Quantized State Feedback Stabilization of Nonlinear Systems under Denial-of-Service

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Abstract: This paper studies the resilient control of networked systems in the presence of cyber attacks. In particular, we consider the state feedback stabilization problem for nonlinear systems when the state measurement is sent to the controller via a communication channel that only has a finite transmitting rate and is moreover subject to cyber attacks in the form of Denial-of-Service (DoS). We use a dynamic quantization method to update the quantization range of the encoder/decoder and characterize the number of bits for quantization needed to stabilize the system under a given level of DoS attacks in terms of duration and frequency. Our theoretical result shows that under DoS attacks, the required data bits to stabilize nonlinear systems by state feedback control are larger than those without DoS since the communication interruption induced by DoS makes the quantization uncertainty expand more between two successful transmissions.

Keywords: Nonlinear systems, Denial-of-Service attacks, Quantization, Cyber-physical systems, Lyapunov function

1. INTRODUCTION

This paper studies the resilient control of Cyber-physical systems (CPSs) under cyber attacks initiated by adversarial attackers. In particular, we consider the problem of stabilizing nonlinear systems by state feedback, where the measurement of the process state is transmitted through a communication channel with limited data rate and DoS attacks, which disrupt and block communication over channels temporarily. This implies that the state measurement should first be quantized before transmission and some packets carrying the state information may not be received by the receiver under DoS attacks.

For the stability of networked control systems, it is well recognized that there are fundamental limitations on the communication data rate (Liberzon, 2003; Liberzon and Hespanha, 2005; Nair et al., 2004; Tatakonda and Mitter, 2004). It should be mentioned that stabilization under stochastic packet dropping has been studied by assuming that the packet dropping follows certain probability distributions (Amin et al., 2009; Gupta et al., 2009). In Okano and Ishii (2014); You and Xie (2011), communication constraints on both data rate and such stochastic packet losses have been studied. However, as malicious attackers can schedule the DoS attacks deliberately, the obtained results would not be applicable when the packet drops are induced by DoS attacks. This poses new challenges in theoretical analysis and controller design.

For nonlinear systems, system stabilization under state quantization has been intensively studied in the literature (see, e.g., De Persis (2006); De Persis and Isidori (2004); Nair et al. (2004)). In general, dynamic quantization methods are proposed to achieve asymptotic stabilization. It has been shown that quantized feedback stabilizability of nonlinear systems relies on properties of the closed-loop systems without state quantization, for example, input-to-state stability (Liberzon, 2003; Liberzon and Hespanha, 2005), integral input-to-state stabilizability (De Persis, 2006), or only state feedback stabilizability (De Persis and Isidori, 2004). Moreover, the work Nair et al. (2004) has shown that a nonlinear system is locally uniformly asymptotically stabilizable if and only if the data rate exceeds the plant’s local topological feedback entropy at the equilibrium.

More recently, system stabilization under DoS attacks has drawn the attention of researchers. Specifically, in De Persis and Tesi (2015), the authors have proposed a deterministic framework to model DoS attack signals by characterizing their frequency and duration. There, it is proved that the closed-loop system is stable if the accumulation of system's stable mode during DoS-free time outperforms the counterpart of unstable mode under DoS signals. Although following this framework, many problems on control under DoS have been investigated by, e.g., Lu and Yang (2018); Feng and Tesi (2017); Cetinkaya et al. (2019), very little attention has been paid
to nonlinear systems, especially considering the generality of nonlinear systems in the real applications (De Persis and Tesi, 2016).

We emphasize that it is more challenging to stabilize systems considering both limited data rate and DoS attacks as the latter makes the prediction of the sampling time instants difficult, which clearly affects the characterization of the quantization uncertainty. A key question is how to select/how many should be the bits of the quantizer such that quantization error converges to zero eventually. There are several works which investigate stabilization of linear systems with quantized state feedback such as Wakuiki et al. (2020); Ling (2017); Feng et al. (2020). They show that the data rate to stabilize the systems under DoS attacks should be no less than that without DoS amplified by a term related to the frequency and duration of DoS.

However, there are still not many comparable results for nonlinear systems. Even though some papers have studied state estimation over communication channels with finite capacity and packet erasure (Diwadkar and Vaidya, 2013; Sanjaroona et al., 2018), these results are all derived in stochastic scenarios, which as mentioned before, may not be suitable for networked systems under DoS attacks. The recent work of Kato et al. (2020) deals with nonlinear systems from an alternative viewpoint, exposing the limitation of control based on linearization in the presence of DoS.

This paper studies the deterministic stabilization of nonlinear systems with quantized state feedback and DoS attacks under the same framework as in De Persis and Tesi (2015). Specifically, we assume that the communication network has a finite data rate. Hence the state measurement should be quantized before transmission. Due to the presence of DoS attacks, some transmissions may fail. We show that if the number of transmission bits is larger than a value which depends on the system Lipschitz constant and the frequency and duration of DoS attacks, then the system can be stabilized (asymptotically to the origin). Confirming with the intuition, the predicted quantization bits for nonlinear systems stabilization under DoS attacks are larger than those for the attack free case. Our result relies only on the assumption that the system is state feedback stabilizable as in De Persis and Isidori (2004).

**Paper outline.** Section 2 introduces the considered networked system and the DoS model, and provides the problem formulation. In Section 3, we design the encoder and decoder for the state transmission and provide the quantized state feedback controller. In Section 4, we show a result guaranteeing that the quantization error under DoS decreases gradually with a sufficient number of quantization bits. Section 5 proves asymptotic stabilization of the quantized nonlinear system under DoS. Section 6 discusses a simulation example and Section 7 concludes the paper. Due to space limitation, this paper omits all the proofs, which can be found in Shi et al. (2021).

## 2. PROBLEM FORMULATION

**Notation.** Given $a \in \mathbb{R}$, let $\mathbb{Z}$ and $\mathbb{Z}_{\geq a}$ denote the sets of integers, and integers no smaller than $a$, respectively. We let $|a|$ denote its absolute value. Given a vector $b = [b_1 \ b_2 \ \cdots \ b_c]^T \in \mathbb{R}^c$ with $c \in \mathbb{Z}_{\geq 1}$, we let $|b|_\infty$ denote its infinity norm, namely $|b|_\infty = \max_i |b_i|$. Given $d \in \mathbb{R}_{>0}$, we let $B(d)$ denote the hypercube that is centered at the origin and has length of $2d$ for each edge. For a vector $h \in \mathbb{R}^c$ and a real vector set $\mathcal{S}$, we let their sum be $h + \mathcal{S} = \{h + \xi : \xi \in \mathcal{S}\}$. For an exponentially decaying function $e^{-\lambda t}$ where $t \in \mathbb{R}_{\geq 0}$ is the argument and $\epsilon \in \mathbb{R}$, $\lambda \in \mathbb{R}_{>0}$ are constants, we say $\lambda$ is the decay rate of this function. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing and $f(0) = 0$. Furthermore, if $f(a) \rightarrow \infty$ when $a \rightarrow \infty$, then it is said to be of class $\mathcal{K}_\infty$.

### 2.1 Quantized Feedback Stabilization of Nonlinear Systems

Consider the following nonlinear system

$$\dot{x} = f(x, u)$$

(1)

where $x \in \mathbb{R}^n$ denotes the state and $u \in \mathbb{R}^m$ is the control input. Here, $f(\cdot, \cdot)$ is a smooth map and satisfies $f(0,0) = 0$. For this system, we have the following assumption.

**Assumption 1.** There exists a smooth feedback control law given as $u = k(x)$ such that system (1) is globally asymptotically stable.

As a consequence of Assumption 1, there exist a smooth Lyapunov function $V(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and $\alpha(\cdot)$ such that

$$\alpha_1(|x|_\infty) \leq V(x) \leq \alpha_2(|x|_\infty)$$

$$\frac{\partial V}{\partial x} f(x, k(x)) \leq -\alpha(|x|_\infty)$$

(2)

In this system, we assume that the plant and the actuator are co-located while the sensors are remote from the plant controller. The sensors can measure the exact plant state. After each sampling, the sensors send the state system to the controller via a communication channel, which can transmit data with only finite rate. This implies that there should be a quantization mechanism for transmitting the measured state. Hence we assume that the sensor/controller is embedded with the encoder/decoder.

Moreover, the state is sampled and transmitted in discrete time. We let $\{t_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be the sequence of time instants at which the sensors measure, encode and send the state of the plant. During the encoding-decoding process, we assume that the state is sampled in a periodic manner. Hence there exists a constant sampling period $\Delta$ such that $t_i = i\Delta$ for $i \in \mathbb{Z}_{\geq 0}$.

After the quantized state information, the decoder updates its state estimate. We assume that the sensors can finish the state measuring and encoding immediately; and the actuator can decode and apply the control input without delay. We also assume that the communication protocol is acknowledgment (ACK)-based and there are no delays in ACK and state transmissions.

We use dynamic quantization methods for the state transmission, which contains two stages generally. In the **zooming out** stage, the quantization range is enlarged to capture the system state. On the other hand, in the **zooming in** stage, the quantization range and the state estimate error decrease. By the evolution of zooming-in, the estimation error converges to zero asymptotically, and therefore the system is stabilized asymptotically. In this paper, we adopt...
the following assumption that the encoder and decoder have a common knowledge of the initial state.

**Assumption 2.** The initial state satisfies $|x_0|_\infty \leq X$ where $X$ is known by the encoder and the decoder.

### 2.2 Time-constrained Denial-of-Service

In general, communication channels do not only suffer from transmission rate constraints, but also are affected by other problems like noises or packet losses. In this paper, we focus on the case in which the packet drops are induced by DoS. DoS may be caused by legitimate but mass communication or by intentional adversary attacks. In this paper, we do not distinguish them and consider DoS as malicious DoS attacks launched by adversary attackers. In the following, we introduce the DoS model that was first proposed in De Persis and Tesi (2015).

Let $\{h_k\}_{k \in \mathbb{Z}_{\geq 0}}$ with $h_0 \geq t_0$ represent the sequence of time instants when the network changes from nominal status to DoS status. For each transition $h_k$, let $\tau_k \geq 0$ be the length of the DoS interval. Then the $k$th DoS time interval can be represented as $H_k = \{h_k\} \cup [h_k, h_k + \tau_k]$. If $\tau_k = 0$, then $H_k$ is a pulse. Given $\tau, t$ with $\tau \leq t$, let $k(\tau, t)$ denote the number of DoS transitions from absence to presence over $[\tau, t]$. Thus, $\Xi(\tau, t) = \bigcup_{k \in \mathbb{Z}_{\geq 0}} H_k \cap [\tau, t]$ denotes the subset of $[\tau, t]$ where DoS is on. We adopt the following assumptions to characterize the DoS frequency and duration, which are known by the encoder and the decoder.

**Assumption 3.** (DoS Frequency). There exist constants $\eta \in \mathbb{R}_{\geq 0}$ and $\tau_D \in \mathbb{R}_{> 0}$ such that $k(\tau, t) \leq \eta + \frac{t - \tau}{\tau_D}$ for all $\tau, t \geq 0$ with $\tau \geq \tau_D$.

**Assumption 4.** (DoS Duration). There exist constants $\kappa \in \mathbb{R}_{\geq 0}$ and $T \in \mathbb{R}_{> 0}$ such that $\Xi(\tau, t) \leq \kappa + \frac{t - \tau}{T}$ for all $\tau, t \geq 0$ with $\tau \geq T$.

Due to the presence of DoS, the state may not be successfully received by the controller at the nominal transmission times. Let $\{z_t\}_{t \in \mathbb{Z}_{\geq 0}}$ be the sequence of successful transmission instants. The following result characterizes the relation between the successful transmission time instants and the number of successful transmissions.

### Lemma 5. (Feng et al., 2020)

For periodic transmissions with period $\Delta$ and DoS sequences satisfying Assumptions 3 and 4, if $\sigma := 1 - \frac{\tau_D}{\Delta} - \frac{\kappa}{\tau_D} > 0$, then $z_0 \leq (\kappa + \eta \Delta)/\sigma$ and $z_t - z_0 \leq (\Delta + \kappa + \eta \Delta)/\sigma$.

### 3. QUANTIZED STATE FEEDBACK

This section presents the state quantization algorithm and the feedback control. We equip the encoder/decoder with two variables: the state estimate $\hat{\pi}(t) \in \mathbb{R}^n$ and the quantization range $L(t) \in \mathbb{R}_{> 0}$ that upper bounds the infinite norm of the estimation error $e_x(t) = \hat{\pi}(t) - x(t)$. To facilitate the presentation of the encoded state feedback, we first introduce two positive real numbers $W$ and $O > W$, characterizing the evolution regions of the system state $x$ and the state estimate $\pi \in \mathbb{R}^n$, respectively. Also let $U > 0$ be the bound on the control input. Overall, one has

$$|x|_\infty \leq W, \quad |\pi|_\infty \leq O, \quad |u|_\infty \leq U. \quad (3)$$

Later we will prove that the system never leaves these regions under the proposed control scheme. Then let $F$ be the Lipschitz constant such that

$$|f(x, u) - f(\pi, u)|_\infty \leq F|x - \pi|_\infty \quad (4)$$

is valid for all $x$, $\pi$ and $u$ satisfying condition (3). Note by definition, $F$ depends on $W$, $O$ and $U$. How to select the values of $W$, $O$ and $U$ is postponed to Section 5.

The state estimate and quantization range define the quantization region $\mathcal{S}(t)$ given by De Persis and Isidori (2004)

$$\mathcal{S}(t) = [\pi(t) + S(L(t)/2)] \quad (5)$$

which must contain the actual state $x(t)$. Hence the state is located within the quantization region when the estimation error satisfies

$$|e_x(t)|_\infty \leq L(t)/2. \quad (6)$$

Based on the state estimate, the feedback control applied to the plant is given by

$$u(t) = k(\pi(t)). \quad (7)$$

We let the initial condition of $\pi$ be $\pi(t_0) = 0$ and the state estimate between two successful state transmissions evolve as follows

$$\pi(t) = f(\pi(k(\pi(t))), t) \quad t \in [z_t, z_{t+1}] \quad (8)$$

As the system evolves, the state estimate uncertainty may increase. We need to update $S(t)$ such that the state is always inside it since otherwise overflow will occur. We classify the updates of $S(t)$ into two cases based on whether the time $t$ is before or after the instant of the first successful transmission $t_0$.

#### Case a) $t < t_0$

Before $t_0$, all the transmission attempts fail. In view of (8), $\pi(t_0) = 0$ implies $\pi(t) = 0$ and $u(t) = 0, \forall t \in [t_0, z_0]$.

Let $L(t_0) = 2X$ be the initial value of the quantization range. For $t \in [t_0, z_0]$, the quantization range updates as

$$L(t) = 2\phi_{\max}(t), \quad \text{where} \quad \phi_{\max}(t) = \max_{\pi(t) \in \mathbb{R}^n} \max_{\exists \xi(t) \leq \xi(t) \leq \xi(t)} |\phi(\hat{x}(t_0), t')|_\infty \quad (9)$$

with $\phi(\hat{x}(t_0), t', t_0)$ denoting the solution of system (1) at time $t'$ under initial state condition $\hat{x}(t_0)$ and zero control input. Note that $\phi_{\max}(t)$ is accessible to both the encoder and decoder given the range of the initial state $X$ and the system dynamics, and hence it is possible for the encoder/decoder to update $L(t)$ as (9).

#### Case b) $t \geq t_0$

At $z_t$ with $t \in \mathbb{Z}_{\geq 0}$, the quantization region $\mathcal{S}(z_t)$ is characterized by $\pi(z_t)$ and $L(z_t)$. For $t \in [z_t, z_{t+1}]$, the quantization range evolves according to the continuous-time dynamics

$$\dot{L}(t) = FL(t). \quad (11)$$

Assume that at $t = z_{t+1}$, the system state is still within the quantization region. The sensor converts the state information to $nR$ bits by partitioning each edge of the quantization region $\mathcal{S}(z_{t+1})$ into $2^R$ segments. This partition yields $2^nR$ sub-hypercubes and each of them can be represented by an $nR$ bits digital number. By sending a specific digital number, the decoder is informed about the sub-hypercube in which the state $x(z)\in [z_{t+1})$ lies. The encoder
and decoder then take the centroid of this sub-hypercube as $\overline{T}(z_{t+1})$. Meanwhile, the quantization range at $t = z_{t+1}$ updates as

$$L(z_{t+1}) = L(z_{t+1}^-)/2^R. \quad (12)$$

This update implies that after a successful state transmission at $z_{t+1}$, the estimate error satisfies

$$|e_x(z_{t+1})|_\infty \leq L(z_{t+1}^-)/2^{R+1} = L(z_{t+1})/2. \quad (13)$$

4. ASYMPTOTIC ESTIMATION WITH DOS

As mentioned before, the estimate uncertainty may enlarge between two successful transmissions. When there are DoS attacks in the network, the communication becomes aperiodic. Hence, the estimate uncertainty expands unpredictably and may enlarge more than that of the nominal transmission. To compensate the additional expansion of the estimate uncertainty, we need to partition the quantization region $S(t)$ into a larger number of sub-hypercubes.

This intuition is formalized as the key result below. It shows that although the communication is affected by DoS, if the number of bits used in the quantization is sufficiently large, then the state can be estimated by the decoder asymptotically, under the assumption of the boundedness of state and control input.

**Lemma 6.** Given any $x \in \mathbb{R}_{>0}$, let $\varpi_0 = (\kappa + \eta \Delta)/\sigma \geq z_0$ and choose $W$ and $O$ as

$$W > \phi_{\text{max}}(\varpi_0), \quad O = \alpha_1^{-1} \circ \alpha_2(2W), \quad (14)$$

respectively. Suppose that the solution of system (1) with initial condition $|x(0)|_\infty \leq X$ and control input $u(\cdot)$ for which $|u(t)|_\infty \leq U$ for all $t \geq 0$ satisfies $|x(t)| \leq W$ for all $t \geq 0$. If the number of the quantization bits $B = nR$ is chosen such that

$$R > \max \left\{ \frac{F \Delta}{\sigma \ln(2)} \frac{F(\kappa + \eta \Delta)}{\sigma \ln(2)} \right\}, \quad (15)$$

then the estimate $\varpi(\cdot)$ generated by the decoder exists for all $t \geq 0$ and satisfies $|\varpi(t)|_\infty \leq O$. Moreover, the state estimate error satisfies the inequalities

$$|e_x(t)|_\infty \leq \phi_{\text{max}}(t) < W, \quad \forall t \in [0, \varpi_0] \quad \text{(16)}$$

$$|e_x(t)|_\infty \leq \gamma \lambda^{t+1}, \quad \forall t \in [\varpi_0, \varpi_1] \quad \text{(17)}$$

with $\gamma = We^{e^{(F \Delta + \eta \Delta)}} / \sigma$, $\lambda = e^{(F \Delta / \sigma - R \ln(2))} < 1$. $lacksquare$

By this result, the estimation error may increase between two successful information transmissions, possibly with DoS in between. However, from (17), we observe that as the number of successful transmissions increases, the estimate error eventually decays to zero. This implies that the state is asymptotically reconstructed by the decoder.

The bound on the estimation error in Lemma 6 is characterized by the number of successful transmissions $l$ for $l \geq 0$. To compare the asymptotic estimation result with that in De Persis and Isidori (2004) for the case without DoS and to discuss the side-effects of DoS on the number of quantization bits, we reform the bounds on the norm of the estimation error (16) and (17) in Lemma 6 with time $t$ as the argument. This leads us to the following corollary.

**Corollary 7.** Under the conditions of Lemma 6, we have $|e_x(t)|_\infty \leq c e^{-\omega t}$, where $c = We^{e(2 \ln(2) - F \Delta / \sigma - (\kappa / \sigma + \eta \Delta))}$ and $\omega = \ln(2) / \sigma - F \Delta$. $lacksquare$

**Remark 8.** It is emphasized that our result on the data rate bound is a generalization of that in the literature. In particular, when no DoS attack is present in the system, the bound reduces to that of Lemma 1 of De Persis and Isidori (2004). Compared with the condition there, we observe that DoS can directly influence the number of quantization bits. Since $R$ is lower bounded as (15) and $\sigma < 1$, the number of bits needed for exponential state estimate is larger than that of the nominal case in De Persis and Isidori (2004). The result regarding $R$ is also intuitive in the sense that when the communication suffers from DoS attacks, the transmission attempts are more likely to be interrupted. This implies that the state estimate uncertainty may expand more between two successful transmissions. Hence we need more bits to compensate the expansion than in the nominal situation in De Persis and Isidori (2004). Moreover, if the frequency and/or duration of DoS increases ($\tau_0$ and/or $T$ is smaller), $\sigma$ decreases and $R$ increases. Hence, to estimate the state under more severe DoS attacks, we need more bits.

Similar arguments on the relation between DoS and the required communication rate can be found in Wakaiki et al. (2020) and Feng et al. (2020) in the case of linear systems; there, an explicit relation to the minimum data rate result (e.g., Feng et al. (2020, Theorem 2), Wakaiki et al. (2020, Theorem 3.2)) can be established. Note that in (15), $R$ depends linearly on $F$, which suggests that more unstable systems with larger $F$ requires more bits. This implication is in alignment with the linear case.

5. ASYMPTOTIC STABILIZATION UNDER DOS

In the previous section, we have shown that the system state can be estimated asymptotically under a sufficiently large number of bits. This raises the possibility of stabilizing the system by the encoded state feedback. However, the validation of Lemma 6 depends on the assumption that the state always evolves within a set where $|x(t)|_\infty \leq W$. Hence before applying the result in Lemma 6, we need to first select a proper set in which we would like the closed-loop system to evolve; then, we estimate the number of quantization bits which ensures that the quantization error is a sufficiently small value at the time when the state of the system closely approaches the boundary of the selected set. This guarantees that the derivative of the Lyapunov function under the encoded state feedback will remain negative and not influenced by the quantization error. Thus the state will always evolve in the selected set.

With the encoded state feedback control $u(t) = k(\varpi(t))$, the corresponding closed-loop system can be written as:

$$\dot{x}(t) = f(x, k(x)) + f(x, \varpi) - f(x, k(x)) = f(x, k(x)) + g(x, \varpi)(x - \varpi) \quad (18)$$

with $g(x, \varpi)$ given as in De Persis and Isidori (2004).

Let

$$l := \alpha_2(\phi_{\text{max}}(\varpi_0)) \quad (19)$$

and define the level set of states $\Gamma_l := \{ x : V(x) \leq l \}$. The reals $W$ and $U$ introduced in Section 3 and Lemma 6 are then given as follows:

$$W = \alpha_1^{-1}(l + \delta), \quad O = \alpha_1^{-1} \circ \alpha_2(2W)$$

$$U = \max_{x:|x|_\infty \leq O} |k(x)|_\infty \quad (20)$$
with $\delta$ being an arbitrary positive number. The Lipschitz constant $F$ is chosen as in Section 3.

Set

$$M := \max_{z \in \mathcal{B}^2(0), \pi \in \mathcal{B}(0)} \left| \frac{\partial V}{\partial x} (g(x, \pi)) \right| .$$

(21)

In the following lemma, we show that the system state satisfies $|x(t)|_{\infty} \leq W$ in a non-empty time interval.

**Lemma 9.** If the number of the quantization bits is chosen as $B = nR$, with $R$ satisfying the condition (15), then there exists a finite time $\theta := z_0 + \delta/(M\gamma)$ such that, for all $t \in [0, \theta)$, $x(t) \in \Gamma_{t+\delta}$ and

$$\frac{\partial V}{\partial x} f(x(t), k(\pi(t))) \leq -\alpha(|x(t)|_{\infty}) + M|x(t) - \pi(t)|_{\infty} .$$

The next result shows that the quantization error at time $t = \theta$ can be guaranteed to be small enough as long as the number of quantization bits is sufficiently large. Let $K = \max\{\frac{\delta \gamma}{M\gamma \Delta} - \frac{\kappa}{\Delta} - \eta, 1\}$.

**Lemma 10.** For any $\varepsilon > 0$, there exists a number of quantization bits $B = nR$ with $R > 0$ satisfying (15) and the following condition

$$R \geq (F\Delta)/\sigma + (\ln(\gamma/\varepsilon))/(K\ln 2)$$

such that for all $t \in [0, \theta]$ with $\theta > z_0$, (16) and (17) with $z_t = \max\{z_t : z_t \leq \theta\}$ hold. And the quantization error at $t = \theta$ satisfies $|\varepsilon(x)|_{\infty} < \varepsilon$. Moreover, if $x(t) \in \Gamma_{t+\delta}$ for $t \in [\theta, \theta]$ with any $\theta > \theta$, then $|\varepsilon_t(x)|_{\infty} < \varepsilon, \forall t \in [\theta, \theta]$.

As discussed earlier, intuitively we may expect that if the quantization error is sufficiently small, the negativness of the Lyapunov function derivative will not be influenced (the derivative of the Lyapunov function is semi-negative in the selected set that contains the origin), and hence the system will evolve in the set. The next lemma establishes a concrete result following this intuition.

**Lemma 11.** Given a positive $\rho < l + \delta$, if the number of quantization bits $B = nR$ is such that

$$R > \max\{\frac{F\Delta}{\rho \ln 2}, \frac{\ln(\kappa + \eta\Delta)}{\ln 2}\}$$

(23)

and

$$R \geq \frac{F\Delta}{\rho \ln 2} + \frac{\ln(\kappa + \eta\Delta)}{\ln 2}$$

(24)

then $x(t) \in \Gamma_{t+\delta}$, and (16) and (17) hold for all $t \geq 0$.

Lemmas 9–11 and Lemma 6 together show that the state $x(t)$ and the estimate $\pi(t)$ can always evolve within the prescribed evolution sets by using a sufficiently large number of quantization bits, which proves the hypothesis mentioned before. By Lemma 6, this indicates that the state estimate $\pi(t)$ approaches the state $x(t)$ asymptotically and paves the way to show that the system can be stabilized by the encoded state feedback. We now arrive at the main result of this paper.

**Theorem 2.** Consider the nonlinear process in (1) with control actions in (7) and (8) under periodic transmission interval $\Delta$. Suppose that Assumptions 1 and 2 hold and the DoS attacks characterized in Assumptions 3 and 4 satisfy $\sigma = 1 - \frac{1}{\tau} - \frac{\Delta}{\tau D} > 0$. Let $\zeta_0 = (\kappa + \eta\Delta)/\sigma$ and $l = \alpha_2(\phi_{\text{max}}(\zeta_0))$, with $\phi_{\text{max}}$ given in (10), then for any $\delta > 0$ and arbitrary $\rho < l + \delta$, if the number of quantization bits $B = nR$ with $R$ satisfies (23) and (24), the closed-loop system is stable.

**Remark 13.** By (23) and (24), the lower bound on $R$ is always larger than $\frac{F\Delta}{\rho \ln 2}$. Compared with the lower bound in Liberon and Hespanha (2005) that assumed ISS of the system to encoding errors and did not consider DoS attacks, our lower bound on $R$ is at least $1/\delta$ times larger, under the same value of $F\Delta$. Without considering the DoS attacks and hypothesizing the system is state-feedback stabilizable, De Persis and Isidori (2004) (Proposition 1) also derived two lower bounds on $R$, where the first one is $\frac{F\Delta}{\rho \ln 2}$. With a same bound on the initial condition $X$, and $O$ given in (20) are always no less than those in De Persis and Isidori (2004), hence $F$ in this paper is no less than that in De Persis and Isidori (2004). This shows that the lower bound on $R$ in (23) is always larger than the first lower bound on $R$ in De Persis and Isidori (2004), provided that the sampling period is the same. The second lower bound on $R$ in De Persis and Isidori (2004) is $\frac{\ln(\kappa + \eta\Delta)}{\ln 2}/(2/\rho \ln 2)$ since $\gamma > W > X$. The lower bound on $R$ in (24) would be smaller than the second lower bound on $R$ in (24) in (De Persis and Isidori, 2004, Proposition 1). For the case $\frac{\sigma}{\kappa \Delta} - \frac{\kappa}{\Delta} - \eta < 1$, it is difficult to compare the lower bound on $R$ in (24) with the second bound in (De Persis and Isidori, 2004, Proposition 1).

6. NUMERICAL SIMULATIONS AND DISCUSSION

In this section, we present a simulation example. We consider the system dynamics in De Persis and Isidori (2004) as

$$\dot{x} = x^2 - x^3 + u, \quad (25)$$

which is not asymptotically stable without control. Let $u = -\frac{1}{2}x^2$ and the candidate Lyapunov function be $V(x) = \frac{1}{4}x^2$. Then $\alpha_1(x) = \alpha_2(x) = \frac{1}{2}x^2$ and $\dot{V}(x) = x^3 - x^4 - \frac{1}{4}x^2 \leq -x^3 = -2V(x)$. Hence the controller can stabilize the system and $\alpha(x) = 2\alpha_1(x) = x^2$.

Let $x(0) = 0.75$, $X = 0.9$ and sampling period $\Delta = 0.1$ second. The parameters of the DoS signal are set as $\kappa = 0.300$, $\eta = 1.300$, $T = 2.222$, $\tau_D = 0.714$ Hence $\sigma = 1 - \frac{1}{T} - \frac{\Delta}{\tau D} = 0.41$ and $\zeta_0 = \frac{\sigma + \Delta}{\sigma} = 0.8985$. From (10), (19) and (20), to estimate $W, O$ and $U$, we need to find $\phi_{\text{max}}(\zeta_0)$. This is obtained by simulating the uncontrolled system to $t = \zeta_0$ with different initial state $x(0) \in B(0.65)$ and take the maximal absolute value of all the solutions. Simulation shows that $\phi_{\text{max}}(\zeta_0) = 0.8$, hence $l = 0.32$. Choosing $\delta = 0.0001$ leads to $W \approx 0.8$, $O \approx 1.6$ and $U \approx 2$. Then by (25), one can verify that for any $x, \pi$ and $O$ satisfying $|x| \leq 0.8$, $|\pi| \leq 1.6$ and $|u| \leq 1.6$, $|x^3 - x^4 + u - x^2 + x^4 - u|_{\infty} \leq 6.88(|x - \pi|)$, which shows that the Lipschitz constant $F$ is 6.88. In the simulation for the closed-loop system, we let $F = 7$. Then by $\gamma$ in Lemma 6, $\gamma = 4311.1$. Now consider the constant $M$ in (21). By (18), (25) is an affine function of $u, g(x, \pi) = \frac{5}{4}$. Hence $M = \max\{|x|_{\leq 1}, |\pi|_{\leq 2} \frac{5}{4}|x|\} = 1$.

We are ready to compute a lower bound on the quantization bit $R$, above which the system can be stabilized to the origin. By (24), if we select $\rho = 0.32$, a lower bound for $R$ can be computed as 16. This bound is quite conservative, as in the simulation, if we select $R = 2,$
the closed-loop system may still be stable. The simulation result is shown in Fig. 1, where the shaded areas represent the time intervals when the DoS attacks are launched. We can see that after the first successful transmission, the state decreases to zero, and $L(t)$ increases over the time intervals when there is no successful transmission and jumps to a smaller value at the successful transmissions.

![Graph](image)

**Fig. 1.** System evolution with the quantized state-feedback control under DoS with $R = 2$

The conservativeness of the theoretical lower bound of the quantization bits may come from two reasons: a) the lower bound is derived based on the worst case where the inter-times of successful transmissions are always equal to the upper bound implied from $z_f - z_0 \leq (\ell \Delta + \kappa + \eta \Delta)/\sigma$ in Lemma 5, while this is not the case in the simulation; b) the estimate uncertainty does not enlarge as fast as that corresponding to the Lipschitz constant $F$.

7. CONCLUSIONS

In this paper, we have designed a quantized state feedback controller to stabilize nonlinear systems, where the state transmissions are subject to DoS attacks. For state-feedback stabilizable nonlinear systems, we have shown that if the quantization bits are above a value which depends on the frequency and duration of the DoS signals, the systems can always be stabilized by the proposed quantized state feedback. As the bound in this paper is derived for general nonlinear systems, it may be conservative for certain classes of nonlinear systems, for example, systems whose dynamics are affine functions of the control inputs, as shown in the simulation.

REFERENCES


