi(t, t_0, \xi^0, \bar{X}_1(\cdot), D(\cdot), \sigma(\cdot)) \) denote the trajectory, of the system

\[
\dot{\xi} = F_{\sigma(t)}(\bar{X}_1(t), \xi, D(t))
\]  

(40)

that is defined by the input \( (\bar{X}_1(\cdot), D(\cdot)) \), by the signal \( \sigma(\cdot) \) and by the initial condition \( \xi(t_0) = \xi^0 \). Following [3] and [35], we also assume that the \( \xi \)-subsystem (40) of the entire system (39) satisfies the following conditions:

1. For every \( \sigma \in \{1, \ldots, M\} \), each \( F_{\sigma}(\cdot, \cdot, \cdot) \) is of class \( C^1 \) and \( F_{\sigma}(0, 0, 0) = 0 \).

Fig. 6. No deadzones, small disturbances.
(II) System (40) is UISpS w.r.t. $(\mathcal{X}_1(\cdot), D(\cdot))$ as the input with known and locally linear around the origin gain $\tilde{y}(\cdot)$, i.e., by definition there are $d \geq 0$, $\tilde{\beta}(\cdot, \cdot) \in \mathcal{K} \mathcal{L}$ and $\tilde{y}(\cdot) \in \mathcal{K}_\infty$ such that for each $t_0 \in \mathbb{R}$, each $\xi^0 \in \mathbb{R}^N$, each $\mathcal{X}_1(\cdot) \in L_\infty(\mathbb{R}^N; \mathbb{R}^{m_1+\ldots+m_N})$, each $D(\cdot) \in L_\infty(\mathbb{R}^N; \mathbb{R}^l)$, each piecewise constant $\sigma(\cdot)$, and each $t \geq t_0$ we have:

$$|\xi(t, t_0, \xi^0, \mathcal{X}_1(\cdot), D(\cdot), \sigma(\cdot))| \leq \max\{\tilde{\beta}(|\xi^0|, t-t_0); \tilde{\gamma}(\|\mathcal{X}_1(\cdot)\|_{L_\infty[t_0, \infty]}); \tilde{\gamma}(\|D(\cdot)\|_{L_\infty[t_0, \infty]}); d\}$$

and such that for some small enough $\tilde{\epsilon} > 0$ and some $c_1 > 0$ we have

$$\tilde{y}(\epsilon) = c_1 \epsilon, \quad \text{whenever } \epsilon \in [0, \tilde{\epsilon}].$$

**Remark 11.** As in [35], let us note that Assumption (II) is a quite natural assumption motivated by [3]. For instance, Assumption (II) holds, if system (40) does not have switching signals, i.e., $F_\sigma(\mathcal{X}_1, \xi, D) = F(x_1, \xi, D)$, and $\dot{\xi} = F(\mathcal{X}_1, \xi, D)$ is ISS w.r.t. $(\mathcal{X}_1, D)$ as the input, and $\frac{\partial F}{\partial \xi}(0, 0, 0)$ is an asymptotically stable matrix (i.e. if Assumption (H2) from [3] holds true).

Then the following extension of our main Theorem 1 also holds true.

**Theorem 4.1.** Under Assumptions (I),(II),(AES),(ApRI), there is a decentralized feedback in the form $u_i = u_i(X_{i,v})$, $i = 1, \ldots, N$ of class $C^1$ with $u_i(0) = 0$ such that the closed-loop system (39) with $u_i = u_i(X_{i,v})$ is UISpS at $X^* = 0$. 

![Fig. 7. Medium deadzones, no disturbances.](image-url)
2. If, in addition, $R_{i,j}(0) = 0$, for all $i = 1, \ldots, N$, $j = 1, \ldots, \nu_i$, as in Item 2 of our main Theorem 1, and if $d = 0$ in system (41), i.e., system (40) is UISS, then there is a decentralized feedback $u_i = u_i(X_{i,v_i})$, $i = 1, \ldots, N$ of class $C^1$ with $u_i(0) = 0$ such that the closed-loop system (39) with $u_i = u_i(X_{i,v_i})$ is UISS at $X^* = 0$. In particular, this is true under Assumptions (AES),(ARI).

(Note that the only difference in (AES) is that the domain of each $\pi_{i,d}(\cdot)$ is extended, since system (39) has the additional state $\xi$, i.e., system (39) is an extension of system (3), but the formulations of (AES),(ApRI),(ARI) are formally the same as for system (3)).

The proof of Theorem 4 is the same as the proof of our main Theorem 1 and therefore it is omitted, in particular, it is omitted, because our current notation used in the proof of our main Theorem 1 is already rather complex even without introducing the dynamic uncertainty $\xi \in \mathbb{R}^\nu$. More specifically, the main part of the proof of Theorem 4 is again Theorem 3, the formulation and the proof of Theorem 3 are not changed, and only Section 5 with Lemma 1 should be updated with introducing more symbols and variables.

8. Conclusion

We have shown the efficiency of the trajectory-based small-gain theorem [2] in uniform decentralized stabilization of complex networks of nonlinear switching systems with unknown switchings. This is an advance in comparison with [17], where the stabilizer is “centralized”, and with [1], where each agent has right invertible input–output maps.
References


