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OBSERVABILITY CONDITIONS BY POLYNOMIAL IDEALS

Y. Kawano and T. Ohtsuka

ABSTRACT

In this paper, we give global and local observability conditions of polynomial systems. Each observability is defined in terms of distinguishability of a pair of initial states. The set of pairs of initial states are represented as an affine variety. By using this affine variety, a global observability condition is derived. Then, a local observability condition is presented based on the minimal decomposition of the affine variety and dimension theory.

Key Words: Polynomial systems, observability, algebraic geometry.

1. INTRODUCTION

Observability is one of the most basic properties in systems and control. Kalman's observability rank condition has been generalized to nonlinear local observability, which is a sufficient condition [1]. In contrast to local observability, global observability has not been studied intensively even though global observability is also an important property. By restricting the class of nonlinear systems to polynomials, global observability conditions have been derived [2–5] which are described in terms of commutative algebra and algebraic geometry. On the basis of the Hilbert basis theorem [6–9] in commutative algebra, it has been shown that the number of computations required to verify global observability is finite.

In this paper, on the basis of the results for global observability [3,5], we present a sufficient condition for local observability of polynomial systems, and the number of computations required to verify each condition is shown to be finite. Note that our condition can be used for testing local observability of a system not satisfying the conventional sufficient condition. The main difficulty in testing for local observability is to show the existence of a suitable neighborhood of each initial state such that the corresponding output is different from the outputs corresponding to the other initial states in the neighborhood. This is the main difference between deriving global and local observability conditions. To overcome this diffi-

culty, in this paper, we exploit the minimal decomposition and dimension theory of algebraic geometry.

To derive the local observability condition, first we give a global observability condition in an affine variety, which means that a system is globally observable for all initial states in the affine variety. Since the entire space or an initial state is a specific affine variety, our criterion contains the criteria for global observability for all initial states [3,10] and at an initial state [5] as particular cases. Our global observability conditions are described by a finite set of polynomials owing to the Hilbert basis theorem. Next, starting from the global observability condition, by exploiting the minimal decomposition of an affine variety [6–8], we derive a local observability condition in an affine variety, which is an extended condition of that in the preliminary versions of this paper [11,12]. Moreover, by exploiting dimension theory of algebraic geometry [6,8], we show that our local observability condition can be verified in a finite computation. In addition, over the field of complex numbers, our condition becomes a necessary and sufficient condition.

Note that local observability analysis of this paper is partly based on two conference papers [11,12], which generalize the results of a journal paper [5]. In [5], local observability conditions at an initial state are derived, but they cannot be used for testing for local observability at an infinite set of initial states, e.g., for all initial states. That is, the papers [11,12] study condition (12) and provide Theorem IV.1 (2a) of this paper in case $V = \mathbb{R}^n$. As we can see in the statement of Theorem IV.1 (2a), for $V = \mathbb{R}^n$, condition (12) is a sufficient condition in $\mathbb{R}^n \setminus \hat{V}$. Therefore, only by using Theorem IV.1 (2a), i.e. results in [11,12] we cannot verify local observability in $\mathbb{R}^n$. To verify this, we need to study local observability in $\hat{V}$. This is the motivation of this paper. As the first step of this paper, we clarify that $\hat{V}$ is an affine variety as shown in Theorem IV.1 (2b) of this paper, which is not clarified in [11,12]. Based on this fact, we start to study local

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Table I. Conditions for Local Observability.

	local observability
at an initial state $\{x_0\} \subset \mathbb{R}^n$	[5]
in $\mathbb{R}^n \setminus \hat{V}$	[11,12]
in an affine variety $V \subset \mathbb{R}^n$	this paper

observability in affine variety and finally present a sufficient condition. In summary, for local observability, the contribution of this paper compared to the previous works may be summarized as in Table I.

The remainder of this paper is organized as follows. In Section II, the class of polynomial systems discussed in this paper and definitions of global and local observability are given. In Section III, we derive a global observability condition. Section IV is dedicated to local observability. Concluding remarks are given in Section V, and some technical results are shown in the Appendices.

Notation. The polynomial ring over the field of real numbers $\mathbb{R}$ with variables ξ_i, η_i ($i = 1, 2, \dots, n$) is denoted by $\mathbb{R}[\xi, \eta] (= \mathbb{R}[\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n])$. The ideal generated by $p_1, p_2, \dots, p_r \in \mathbb{R}[\xi, \eta]$ is denoted as $\langle p_1, \dots, p_r \rangle \subset \mathbb{R}[\xi, \eta]$. The set $\mathbf{V}(K) \subset \mathbb{R}^n \times \mathbb{R}^n$ represents the affine variety defined by an ideal K in $\mathbb{R}[\xi, \eta]$, which is the set of common zeros of all polynomials in ideal $K \subset \mathbb{R}[\xi, \eta]$. Unless ideal K is important, we just say $\mathbf{V}(K)$ as an affine variety V without mentioning ideal K . For more details, see Appendix VI.

II. PROBLEM SETTING

Consider a continuous-time polynomial system

$$\Sigma \begin{cases} dx(t)/dt = f(x(t)), & x(0) = x_0, \\ y(t) = h(x(t)), \end{cases}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ denote the state and output, respectively; $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $h: \mathbb{R}^n \rightarrow \mathbb{R}$ are polynomial mappings. The output of a system Σ only depends on time $t \geq 0$ and the initial state $x_0 \in \mathbb{R}^n$, and thus we denote the output trajectory at time $t \geq 0$ starting from x_0 as $y(t; x_0)$ if it exists. Since f and h are analytic, $y(t; x_0)$ is an analytic function of t and x_0 in a suitable domain.

We consider the following observability.

Definition II.1. [1] A pair of initial states $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ of a system Σ is distinguishable if there exists a time $t \geq 0$ such that $y(t; \xi) \neq y(t; \eta)$ holds. If a pair (ξ, η) is not distinguishable, the pair is said to be indistinguishable.

Definition II.2. A system Σ is globally observable at an initial state $x_0 \in \mathbb{R}^n$ if x_0 is distinguishable from any $\xi \in \mathbb{R}^n \setminus \{x_0\}$. Moreover, a system Σ is said to be globally observable in $V \subset \mathbb{R}^n \setminus \{0\}$ if the system is globally

observable at any $x_0 \in V$. If $V = \mathbb{R}^n$, the system is said to be globally observable.

Definition II.3. A system Σ is locally observable at an initial state $x_0 \in \mathbb{R}^n$ if there exists a neighborhood of x_0 denoted as $U(x_0) \subset \mathbb{R}^n$, such that x_0 is distinguishable from any $\xi \in U(x_0) \setminus \{x_0\}$. Moreover, a system Σ is said to be locally observable in $V \subset \mathbb{R}^n \setminus \{0\}$ if the system is locally observable at any $x_0 \in V$. If $V = \mathbb{R}^n$, the system is said to be locally observable.

The following observability rank condition is a well-known criterion for local observability [1].

Proposition II.4 (Observability rank condition). A system Σ is locally observable if there exists a certain natural number N such that

$$\text{rank} \frac{\partial \mathcal{O}}{\partial x}(x_0) = n, \\ \mathcal{O}(x) = \begin{bmatrix} h(x) & L_f h(x) & \cdots & L_f^N h(x) \end{bmatrix}^T$$

holds for all $x_0 \in \mathbb{R}^n$, where $L_f^0 h(x) = h(x)$ and $L_f^{(i+1)} h(x) := (\partial L_f^i h(x) / \partial x) f(x)$ ($i = 0, 1, \dots$).

To check the observability rank condition, it is needed to determine whether or not there exists an initial state $x_0 \in \mathbb{R}^n$ where $\text{rank}(\partial \mathcal{O}(x_0) / \partial x) < n$ holds. If such an initial state x_0 exists, it is obtained by solving a set of algebraic equations. Let $g_i \in \mathbb{R}[x]$ ($i = 1, 2, \dots, s$) be the n th minors of $\partial \mathcal{O} / \partial x \in (\mathbb{R}[x])^{n \times N}$. If x_0 is an element in $\{x_0 \in \mathbb{R}^n : g_i(x_0) = 0 \text{ for all } i = 1, \dots, s\}$, this x_0 satisfies $\text{rank}(\partial \mathcal{O}(x_0) / \partial x) < n$, and vice versa. $\{x_0 \in \mathbb{R}^n : g_i(x_0) = 0 \text{ for all } i = 1, \dots, s\}$ are the common zeros of the set of algebraic equations.

The observability rank condition is only a sufficient condition for local observability in general. Even for the following simple, one-dimensional and locally observable polynomial system, the observability rank condition does not hold.

Example II.5. Consider the following one-dimensional polynomial system

$$\begin{cases} dx(t)/dt = x(t)/2, & x(0) = x_0, \\ y(t) = x^2(t). \end{cases}$$

First, we check the observability rank condition for this system. The mapping $\mathcal{O}$ is computed as

$$\mathcal{O}(x) = \begin{bmatrix} x^2 & x^2 & \cdots & x^2 \end{bmatrix}^T,$$

and consequently, we have

$$\frac{\partial \mathcal{O}}{\partial x}(x) = [2x \ 2x \ \cdots \ 2x]^T.$$

At the origin, the rank of the Jacobian matrix $\partial \mathcal{O}(0)/\partial x$ is zero. Therefore, the observability rank condition does not hold. However, it is possible to show that the system satisfies the definition of local observability. To check Definition II.3, we obtain the pairs of indistinguishable initial states. From Lemma III.1 below, the set of pairs of indistinguishable initial states is obtained as

$$U := \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi^2 = \eta^2\}.$$

Next, we derive the initial states indistinguishable from the initial state $x_0 \in \mathbb{R}$. These initial states are characterized by the intersection $U \cap (\mathbb{R} \times \{x_0\}) \subset \mathbb{R} \times \mathbb{R}$ computed as follows:

$$\begin{aligned} U \cap (\mathbb{R} \times \{x_0\}) &= \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi^2 = \eta^2\} \cap (\mathbb{R} \times \{x_0\}) \\ &= \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi = \eta\} \\ &\quad \cup \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi = -\eta\} \cap (\mathbb{R} \times \{x_0\}) \\ &= \{(\xi, x_0) \in \mathbb{R} \times \{x_0\} : \xi = x_0\} \\ &\quad \cup \{(\xi, x_0) \in \mathbb{R} \times \{x_0\} : \xi = -x_0\}. \end{aligned} \quad (1)$$

Note that x_0 in (1) is fixed. The projection of $U \cap (\mathbb{R} \times \{x_0\})$ on η space is $\{x_0\} \subset \mathbb{R}$, and that on ξ space is $\{x_0, -x_0\} \subset \mathbb{R}$, whose elements are indistinguishable from x_0 . That is, x_0 is indistinguishable from $\xi = x_0$ and $\xi = -x_0$. For instance, $x_0 = 0$ is indistinguishable from $\xi = x_0 = 0$ and $\xi = -x_0 = 0$. That is, $x_0 = 0$ is distinguishable from any $\xi \in \mathbb{R} \setminus \{0\}$. From Definition II.2, the system is globally observable at the origin. Finally, we test for global and local observability at a nonzero initial state based on Definitions II.2 and II.3. For any nonzero initial state x_0 , we have $x_0 \neq -x_0$. Since x_0 is indistinguishable from $-x_0$, the system is not globally observable at any nonzero initial state. Next, for any nonzero initial state $x_0 \in \mathbb{R}$, it can be shown the existence of a neighborhood $U(x_0) \subset \mathbb{R}^n$ such that x_0 is distinguishable from any $\xi \in U(x_0) \setminus \{x_0\}$, i.e., $\{x_0, -x_0\} \cap U(x_0) = \{x_0\}$ holds. From Definition II.3, the system is locally observable at any nonzero initial state. In summary, the system is locally observable even though the observability rank condition does not hold.

Example II.5 demonstrates that even for polynomial systems, the observability rank condition is only a sufficient condition. Note that, in Example II.5, for a system not satisfying the rank condition, the definitions of global and local observability can be checked. Thus, in this paper, we consider to derive the global and local

observability conditions of a polynomial system Σ based on their definitions.

III. GLOBAL OBSERVABILITY

In this section, we derive a necessary and sufficient condition for global observability in an affine variety of a polynomial system Σ based on Definition II.2. First, we show that the set of pairs of indistinguishable initial states can be represented by an affine variety. By using this affine variety, a global observability condition is presented.

Pairs of indistinguishable initial states of a system Σ are characterized by Lie derivatives of the output function [5].

Lemma III.1. A pair of initial states $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ of a system Σ is indistinguishable if and only if

$$L_f^i h(\xi) = L_f^i h(\eta), \quad i = 0, 1, \dots \quad (2)$$

If f and h are restricted to polynomials, the Lie derivatives $L_f^i h(x)$ ($i = 0, 1, \dots$) are also polynomials. Thus, the pairs of indistinguishable initial states $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ of a polynomial system Σ are characterized by the common zeros of polynomials. However, Lemma III.1 does not give an upper bound on the number of computations of Lie derivatives. Thus, by only exploiting Lemma III.1, it is difficult to find the set of pairs of indistinguishable initial states. In [3,5,10], it has been shown that this set can be obtained as the common zeros of a finite set of polynomials. On the basis of these results, we derive the global observability condition in an affine variety.

From condition (2), we define an ideal $J_k \subset \mathbb{R}[\xi, \eta]$ ($k = 0, 1, \dots$) as follows:

$$J_k := \langle h(\xi) - h(\eta), \dots, L_f^k h(\xi) - L_f^k h(\eta) \rangle. \quad (3)$$

The affine variety $\mathbf{V}(J_k) \subset \mathbb{R}^n \times \mathbb{R}^n$ denotes the set of $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying

$$L_f^i h(\xi) = L_f^i h(\eta) \quad (i = 0, 1, \dots, k).$$

From (3), an ascending chain of ideals $J_0 \subset J_1 \subset \dots$ is obtained, and the chain has the strict inclusion property [5]. According to the Hilbert basis theorem [6–9], there exist a certain natural number N and ideal $\mathcal{J} \subset \mathbb{R}[\xi, \eta]$ such that

$$J_0 \subsetneq J_1 \subsetneq \dots \subsetneq J_N = J_{N+1} = \dots =: \mathcal{J}. \quad (4)$$

Consequently, the affine variety $\mathbf{V}(\mathcal{J}) \subset \mathbb{R}^n \times \mathbb{R}^n$ represents the set of pairs of indistinguishable initial states $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$.

Remark III.2. The ideal $\mathcal{J} \subset \mathbb{R}[\xi, \eta]$ in (4) can always be obtained by using the reduced Gröbner basis of an ideal. If we give a monomial ordering for the polynomials of the polynomial ring $\mathbb{R}[\xi, \eta]$, the reduced Gröbner basis of an ideal is uniquely determined. Thus, if two ideals are equivalent, then they have the same reduced Gröbner basis, and vice versa. By computing the reduced Gröbner basis, it can be found when ascending chain (4) stops.

Next, to describe a global observability condition, we define an ideal $\mathcal{I} \subset \mathbb{R}[\xi, \eta]$:

$$\mathcal{I} = \langle \xi_1 - \eta_1, \xi_2 - \eta_2, \dots, \xi_n - \eta_n \rangle. \quad (5)$$

Then, we have

$$\mathbf{V}(\mathcal{I}) = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = \eta\}, \quad (6)$$

which can also be expressed as

$$\mathbf{V}(\mathcal{I}) = \bigcup_{x_0 \in \mathbb{R}^n} \{(x_0, x_0)\}. \quad (7)$$

Note that (5) and (6) yield

$$\mathcal{I} = \mathbf{I}(\mathbf{V}(\mathcal{I})). \quad (8)$$

Now, we have a global observability condition.

Theorem III.3. A polynomial system Σ is globally observable in an affine variety $V \subset \mathbb{R}^n$ if and only if

$$\mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times V) = \mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V). \quad (9)$$

Proof. (Sufficiency) If (9) holds, then

$$\mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times \{x_0\}) = \mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times \{x_0\}) \quad (10)$$

holds for all $x_0 \in V$. The intersection $\mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times \{x_0\})$ represents the set

$$\{(\xi, x_0) \in \mathbb{R}^n \times \{x_0\} : \xi \text{ is indistinguishable from } x_0\}.$$

From (7), the right hand side of (10) is $\{(x_0, x_0)\}$. Therefore, $x_0 \in V$ is only indistinguishable from itself. That is, the system Σ satisfies Definition II.2 in V .

(Necessity). We prove this by contraposition. Since $\mathbf{V}(\mathcal{J}) \supset \mathbf{V}(\mathcal{I})$ always holds, we have

$$\mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times V) \supset \mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V). \quad (11)$$

Thus, if (9) does not hold, we obtain $\mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times V) \not\subset \mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V)$. That is, there exists an element $(\xi, x_0) \in (\mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times V)) \setminus (\mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V))$, where $\mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V)$ contains (x_0, x_0) . Thus, the pair $(\xi, x_0) \in \mathbb{R}^n \times V$ is indistinguishable, and $\xi \neq x_0$. That is, the system Σ does not satisfy Definition II.2 in V .

When $V = \{x_0\}$ or $V = \mathbb{R}^n$, global observability condition (9) is equivalent to the condition derived in [5] or [3], respectively. The proof of Theorem III.3 is similar to those in [3,5]. The difference is that we take the intersection with $\mathbb{R}^n \times V$. Actually, this is a slight difference but owing to this difference, we have condition (9) containing both conditions in [3,5] as special cases.

In Theorem III.3, to test for global observability in an affine variety $V \subset \mathbb{R}^n$, two affine varieties, $\mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times V) \subset \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V) \subset \mathbb{R}^n \times \mathbb{R}^n$, are computed. These two affine varieties are important when testing for global and local observability. Hereafter, for the sake of simplicity, we define $V^{\mathcal{J}} := \mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times V)$ and $V^{\mathcal{I}} := \mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V)$.

IV. LOCAL OBSERVABILITY

In this section, we give a sufficient condition for local observability in an affine variety of a polynomial system Σ , which can be verified in a finite time. This condition is obtained based on the following theorem proved in Appendix 6.2.

Theorem IV.1. We consider the ideals $\mathcal{J} \subset \mathbb{R}[\xi, \eta]$ and $\mathcal{I} \subset \mathbb{R}[\xi, \eta]$ defined in (4) and (5), respectively. Let $V \subset \mathbb{R}^n$ be an irreducible variety, and let $V^{\mathcal{I}} := \mathbf{V}(\mathcal{I}) \cap (\mathbb{R}^n \times V) \subset \mathbb{R}^n \times \mathbb{R}^n$ and $V^{\mathcal{J}} := \mathbf{V}(\mathcal{J}) \cap (\mathbb{R}^n \times V) \subset \mathbb{R}^n \times \mathbb{R}^n$. Then, we have the following.

1. $V^{\mathcal{I}}$ is an irreducible variety.
2. If the minimal decomposition of $V^{\mathcal{J}}$ is in the form

$$V^{\mathcal{J}} = V^{\mathcal{I}} \cup X, \quad (12)$$

where $X = X_1 \cup \dots \cup X_s$, then one has the following.

- (a) Let $\hat{V} := \{x_0 \in \mathbb{R}^n : (x_0, x_0) \in V^{\mathcal{I}} \cap X\}$. A polynomial system Σ is locally observable in $V \setminus \hat{V}$.
- (b) $\hat{V}$ is an affine variety, and $\dim V > \dim \hat{V}$, where $\dim V$ is the dimension of affine variety V defined in Definition A.8.

Theorem IV.1 can be used for testing for local observability in not only an irreducible variety but also an affine variety (which is not necessarily irreducible). To test for local observability in an affine variety $V \subset \mathbb{R}^n$, we compute the minimal decomposition $V = V_1 \cup V_2 \cup \dots \cup V_r$. Then, every V_i ($i = 1, \dots, r$) is irreducible, and thus we can exploit Theorem IV.1. Here, we summarize how to use Theorem IV.1 to test local observability in V_i . First, for all V_i ($i = 1, \dots, r$), we compute two affine varieties $V_i^{\mathcal{J}}, V_i^{\mathcal{I}} \subset \mathbb{R}^n \times \mathbb{R}^n$. From Theorem IV.1 (2a),

the system is locally observable in $V_i \setminus \hat{V}_i$ ($i = 1, \dots, r$), where $\hat{V}_i := \{x_0 \in \mathbb{R}^n : (x_0, x_0) \in V_i^I \cap X_i\}$. This $\hat{V}_i$ is an affine variety according to Theorem IV.1 (2b). Next, we need to evaluate local observability in the affine variety $\hat{V}_i$ ($i = 1, \dots, r$). To test for local observability in $\hat{V}_i$, we can again exploit Theorem IV.1. In summary, local observability in V is verified by recursively utilizing Theorem IV.1. Moreover, based on Theorem IV.1 (2b), Theorem IV.4 below shows that the recursion terminates in finite steps. Actually, in the following example, local observability for all initial states is tested by repeatedly checking the condition in Theorem IV.1, and the number of repetitions is finite.

Example IV.2. In this example, Theorem IV.1 is exploited to check local observability of the system considered in Example II.5. In Example II.5, to test for local observability, the existence of a suitable neighborhood of each initial state, in which the initial state is only indistinguishable from itself, is discussed. However, it is difficult to find such neighborhoods in general. Here, by utilizing Theorem IV.1, we test for local observability without directly considering the existences of such neighborhoods. For the system in Example II.5, we showed that the following $U \subset \mathbb{R} \times \mathbb{R}$ represents the set of pairs of indistinguishable initial states.

$$U = \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi^2 = \eta^2\}$$

First, we test for local observability for all initial states by exploiting Theorem IV.1. In this case, we choose $V \subset \mathbb{R}$ of Theorem IV.1 as $\mathbb{R}$, and thus we denote $\mathbb{R}^J := U$. The minimal decomposition of $\mathbb{R}^J$ is

$$\mathbb{R}^J = \mathbb{R}^I \cup X_1, \quad X_1 := \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi = -\eta\},$$

where $I := \langle \xi - \eta \rangle \subset \mathbb{R}[\xi, \eta]$. Therefore, condition (12) holds. We compute $\hat{V} \subset \mathbb{R}$ in Theorem IV.1 (2a), which is the set

$$\begin{aligned} \hat{V} &= \{x_0 \in \mathbb{R} : (x_0, x_0) \in \mathbb{R}^I \cap X_1\} \\ &= \{x_0 \in \mathbb{R} : (x_0, x_0) \in \{(0, 0)\}\} = \{0\}. \end{aligned}$$

Thus, the system is locally observable in $\mathbb{R} \setminus \{0\}$. Actually, Example II.5 showed that the system is locally observable in $\mathbb{R} \setminus \{0\}$. In contrast to Example II.5, in this example, without directly showing the existence of a suitable neighborhood of each initial state, local observability in $\mathbb{R} \setminus \{0\}$ is verified. Since at this stage Theorem IV.1 (2a) does not guarantee local observability in $\hat{V} = \{0\} \subset \mathbb{R}$ yet, it is necessary to test for local observability in $\{0\}$. To this end, we again exploit Theorem IV.1. Let $V \subset \mathbb{R}$ of Theorem IV.1 be $\{0\}$. Then, $\{0\}^I := \mathbf{V}(I) \cap (\mathbb{R} \times \{0\}) = \{(0, 0)\}$, and we have

$$\begin{aligned} \{0\}^J &:= \mathbf{V}(J) \cap (\mathbb{R} \times \{0\}) \\ &= \{(\xi, \eta) \in \mathbb{R} \times \mathbb{R} : \xi^2 = \eta^2\} \cap (\mathbb{R} \times \{0\}) \\ &= \{(0, 0)\} = \{0\}^I. \end{aligned}$$

Therefore, condition (12) holds, where $X \subset \mathbb{R} \times \mathbb{R}$ in (12) is the empty set. Finally, $\hat{V}$ in Theorem IV.1 (2a) is obtained as $\hat{V} := \{x_0 \in \mathbb{R} : (x_0, x_0) \in \mathbf{V}(I) \cap \emptyset\}$, which is the empty set. Therefore, the system is locally observable in $\{0\} \setminus \emptyset = \{0\}$. In summary, the system is locally observable for all initial states. Local observability is thus checked without computing a suitable neighborhood of each initial state in finite time. Note that the system does not satisfy the local observability rank condition at the origin.

In Example IV.2, we repeat four procedures. First, we compute the minimal decomposition of $V = V_1 \cup \dots \cup V_r$ and, second, we compute pairs of affine varieties $V_i^I, V_i^J \subset \mathbb{R}^n \times \mathbb{R}^n$ ($i = 1, \dots, r$). Third, we compute the minimal decomposition of each V_i^J ($i = 1, \dots, r$) and, fourth, we verify the condition in Theorem IV.1 in each V_i ($i = 1, \dots, r$). The process of testing for local observability in an affine variety is summarized in the following Algorithm IV.3, in which these four procedures are repeatedly executed. Then, we show that Algorithm IV.3 terminates finitely. Moreover, on the basis of Algorithm IV.3, we present a sufficient condition for local observability.

Algorithm IV.3.

Input: An affine variety $V \subset \mathbb{R}^n$.

Output: An affine variety $V_0 \subset \mathbb{R}^n$.

1. $V_0 := \emptyset \subset \mathbb{R}^n$, $\hat{V} := \emptyset \subset \mathbb{R}^n$.
2. Compute the minimal decomposition of V as $V_1 \cup \dots \cup V_r \subset \mathbb{R}^n$.
3. For $i = 1, \dots, r$,
 - (a) $V_i^I := \mathbf{V}(I) \cap (\mathbb{R}^n \times V_i)$, $V_i^J := \mathbf{V}(J) \cap (\mathbb{R}^n \times V_i) \subset \mathbb{R}^n \times \mathbb{R}^n$.
 - (b) Compute the minimal decomposition of V_i^J . If V_i^J can be decomposed as $V_i^J = V_i^I \cup X_{i,1} \cup \dots \cup X_{i,s_i}$, go to (3c). Otherwise, go to (3e).
 - (c) Denote $X_i := X_{i,1} \cup \dots \cup X_{i,s_i}$ and $\hat{V}_i := \{x_0 \in \mathbb{R}^n : (x_0, x_0) \in V_i^I \cap X_i\}$.
 - (d) $\hat{V} := \hat{V} \cup \hat{V}_i$, and go to (3f).
 - (e) $V_0 := V_0 \cup V_i$.
 - (f) $i := i + 1$.
4. If $\hat{V} = \emptyset$, output V_0 . Otherwise, $V := \hat{V}$.
5. $\hat{V} := \emptyset$, and go back to (2).

Theorem IV.4. Algorithm IV.3 terminates finitely. Moreover, the number of repetitions of steps (2) – (5) is at most n .

Proof. First, we show that the Krull dimension of $\hat{V} \subset \mathbb{R}^n$ at step (4) is less than that of $V \subset \mathbb{R}^n$ at step (2). Suppose that the condition at step (3b) holds for $i = 1, \dots, k$ ($k \leq r$) without loss of generality. Theorem IV.1 (2b) implies $\dim V_i > \dim \hat{V}_i$ ($i = 1, \dots, k$). From Proposition A.11, $\dim V = \max\{\dim V_1, \dots, \dim V_r\}$ holds, and from step (33d), $\dim \hat{V} = \max\{\dim \hat{V}_1, \dots, \dim \hat{V}_k\}$ holds. Therefore, we have $\dim V > \dim \hat{V}$. In other words, after executing steps (2) – (5) once, the Krull dimension of $\hat{V}$ at step (4) decreases.

Consider the given input of the algorithm $V \subset \mathbb{R}^n$ for which $\dim V \leq \dim \mathbb{R}^n = n$ holds. By executing steps (2) – (5) n times at most, eventually the Krull dimension of $\hat{V}$ at step (4) becomes -1 . That is, $\hat{V}$ at step (4) becomes the empty set. Therefore, Algorithm IV.3 terminates finitely.

Theorem IV.5. A polynomial system Σ is locally observable in an affine variety $V \subset \mathbb{R}^n$ if the output of Algorithm IV.3, $V_0 \subset \mathbb{R}^n$, is the empty set.

Proof. We consider step (3). If the output V_0 is the empty set, we do not execute step (3e) for all $i = 1, 2, \dots, r$. That is, the condition at step (3b) holds. From Theorem IV.1 (2a), the system is locally observable in $V_i \setminus \hat{V}_i$ ($i = 1, \dots, r$), where $\hat{V}_i$ is defined at step (3c). From the definitions of V at step (2) and $\hat{V}$ at step (4), the system is locally observable in $V \setminus \hat{V}$.

Let $\tilde{V}_1 \subset \mathbb{R}^n$ be the input V of the algorithm. We denote $\hat{V} \subset \mathbb{R}^n$ at step (4) as $\tilde{V}_2$. Then, from Theorem IV.1 (2a), the system is locally observable in $\tilde{V}_1 \setminus \tilde{V}_2$. If $\tilde{V}_2$ is the empty set, the system is locally observable in $\tilde{V}_1$. Suppose that $\tilde{V}_2$ is not the empty set. Then, from step (4), we execute steps (2) – (5) for $V := \tilde{V}_2$. We denote $\hat{V} \subset \mathbb{R}^n$ at step (4) as $\tilde{V}_3$. Then, the system is locally observable in $\tilde{V}_2 \setminus \tilde{V}_3$ and consequently in $(\tilde{V}_1 \setminus \tilde{V}_2) \cup (\tilde{V}_2 \setminus \tilde{V}_3) = \tilde{V}_1 \setminus \tilde{V}_3$. In a similar manner, by denoting $\hat{V} \subset \mathbb{R}^n$ at step (4) in each iteration as $\tilde{V}_i$ ($i = 1, 2, \dots$), the system is locally observable in $\tilde{V}_1 \setminus \tilde{V}_i$ ($i = 1, 2, \dots$). From the proof of Theorem IV.4, there exists a positive integer $N \leq n$ such that $\tilde{V}_N = \emptyset$. Therefore, the system is locally observable in $V := \tilde{V}_1$.

Remark IV.6. Over the field of complex numbers $\mathbb{C}$, condition $V_0 = \emptyset$ becomes a necessary and sufficient condition for local observability. Over $\mathbb{C}$, we have stronger results. For instance, an irreducible variety is connected in Euclidean topology [13] in contrast over $\mathbb{R}$. Due to this fact, it is possible to prove that (12) is a necessary condition for local observability. For more details, see Appendix 2.1. In Algorithm IV.3, if V_0 is not the empty set, we execute step (3e) for some i . That is, there exists $V_i \subset \mathbb{C}^n$ not satisfying the condition at step (3b), i.e.

necessary condition (12). Thus, the system is not locally observable in such V_i .

Theorem IV.5 gives a sufficient condition for local observability in an affine variety $V \subset \mathbb{R}^n$. To check this condition, we execute Algorithm IV.3, and the output $V_0 \subset \mathbb{R}^n$ of Algorithm IV.3 is the set of initial states where the sufficiency is not guaranteed. From the proof of Theorem IV.4, the number of repetitions of steps (2) – (5) in Algorithm IV.3 is no more than the dimension of the polynomial system Σ . In steps (2) – (5), we iteratively check the condition in Theorem IV.1. Therefore, the sufficient condition in Theorem IV.5 can be verified by repeatedly checking the condition in Theorem IV.1, and the number of repetitions is finite.

V. CONCLUSION

In this paper, we considered global and local observability in an affine variety of polynomial systems. Global and local observability conditions were obtained in terms of commutative algebra and algebraic geometry. According to our local observability conditions, local observability can be evaluated by repeatedly checking a condition for local observability. The local observability condition is derived by exploiting the minimal primary decomposition of an affine variety. Moreover, owing to dimension theory of commutative algebra, the number of repetitions required to verify the condition is finite. In this paper, we considered a single-output autonomous system. However, every result of this paper can be readily extended to a multiple-input and multiple-output (MIMO) system by exploiting multiple-experiment observability [2]. Also for MIMO discrete-time polynomial systems, observability conditions can be derived in similar forms. To extend our results to such systems, we need to obtain the sets of indistinguishable initial states as affine varieties for such systems. In other words, our results can be extended to any system for which the sets of indistinguishable initial state is described by an affine variety. For the computation of the sets of indistinguishable initial states for MIMO continuous-and discrete-time systems, see [4,10], respectively.

VI. APPENDIX

6.1 Definitions and Properties of commutative algebra and algebraic geometry

Here, we give definitions and propositions of commutative algebra and algebraic geometry [6–9,14] for self containedness.

An ideal I of the polynomial ring $\mathbb{R}[\xi]$ has the following properties for any $a, b \in I$ and $c \in \mathbb{R}[\xi]$:

- (1) $a + b \in I$,
- (2) $ca \in I$.

Let $p_1, p_2, \dots, p_s$ be in $\mathbb{R}[\xi]$. Then the set $\langle p_1, \dots, p_s \rangle$, defined as

$$\langle p_1, \dots, p_s \rangle = \left\{ \sum_{i=1}^s a_i p_i : a_i \in \mathbb{R}[\xi], i = 1, 2, \dots, s \right\},$$

is called the ideal generated by $p_1, \dots, p_s$.

If I and J are ideals in $\mathbb{R}[\xi]$, then the sum of I and J , denoted by $I + J$, is the set

$$I + J = \{a + b : a \in I \text{ and } b \in J\},$$

which is also an ideal in $\mathbb{R}[\xi]$. The intersection $I \cap J$ of two ideals I and J in $\mathbb{R}[\xi]$ is the set of polynomials which belong to both I and J and is an ideal in $\mathbb{R}[\xi]$. Let $I \subset \mathbb{R}[\xi]$ be an ideal. The radical of I , denoted by $\sqrt{I}$, is the set

$$\{f : f^m \in I \text{ for some integer } m \geq 1\}.$$

The radical $\sqrt{I}$ is an ideal in $\mathbb{R}[\xi]$ containing I . An ideal I is radical if $I = \sqrt{I}$.

An ideal $p \subset \mathbb{R}[\xi]$ is said to be prime if $ab \in p$ implies $a \in p$ or $b \in p$. An ideal $q \subset \mathbb{R}[\xi]$ is said to be primary if $ab \in q$ implies $a \in q$ or $b^s \in q$ for a positive integer s . The radical of a primary ideal is the smallest prime ideal containing the primary ideal. Every prime ideal is a radical ideal. A primary decomposition of an ideal $I \subset \mathbb{R}[\xi]$ is an expression of I as an intersection of primary ideals: $I = q_1 \cap \dots \cap q_s$, where s is a positive integer. It is called minimal or irredundant if $\sqrt{q_i} (i = 1, \dots, s)$ are all distinct and $q_i \not\supset \bigcap_{j \neq i} q_j$. It can be shown that every ideal $I \subset \mathbb{R}[\xi]$ has the minimal primary decomposition.

The affine variety of polynomials $p_1, \dots, p_s \in \mathbb{R}[\xi]$ is defined as

$$\mathbf{V}(p_1, \dots, p_s) = \{\xi \in \mathbb{R}^n : p_i(\xi) = 0, i = 1, \dots, s\}.$$

The affine variety of an ideal $I \subset \mathbb{R}[\xi]$ is defined as

$$\mathbf{V}(I) = \{\xi \in \mathbb{R}^n : p(\xi) = 0, \forall p \in I\},$$

and thus $\mathbf{V}(I)$ represents the set of common zeros of all polynomials in I . If $p_1, \dots, p_s \in \mathbb{R}[\xi]$ are generators of the ideal $I \subset \mathbb{R}[\xi]$, we have

$$\mathbf{V}(I) = \mathbf{V}(p_1, \dots, p_s).$$

An affine variety $V \subset \mathbb{R}^n$ is said to be irreducible if $V = V_1 \cup V_2$ implies $V = V_1$ or $V = V_2$, where $V_1, V_2 \subset \mathbb{R}^n$ are affine varieties. Let $V \subset \mathbb{R}^n$ be an affine variety. A decomposition $V = V_1 \cup \dots \cup V_s$, where each V_i is an irreducible variety, and s is positive integer, is called minimal (or, sometimes, an irredundant union) if $V_i \not\subset V_j$ for $i \neq j$. It can be shown that every affine variety has the minimal decomposition.

Let $V \subset \mathbb{R}^n$ be an affine variety. We define the following set:

$$\mathbf{I}(V) = \{p \in \mathbb{R}[\xi] : p(\xi) = 0, \forall \xi \in V\},$$

which is a radical ideal. For every ideal $I \subset \mathbb{R}[\xi]$, we have $I \subset \mathbf{I}(\mathbf{V}(I))$.

Next, we state some technical lemmas used in this paper.

Proposition A.1. Let $I, J \subset \mathbb{R}[\xi]$ be ideals, and let $V, W \subset \mathbb{R}^n$ be affine varieties. Then, the following hold.

1. $\mathbf{V}(I) \subset \mathbf{V}(J)$ if $I \supset J$.
2. $V \supset W$ if and only if $\mathbf{I}(V) \subset \mathbf{I}(W)$.

Proposition A.2. Let $I, J \subset \mathbb{R}[\xi]$ be ideals, and let $V, W \subset \mathbb{R}^n$ be affine varieties. Then, the following hold.

1. $\mathbf{V}(I + J) = \mathbf{V}(I) \cap \mathbf{V}(J)$.
2. $\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J)$.
3. $\mathbf{I}(V \cup W) = \mathbf{I}(V) \cap \mathbf{I}(W)$.
4. $\mathbf{I}(V \cap W) = \sqrt{\mathbf{I}(V) + \mathbf{I}(W)}$.

Proposition A.3. An affine variety $V \subset \mathbb{R}^n$ is irreducible if and only if $\mathbf{I}(V) \subset \mathbb{R}[\xi]$ is a prime ideal.

Proposition A.4. Let $\varphi : \mathbb{R}[\xi, \eta] \rightarrow \mathbb{R}[\xi]$ be a ring homomorphism. If an ideal $p \subset \mathbb{R}[\xi]$ is prime, $\varphi^{-1}(p) \subset \mathbb{R}[\xi, \eta]$ is a prime ideal.

Proposition A.5. Let $I, J \subset \mathbb{R}[\xi]$ be ideals and $I \subset J$. Then, $(\mathbb{R}[\xi]/I)/(J/I) \simeq \mathbb{R}[\xi]/J$ holds.

Proposition A.6. Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ be irreducible varieties. Then $V \times W \subset \mathbb{R}^n \times \mathbb{R}^m$ is an irreducible variety.

Proposition A.7. Let $V \subset \mathbb{R}^n$ and $W_i \subset \mathbb{R}^n$ ($i = 1, 2, \dots, s$) be irreducible varieties. If $V \not\subset W_i$ ($i = 1, 2, \dots, s$), then $V \not\subset W_1 \cup \dots \cup W_s$.

Proof. From Proposition A.1 2), $\mathbf{I}(V) \not\subset \mathbf{I}(W_i)$ ($i = 1, \dots, s$) holds. Let $a_i \in \mathbf{I}(W_i) \setminus \mathbf{I}(V)$ ($i = 1, \dots, s$). Then $a_1 \cdots a_s \in \mathbf{I}(W_1) \cap \dots \cap \mathbf{I}(W_s)$ holds. From Proposition A.3, $\mathbf{I}(V)$ is prime, and thus $a_1 \cdots a_s \notin \mathbf{I}(V)$ holds.

Therefore, we have $\mathbf{I}(V) \not\supseteq \mathbf{I}(W_1) \cap \cdots \cap \mathbf{I}(W_s)$, and from Proposition A.2 3) $\mathbf{I}(V) \not\supseteq \mathbf{I}(W_1 \cup \cdots \cup W_s)$. From Proposition A.1 2), $V \not\subseteq W_1 \cup \cdots \cup W_s$ holds.

Next, we give properties of the Krull dimension.

Definition A.8. The Krull dimension of an irreducible variety X is defined by the least upper bound of the length of the chain of any irreducible variety $X_i \neq \emptyset$; $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$, and the least upper bound is denoted by $\dim X$. The Krull dimension of the empty set is defined as -1 .

Definition A.9. The Krull dimension of a ring $\mathbb{R}[\xi]$ is denoted by the least upper bound of the length of the chain of prime ideals $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$, and the least upper bound is denoted by $\dim \mathbb{R}[\xi]$. Moreover, the Krull dimension of an ideal $I \subset \mathbb{R}[\xi]$ is denoted by $\dim I := \dim \mathbb{R}[\xi]/I$.

Proposition A.10. The Krull dimension of an affine variety $V \subset \mathbb{R}^n$ is equivalent to the Krull dimension of an ideal $\mathbf{I}(V)$.

Proposition A.11. Let $V \subset \mathbb{R}^n$ be an affine variety, and let $V = V_1 \cup V_2 \cup \cdots \cup V_s$, where $V_i \subset \mathbb{R}^n$ ($i = 1, \dots, s$) is an affine variety, and s is a positive integer. Then, $\dim V = \max\{\dim V_1, \dots, \dim V_s\}$ holds.

Proposition A.12. Let $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ be affine varieties. Then, $\dim(V \times W) = \dim V + \dim W$.

Proposition A.13. Let $V, W \subset \mathbb{R}^n$ be affine varieties. Then, $V \supset W$ implies $\dim V \geq \dim W$. If V is irreducible, then $V \supsetneq W$ implies $\dim V > \dim W$.

Proposition A.14. Let $U \subset \mathbb{R}^n$ be an irreducible variety, and let $V \subset U$ and $W \subset U$ be irreducible varieties. Then, for each component of the minimal decomposition of $V \cap W \neq \emptyset$, represented by Z , $\dim Z \geq \dim V + \dim W - \dim U$ holds.

6.2 Proofs of Theorem IV.1

Here, Theorem IV.1 is proved. To this end, some lemmas are given.

Lemma B.1. A polynomial system Σ is locally observable at $x_0 \in \mathbb{R}^n$ if and only if there exists a neighborhood $U(x_0) \subset \mathbb{R}^n$ such that

$$\mathbf{V}(\mathcal{J}) \cap (U(x_0) \times \{x_0\}) = \{(x_0, x_0)\}, \quad (13)$$

where $U(x_0)$ is an open subset in the Euclidean topology.

Proof. $\mathbf{V}(\mathcal{J}) \cap (U(x_0) \times \{x_0\})$ is the following set.

$$\begin{aligned} & \{(\xi, x_0) \in \mathbb{R}^n \times \mathbb{R}^n : \\ & \xi \text{ is indistinguishable from } x_0 \text{ in } U(x_0)\} \end{aligned}$$

Therefore, if there exists $U(x_0)$ satisfying (13), the system Σ satisfies Definition II.3, and vice versa. $\square$

Lemma B.2. Let $X \subset \mathbb{R}^n \times \{x_0\} \subset \mathbb{R}^n \times \mathbb{R}^n$ be an affine variety. Then, there exists an affine variety $Y \subset \mathbb{R}^n$ such that $X = Y \times \{x_0\}$ holds.

Proof. From the definition of X , there exists a set $Y \subset \mathbb{R}^n$ such that $X = Y \times \{x_0\}$ holds. Since X is a closed subset in the Zariski topology and vice versa, the closure of X , denoted by $\overline{X}$, is X . Thus, we have

$$Y \times \{x_0\} = \overline{Y \times \{x_0\}} = \overline{Y} \times \overline{\{x_0\}} = \overline{Y} \times \{x_0\},$$

which yields $Y = \overline{Y}$, i.e., Y is an affine variety.

We next prove Theorem IV.1.

Proof. [Theorem IV.1 (1)] Owing to Proposition A.3, it suffices to show that $\mathbf{I}(V^I)$ is prime. From Proposition A.2 (4) and (8), we obtain

$$\begin{aligned} \mathbf{I}(V^I) &= \mathbf{I}(\mathbf{V}(I) \cap (\mathbb{R}^n \times V)) \\ &= \sqrt{\mathbf{I}(\mathbf{V}(I)) + \mathbf{I}(\mathbb{R}^n \times V)} \\ &= \sqrt{I + \mathbf{I}(V)\mathbb{R}[\xi, \eta]}, \end{aligned} \quad (14)$$

where $\mathbf{I}(V)$ is in $\mathbb{R}[\eta]$. We show that $\sqrt{I + \mathbf{I}(V)\mathbb{R}[\xi, \eta]}$ is prime. First, since V is irreducible, from Proposition A.3, $\mathbf{I}(V) \subset \mathbb{R}[\eta]$ is prime. Let $\varphi : \mathbb{R}[\xi, \eta] \rightarrow \mathbb{R}[\eta]$ be the mapping substituting $\xi = \eta$.

$\mathbb{R}[\xi, \eta] \ni \xi \mapsto \eta \in \mathbb{R}[\eta]$, which is a surjective ring homomorphism whose kernel is I . Thus, from Proposition A.4, $\varphi^{-1}(\mathbf{I}(V)) = I + \mathbf{I}(V)\mathbb{R}[\xi, \eta]$ is prime. Note that a prime ideal is radical, i.e., we have

$$\sqrt{I + \mathbf{I}(V)\mathbb{R}[\xi, \eta]} = I + \mathbf{I}(V)\mathbb{R}[\xi, \eta]. \quad (15)$$

This completes the proof.

Proof. [Theorem IV.1 (2a)] Let $x_0 \in V \setminus \hat{V}$. Then, we have $\{(x_0, x_0)\} \not\subset X$, and consequently,

$$\{(x_0, x_0)\} \not\subset X \cap (\mathbb{R}^n \times \{x_0\}).$$

Lemma B.2 implies the existence of an affine variety $Y \subset \mathbb{R}^n$ satisfying

$$Y \times \{x_0\} = X \cap (\mathbb{R}^n \times \{x_0\}), \quad (16)$$

and thus, we obtain

$$\{(x_0, x_0)\} \not\subset Y \times \{x_0\}.$$

That is, $\{x_0\}$ is not contained in Y . Affine variety Y is a closed subset in the Zariski topology, which is also a closed subset in the Euclidean topology [15]. Thus, $\mathbb{R}^n \setminus Y$ containing $\{x_0\}$ is an open subset of $\mathbb{R}^n$ in the Euclidean topology. Therefore, there exists a neighborhood $U(x_0) \subset \mathbb{R}^n$ such that $Y \cap U(x_0)$ is the empty set. In summary, (12) and (16) lead to

$$\begin{aligned} & \mathbf{V}(\mathcal{J}) \cap (U(x_0) \times \{x_0\}) \\ &= \mathbf{V}(\mathcal{J}) \cap ((\mathbb{R}^n \times V) \cap (U(x_0) \times \{x_0\})) \\ &= (V^I \cup X) \cap (U(x_0) \times \{x_0\}) \\ &= (V^I \cap (U(x_0) \times \{x_0\})) \cup (X \cap (U(x_0) \times \{x_0\})) \\ &= (V^I \cap (U(x_0) \times \{x_0\})) \\ &\quad \cup (X \cap ((U(x_0) \times \{x_0\}) \cap (\mathbb{R}^n \times \{x_0\}))) \\ &= (V^I \cap (U(x_0) \times \{x_0\})) \\ &\quad \cup ((Y \times \{x_0\}) \cap (U(x_0) \times \{x_0\})) \\ &= (V^I \cap (U(x_0) \times \{x_0\})) \cup ((Y \cap U(x_0)) \times \{x_0\}) \\ &= V^I \cap (U(x_0) \times \{x_0\}). \end{aligned}$$

Because of $V^I \subset \mathbf{V}(\mathcal{I})$, from (7) there exists $\tilde{V} \subset \mathbb{R}^n$ such that $V^I = \bigcup_{\tilde{V}} \{(x_0, x_0)\}$. This $\tilde{V}$ is simply V . Thus, we have $V^I \cap (U(x_0) \times \{x_0\}) = \{(x_0, x_0)\}$ for any $x_0 \in V \setminus \tilde{V}$. In summary,

$$\mathbf{V}(\mathcal{J}) \cap (U(x_0) \times \{x_0\}) = \{(x_0, x_0)\}$$

holds. From Lemma B.1, the system Σ is locally observable in $V \setminus \hat{V}$.

Proof. [Theorem IV.1 (2b)] First, we show that $\hat{V}$ is an affine variety. Let $d : \mathbb{R}^n \ni x_0 \mapsto (x_0, x_0) \in \mathbb{R}^n \times \mathbb{R}^n$, which is a polynomial mapping which is continuous in the Zariski topology [16]. An affine variety is a closed subset and vice versa in the Zariski topology. Therefore, the inverse image $d^{-1}(V^I \cap X) \subset \mathbb{R}^n$ is an affine variety, which is simply $\hat{V}$. That is, $\hat{V}$ is an affine variety.

Next, we show that $V^I \not\subset X$. When $X = \emptyset$, this is obvious. Suppose that $X \neq \emptyset$, i.e., there exists $V^I \not\subset X_j$ ($j = t+1, \dots, s$). If $X_0 \cup \dots \cup X_t = V^I$ holds, then from Theorem IV.1 (1), Proposition A.7 and a property of the minimal decomposition: $X_i \not\subset X_j$ ($i \neq j$), we have $V^I \not\subset X$, which implies $V^I \not\supseteq V^I \cap X$. Note that $V^I = \bigcup_{x_0 \in V} \{(x_0, x_0)\}$ and $V^I \cap X = \bigcup_{x_0 \in \hat{V}} \{(x_0, x_0)\}$. Thus, we have $\bigcup_{x_0 \in V} \{(x_0, x_0)\} \not\supseteq \bigcup_{x_0 \in \hat{V}} \{(x_0, x_0)\}$, and consequently $V \not\supseteq \hat{V}$. Since V is irreducible, from Proposition A.13, $\dim V > \dim \hat{V}$.

6.3 Theorem IV.1 over the field of complex numbers

Hereafter, we consider the field of complex numbers $\mathbb{C}$. Definitions, theorems, lemmas, and propositions used in this paper are applicable over $\mathbb{C}$.

Lemma B.3. Let $X \subset \mathbb{C}^n \times \{x_0\} \subset \mathbb{C}^n \times \mathbb{C}^n$ be an irreducible variety. If $X \not\supseteq \{(x_0, x_0)\}$ holds, then $X \cap (U(x_0) \times \{x_0\}) \not\supseteq \{(x_0, x_0)\}$ holds for any neighborhood $U(x_0) \subset \mathbb{C}^n$ in the Euclidean topology.

Proof. First, $X \supset \{(x_0, x_0)\}$ and $U(x_0) \times \{x_0\} \supset \{(x_0, x_0)\}$ imply $X \cap (U(x_0) \times \{x_0\}) \supset \{(x_0, x_0)\}$. Next, we prove $X \cap (U(x_0) \times \{x_0\}) \not\subset \{(x_0, x_0)\}$ by contradiction. From Lemma B.2, there exists an affine variety $Y \subset \mathbb{C}^n$ such that $X = Y \times \{x_0\}$ holds. The existence of $U(x_0) \subset \mathbb{C}^n$ satisfying $X \cap (U(x_0) \times \{x_0\}) = \{(x_0, x_0)\}$ implies that $Y \cap U(x_0) = \{x_0\}$. Thus, one of the components of the minimal decomposition of Y is $\{x_0\}$ because an irreducible variety is connected in Euclidean topology [13]. Consequently, one of the components of the minimal decomposition of $X = Y \times \{x_0\}$ is $\{x_0\} \times \{x_0\} = \{(x_0, x_0)\}$. Since X is irreducible, the component of the minimal decomposition of X is simply X . Therefore, we have $X = \{(x_0, x_0)\}$. This contradicts $X \not\supseteq \{(x_0, x_0)\}$.

Before proving Theorem B.4, we show that

$$\dim V^I = \dim V. \quad (17)$$

Since $V^I \subset \mathbf{V}(\mathcal{I}) \subset \mathbb{R}^n \times \mathbb{R}^n$, we have, from Proposition A.1 (2) and (8), $\mathbf{I}(V^I) \supset \mathbf{I}(\mathbf{V}(\mathcal{I})) = \mathcal{I}$. Also (14) and (15) yield $\mathbf{I}(V^I) = \mathcal{I} + \mathbf{I}(V)\mathbb{R}[\xi, \eta]$, where $\mathbf{I}(V)$ is in $\mathbb{R}[\eta]$. Let $\varphi : \mathbb{R}[\xi, \eta] \rightarrow \mathbb{R}[\eta]$ be the mapping substituting $\xi = \eta$, whose kernel is $\mathcal{I}$. From Proposition A.5, we obtain

$$\begin{aligned} & \mathbb{R}[\xi, \eta]/\mathbf{I}(V^I) \\ & \simeq (\mathbb{R}[\xi, \eta]/\mathcal{I})/(\mathbf{I}(V^I)/\mathcal{I}) \\ & = (\mathbb{R}[\xi, \eta]/\mathcal{I})/((\mathcal{I} + \mathbf{I}(V)\mathbb{R}[\xi, \eta])/\mathcal{I}) \\ & = (\mathbb{R}[\xi, \eta]/\mathcal{I})/(\mathbf{I}(V)\mathbb{R}[\xi, \eta]/\mathcal{I}) \simeq \mathbb{R}[\eta]/\mathbf{I}(V), \end{aligned}$$

which implies $\dim \mathbf{I}(V^I) = \dim \mathbf{I}(V)$. From Proposition A.10, we have (17).

Theorem B.4. We consider the ideals $\mathcal{J} \subset \mathbb{C}[\xi, \eta]$ and $\mathcal{I} \subset \mathbb{C}[\xi, \eta]$ defined in (4) and (5), respectively. Let $V \subset \mathbb{C}^n$ be an irreducible variety, and let $V^I := \mathbf{V}(\mathcal{I}) \cap (\mathbb{C}^n \times V) \subset \mathbb{C}^n \times \mathbb{C}^n$ and $V^J := \mathbf{V}(\mathcal{J}) \cap (\mathbb{C}^n \times V) \subset \mathbb{C}^n \times \mathbb{C}^n$. If a polynomial system Σ is locally observable at some $x_0 \in V$, the minimal decomposition of V^J is obtained in the form

$$V^J = V^I \cup X,$$

where $X = X_1 \cup \dots \cup X_s$.

Proof. Let $V^J = X_0 \cup X_1 \cup \dots \cup X_s$ be the minimal decomposition. First, we show that at least one of $X_j \subset \mathbb{C}^n \times \mathbb{C}^n$ ($j = 0, 1, \dots, s$) contains $V^I \subset \mathbb{C}^n \times \mathbb{C}^n$. From (11), $X_0 \cup \dots \cup X_s = V^J \supset V^I$ holds, where V^I and X_j ($j = 0, 1, \dots, s$) are irreducible. From Proposition A.7, there exists X_j containing V^I .

Let $X_0 \supset V^I$ without loss of generality. We prove that $X_0 = V^I$ if the system Σ is locally observable at some $x_0 \in V$ by contraposition. If $X_0 = V^I$ does not hold, then $X_0 \not\supseteq V^I$. From Proposition A.13 and (17), we obtain

$$\dim X_0 > \dim V^I = \dim V. \quad (18)$$

On the other hand, $X_0 \not\supseteq V^I$ and (7) yield

$$X_0 \cap (\mathbb{C} \times \{x_0\}) \supset V^I \cap (\mathbb{C} \times \{x_0\}) = \{(x_0, x_0)\}$$

for any $x_0 \in V$.

We show that $X_0 \cap (\mathbb{C} \times \{x_0\}) \supset \{(x_0, x_0)\}$ is a strong inclusion property by exploiting Proposition A.14. Let $U \subset \mathbb{C}^n \times \mathbb{C}^n$ be one of the components of the minimal decomposition of $X_0 \cap (\mathbb{C} \times \{x_0\})$ containing $\{(x_0, x_0)\}$, and from Proposition A.7, such a U exists because $\{(x_0, x_0)\}$ is irreducible. Note that X_0 is a subset of $\mathbb{C}^n \times V$ and an irreducible variety. From Proposition A.6, $\mathbb{C}^n \times \{x_0\}$ and $\mathbb{C}^n \times V$ are also irreducible. From Propositions A.12 and A.14, U satisfies

$$\begin{aligned} \dim U &\geq \dim X_0 + \dim(\mathbb{C}^n \times \{x_0\}) - \dim(\mathbb{C}^n \times V) \\ &= \dim X_0 + (\dim \mathbb{C}^n + \dim \{x_0\}) \\ &\quad - (\dim \mathbb{C}^n + \dim V) \\ &= \dim X_0 + \dim \{x_0\} - \dim V \\ &= \dim X_0 - \dim V, \end{aligned}$$

where we use $\dim \{x_0\} = 0$. From (18), $\dim U > 0$. Thus, from Proposition A.13, we also have $U \not\supseteq \{(x_0, x_0)\}$. Since U is in $\mathbb{C} \times \{x_0\}$, Lemma B.3 implies

$$U \cap (U(x_0) \times \{x_0\}) \not\supseteq \{(x_0, x_0)\}$$

for any neighborhood $U(x_0) \subset \mathbb{C}^n$ in the Euclidean topology. From the definitions of X_0 and U , we have

$$\begin{aligned} V(J) \cap (U(x_0) \times \{x_0\}) \\ &\supset X_0 \cap (U(x_0) \times \{x_0\}) \\ &\supset U \cap (U(x_0) \times \{x_0\}) \not\supseteq \{(x_0, x_0)\} \end{aligned}$$

for any $x_0 \in V$ and any neighborhood $U(x_0) \subset \mathbb{C}^n$. From Lemma B.1, the system Σ is not locally observable at any $x_0 \in V$.

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