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A Dynamic Informational-Epistemic Logic

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Abstract. Epistemic logic is usually employed to model two aspects of a situation: the ontic and the epistemic aspects. Truth, however, is not always attainable, and in many cases we are forced to reason only with whatever information is available to us. In this paper, we will explore a four-valued epistemic logic designed to deal with situations of this sort. The technical results include a set of reduction axioms for public announcements, correspondence proofs, and a complete tableau system.

Keywords: Many-valued logics · Epistemic logic
Paraconsistent logics · Public announcements
Multi-agent systems

1 Introduction

Is drinking a glass of red wine per day good for your heart? The answer may be yes or no depending on where you look. We do not dare to try to answer this question here, but we want to offer a logical formalism that can help us understand situations like this, where the available information about a certain topic can be conflicting or incomplete. In most practical settings, obtaining the ultimate truth about anything is out of the question, and one has to deal with whatever information is available to them, which might be incomplete and sometimes even contradictory.

We will carry out our analysis on a four-valued epistemic logic, a variant of the so-called **BK** logic [13], a Belnapian version of modal logic **K**. As an important component of modern dynamic epistemic logics, public announcements will also be examined. In this way, we also intend to contribute to the motivation for the use of many-valued modal logics. As remarked by Fitting in the conclusion of [6], very little has been said about intuitions underlying many-valued modal logics, a situation which seems to persist in the current literature.

In four-valued logics, a proposition p can be, besides true or false, *both* (true and false) or *neither* (true nor false), denoted in this paper by the valuations $V(p) = \{0, 1\}$ and $V(p) = \emptyset$, respectively. One can, as was done by Belnap in his influential paper [2], interpret these truth-values as the status of information possibly coming from several sources. For example, if *both* is the value assigned to p, then this means that some source points to the truth and another to the falsity of p. The value *none* could mean that no information is available about p.

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In this way, the valuation already represents the epistemic level, instead of the ontic level. This was not a problem since Belnap was not dealing with a modal logic. Now, the addition of a modal operator of belief to this logic will create two separate epistemic "layers".

Look at the classical epistemic model of Fig. 1(left). It represents a situation wherein an agent cannot distinguish between the truth and falsity of proposition p, or, equivalently, wherein the agent does not know whether p.

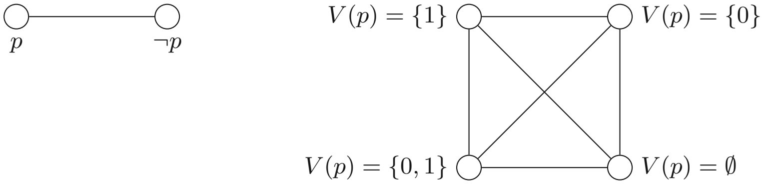

Fig. 1. An epistemic model (left) and a four-valued epistemic model (right).

Now, compare this situation with the four-valued model of Fig. 1(right). What is a plausible interpretation for this model? Here, the agent not only cannot distinguish between worlds where p is *true* or *false*, but also between worlds where it is *neither* true nor false, or *both*. If we adopt an epistemic interpretation of the valuations, what kind of interpretation is left for the operator $\Box$?

As mentioned before, we should think of two layers: the first concerning information, and the second concerning knowledge. The (four-valued) valuation function embodies the informational layer, while the accessibility relations account for the (multi-agent) epistemic layer. For example, we can regard the valuation as representing the information about some propositions stored in a database. The database only registers the information it receives, so it is well possible that at first it receives the information that p is *true*, but subsequently it receives (possibly from another source) the information that p is *false*. In this case the database contains contradictory information about p. The second level (the epistemic/doxastic level), represented by the accessibility relations, may be illustrated, for instance, by the knowledge of a user of this database. The user may be in a state like the one in Fig. 1(right), where she considers it possible that the database is in any of the four possible states regarding p.

Notice that our agents do not possess real knowledge (knowledge about facts), but only a superficial knowledge about information itself – whence we say that one layer concerns *information*, as opposed to reality, and the other concerns *knowledge* about the "informational layer".

This interpretation also makes clear the difference between *none* and *both*, which could otherwise be equally understood as no information or useless information. If the database has *both* as the value of p, deleting information that supports the truth of p would result in p being just *false*, whereas if p was *none* this would have no effect. Similarly, receiving information when p is *none* can lead to a consistent state, but, if p is *both*, receiving new information has no qualitative effect. Such changes in information could be modelled through dynamic operators, but in this paper we only use public announcements (Sect. 5), and with a different purpose.

Another example not involving databases can be given. Let us consider a typical epistemic logic scenario. Anne lives in Groningen, so she knows whether *It is raining in Groningen* ($\Box_a g \vee \Box_a \neg g$). Likewise, Bart lives in Rotterdam and knows whether *It is raining in Rotterdam* ($\Box_b r \vee \Box_b \neg r$). The traditional epistemic model for this situation is depicted in Fig. 2.

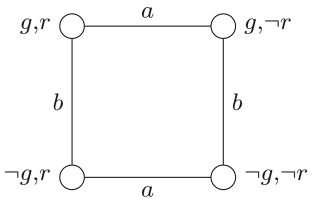

Fig. 2. A classic epistemic model.

Now suppose both of them usually inform themselves of the weather by watching the local television's newscast, and g and r mean that *It will rain in Groningen tonight* and *It will rain in Rotterdam tonight*, respectively. This changes nothing in the model of Fig. 2. However, imagine the situation in which Anne heard that g in the newscast of Channel 1, but $\neg g$ in the newscast of Channel 2. The status of g for Anne is now contradictory, which is denoted by the truth value *both*. Moreover, assuming that Anne is always up to date with the weather news from Channels 1 and 2, she will always be aware of the four-valued status of g. In this example, the sources of information, namely the television channels, play the role of the database. We are not endorsing the position that proposition g can actually be true and false at the same time, but only that there may be different pieces of information available, one supporting the truth and the other the falsity of g.

In this way, the logic preserves the standard meaning of the accessibility relations, namely that of epistemic alternatives (or uncertainty). So, in a state where g was announced to be both true and false, Anne is aware of that. She does not consider a world to be possible where only $\neg g$ was announced, for she already knows this is not the case. Bart, on the other hand, does not have access to Groningen weather in his local newscast, so he considers all of the four values to be possible for g. Now we can have a formula like $\Box_a(g \wedge \neg g)$, meaning that *Anne knows that there is information supporting both the truth and the falsity of* g.

The rest of this paper will explore in detail this logic with two epistemic layers, which we will simply call *four-valued epistemic logic* (FVEL, in short). The remaining content is organized as follows. In Sect. 2 we define the syntax and semantics of the logic, and present some of its basic properties. In Sect. 3 we present a sound and complete tableau system. In Sect. 4 we show some correspondence results concerning classical epistemic logic axioms. In Sect. 5 we add public announcements to FVEL and show that they do not increase expressivity. We also extend the tableau system with rules for public announcements, and prove completeness. In Sect. 6 we give an illustrative example of FVEL in action. To wrap up, we comment on related work in Sect. 7 and conclude with Sect. 8. Some of the proofs can be found in the appendix[1].

2 Four-Valued Epistemic Logic

In this section, we will define the syntax and the semantics of the logical language being examined.

2.1 Syntax

Let P be a countable set of atomic propositions and A a finite set of agents. A well-formed formula φ in our language $\mathcal{L}$ is inductively defined as follows:

$$\varphi ::= p \mid {\sim}\varphi \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi$$

with $p \in P$ and $i \in A$. The following abbreviations will be employed throughout the text: $(\varphi \vee \psi) \overset{\text{def}}{=} \neg(\neg\varphi \wedge \neg\psi)$; $(\varphi \rightarrow \psi) \overset{\text{def}}{=} (\neg\varphi \vee \psi)$; $(\varphi \leftrightarrow \psi) \overset{\text{def}}{=} ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$; $\Diamond_i\varphi \overset{\text{def}}{=} \neg\Box_i\neg\varphi$. Parentheses will be left out when there is no room for ambiguity.

2.2 Semantics

Given the non-empty finite set $A = \{1, 2, ..., n\}$ of agents, an interpretation is a tuple $M = \langle S, R, V \rangle$, where S is a non-empty set of states, $R = \langle R_1, R_2, ..., R_n \rangle$ is an n-tuple of binary relations on S and $V : P \times S \rightarrow 2^{\{0,1\}}$ is a valuation function that assigns to each proposition one of four truth values ($\{0\}$ is *false*, $\{1\}$ is *true*, $\{\}$ is *none* and $\{0, 1\}$ is *both*). Although the results in this paper do not depend on the accessibility relations being equivalence relations, Sect. 4 presents some results that illustrate the effects of restricting R. With $p \in P$, $s \in S$, $i \in A$ and $\varphi, \psi \in \mathcal{L}$, the satisfaction relation $\models$ is inductively defined as follows:

$$\begin{array}{ll}
M, s \models p & \text{iff } 1 \in V(p, s) \\
M, s \models \neg p & \text{iff } 0 \in V(p, s) \\
M, s \models (\varphi \wedge \psi) & \text{iff } M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models \neg(\varphi \wedge \psi) & \text{iff } M, s \models \neg\varphi \text{ or } M, s \models \neg\psi \\
M, s \models \Box_i\varphi & \text{iff } \forall t \in S \text{ s.t. } sR_it, \text{ it holds that } M, t \models \varphi \\
M, s \models \neg\Box_i\varphi & \text{iff } \exists t \in S \text{ such that } sR_it \text{ and } M, t \models \neg\varphi \\
M, s \models {\sim}\varphi & \text{iff } M, s \not\models \varphi \\
M, s \models \neg{\sim}\varphi & \text{iff } M, s \models \varphi \\
M, s \models \neg\neg\varphi & \text{iff } M, s \models \varphi
\end{array}$$

---

[1] Some proofs have been omitted due to space limitations, but are available at https://www.ime.usp.br/~yurids/appendix-dali17.pdf.

Now, we can talk not only about 4-valued atoms but also about 4-valued formulas in general. We define the *extended valuation function* $\overline{V} : \mathcal{L} \times S \rightarrow 2^{\{0,1\}}$ as follows:

$$1 \in \overline{V}(\varphi, s) \text{ iff } M, s \models \varphi$$
$$0 \in \overline{V}(\varphi, s) \text{ iff } M, s \models \neg\varphi$$

Using the above definition, we say that a formula φ has value *both* at s, for example, if and only if $\overline{V}(\varphi, s) = \{0, 1\}$, which is the case when both $M, s \models \varphi$ and $M, s \models \neg\varphi$. Truth and falsity of formulas are evaluated independently, and for that reason we define semantic conditions for each negated formula separately. Even though the semantics of $\neg$ as defined above is non-compositional[2], the connective is still truth-functional, as we will see in the next section.

2.3 Basic Properties

Now we build the truth tables for the truth-functional connectives according to the truth definitions given above (compare truth Tables 1, 2, 3, 4 and 5 below to the ones in [16, p. 146]). *True*, *false*, *none* and *both* are abbreviated to t, f, n and b, respectively.

Table 1. $\neg\varphi$.

φ	n	f	t	b
	n	t	f	b

Table 2. $\sim\varphi$.

φ	n	f	t	b
	t	t	f	f

Table 3. $\varphi \wedge \psi$.

$\varphi \backslash \psi$	n	f	t	b
n	n	f	n	f
f	f	f	f	f
t	n	f	t	b
b	f	f	b	b

Table 4. $\varphi \vee \psi$.

$\varphi \backslash \psi$	n	f	t	b
n	n	n	t	t
f	n	f	t	b
t	t	t	t	t
b	t	b	t	b

Example for Table 1 ($\neg \boldsymbol{b} = \boldsymbol{b}$): $\overline{V}(\varphi, s) = \{0, 1\}$ iff $0 \in \overline{V}(\varphi, s)$ and $1 \in \overline{V}(\varphi, s)$ iff $M, s \models \neg\varphi$ and $M, s \models \varphi$ iff $M, s \models \neg\varphi$ and $M, s \models \neg\neg\varphi$ iff $1 \in \overline{V}(\neg\varphi, s)$ and $0 \in \overline{V}(\neg\varphi, s)$ iff $\overline{V}(\neg\varphi, s) = \{0, 1\}$.

Example for Table 4 ($\boldsymbol{n} \vee \boldsymbol{b} = \boldsymbol{t}$): Recall that disjunction is defined in terms of conjunction and negation. $M, s \models \neg(\neg\varphi \wedge \neg\psi)$ iff $M, s \models \neg\neg\varphi$ or $M, s \models \neg\neg\psi$ iff $M, s \models \varphi$ or $M, s \models \psi$ iff $1 \in \overline{V}(\varphi, s)$ or $1 \in \overline{V}(\psi, s)$, which is true, for $\overline{V}(\psi, s) = \{0, 1\}$. $M, s \models \neg\neg(\neg\varphi \wedge \neg\psi)$ iff $M, s \models \neg\varphi \wedge \neg\psi$ iff $M, s \models \neg\varphi$ and $M, s \models \neg\psi$ iff $0 \in \overline{V}(\varphi, s)$ and $0 \in \overline{V}(\psi, s)$, which is false, for $\overline{V}(\varphi, s) = \emptyset$. Therefore $M, s \models \varphi \vee \psi$ holds, but $M, s \models \neg(\varphi \vee \psi)$ does not, thus $1 \in \overline{V}(\varphi \vee \psi)$ and $0 \notin \overline{V}(\varphi \vee \psi)$, hence $\overline{V}(\varphi \vee \psi) = \{1\}$.

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[2] The semantics would be compositional if we used two support relations $\models^+$ and $\models^-$, as was done in [13]. While these two formalisms have the same expressivity, ours has a larger number of formulas (see more on this comparison in Sect. 7).

Table 5. $\varphi \rightarrow \psi$.

$\varphi \backslash \psi$	n	f	t	b
n	n	n	t	t
f	t	t	t	t
t	n	f	t	b
b	t	b	t	b

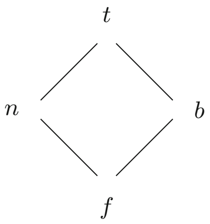

Fig. 3. Lattice L4.

If we leave $\neg$ out, we have classical modal logic, with $\{1\}$ and $\{0, 1\}$ (to which [16] refers as *designated values*) behaving as *true*, and $\emptyset$ and $\{0\}$ (accordingly, *non-designated values*) behaving as *false*.

Moreover, observing these truth tables, we notice that the fragment resulting from leaving $\sim$ and $\Box$ out behaves exactly as *first degree entailment* (**FDE**) [5,16]. Conjunction and disjunction are given by the meet and join, respectively, of the values in the lattice depicted in Fig. 3, called $L4$ in [2]. Now, adding the modal operator to **FDE** we obtain $\mathbf{K}_{\mathbf{FDE}}$, a logic which Priest has studied [16]. He provides a complete tableau system for this logic. Moreover, he shows that this logic contains no validities, as is the case for **FDE** itself.

We can also build the truth tables for the connectives $\vee$ and $\rightarrow$ defined over $\sim$ instead of $\neg$ (which we will denote by $\tilde{\vee}$ and $\tilde{\rightarrow}$, respectively). Despite these connectives being binary functions accepting two four-valued parameters, they behave analogously to their classical (Boolean) counterparts. They can be viewed as a composition of a function that compresses designated values into *true* and non-designated values into *false* (just like the $\sim$ operator itself) with the corresponding Boolean function. In other words, if *or* is classical disjunction and *imp* is classical implication, $x \tilde{\vee} y = or({\sim}{\sim}x, {\sim}{\sim}y)$ and $x \tilde{\rightarrow} y = imp({\sim}{\sim}x, {\sim}{\sim}y)$. It is also relevant to remark that when the operands take on only classical values, both pairs of operators ($\vee$,$\rightarrow$ and $\tilde{\vee}$,$\tilde{\rightarrow}$) behave exactly alike.

Validity. We say that $M \models \varphi$ if and only if $M, s \models \varphi$ for all $s \in S$, where $M = \langle S, R, V\rangle$. A formula φ is valid ($\models \varphi$) if and only if $M \models \varphi$ for all models M. A frame is a pair $\mathcal{F} = \langle S, R\rangle$. We say a formula φ is valid in a frame $\mathcal{F} = \langle S, R\rangle$, that is, $\mathcal{F} \models \varphi$, if and only if, for all valuations V, it holds that $M \models \varphi$, where $M = \langle S, R, V\rangle$ (and we say M is based on frame $\mathcal{F}$). If for all models M and all states s it is the case that $M, s \models \Sigma$ implies $M, s \models \varphi$, we say that $\Sigma \models \varphi$ (φ is a logical consequence of Σ).

We can define $\top$, a validity, as $\top \overset{\text{def}}{=} (p \vee {\sim}p)$. While **FDE** has no validities, FVEL has an infinity of them, including $\top$. Moreover, all propositional tautologies (built with $\sim$) are still validities in FVEL, as expected, but there are other valid formulas with both $\sim$ and $\neg$, such as ${\sim}p \vee \neg{\sim}p$.

Equivalence. Logical equivalence (sameness in truth value) cannot be expressed by $\varphi \leftrightarrow \psi$ in FVEL. Look at Table 6. The diagonal should be *designated*, and the rest *non-designated.* In fact, in this case even the logical equivalence connective ($\tilde{\leftrightarrow}$) derived using ${\sim}$ instead of $\neg$ does not give a truth table which is designated in the diagonal and non-designated everywhere else, for it treats b and t as equals (and the same goes for f and n), resulting in a weaker type of equivalence.

Table 6. $\varphi \leftrightarrow \psi$.

$\varphi \backslash \psi$	n	f	t	b
n	n	n	n	t
f	n	t	f	b
t	n	f	t	b
b	t	b	b	b

Table 7. φ^n, φ^f, φ^t and φ^b.

φ	φ^n	φ^f	φ^t	φ^b
n	t	f	f	f
f	f	t	f	f
t	f	f	t	f
b	f	f	f	t

The reason for adding the classical negation (${\sim}$) to a language which already has a negation operator ($\neg$) is that this increases the expressivity of the language[3]. For instance, we can now define formulas discriminating which of the four truth values a formula φ has: $\varphi^n \overset{\text{def}}{=} ({\sim}\varphi \wedge {\sim}\neg\varphi)$; $\varphi^f \overset{\text{def}}{=} {\sim}{\sim}({\sim}\varphi \wedge \neg\varphi)$; $\varphi^t \overset{\text{def}}{=} {\sim}{\sim}(\varphi \wedge {\sim}\neg\varphi)$; $\varphi^b \overset{\text{def}}{=} {\sim}{\sim}(\varphi \wedge \neg\varphi)$. As can be seen in Table 7, φ^i is true if and only if φ has truth value i, for $i \in \{n, f, t, b\}$, and false otherwise. Using these connectives, it is easy to see that a stronger notion of logical equivalence can be expressed in FVEL:

$$\varphi \Leftrightarrow \psi \overset{\text{def}}{=} (\varphi^n \wedge \psi^n) \vee (\varphi^f \wedge \psi^f) \vee (\varphi^t \wedge \psi^t) \vee (\varphi^b \wedge \psi^b)$$

Since this formula is complex and difficult to evaluate, we will often favor the use of the metalanguage operator $\equiv$, defined by:

$$\varphi \equiv \psi \overset{\text{def}}{=} (M, s \models \varphi \text{ iff } M, s \models \psi) \text{ and } (M, s \models \neg\varphi \text{ iff } M, s \models \neg\psi),$$
$$\text{for all models } M \text{ and all states } s.$$

The formula $\varphi \Leftrightarrow \psi$ is designated if and only if φ and ψ have the same truth value. We can use $\Leftrightarrow$ and $\equiv$ interchangeably for it holds that:

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[3] Interestingly, Girard and Tanaka [8] show that the standard definition of $p \rightarrow q$ as $\neg p \vee q$ does not suffice to prove reduction axioms for public announcements when working with an epistemic extension of Priest's three-valued Logic of Paradox. To circumvent that, they introduced an alternative implication. Our classical negation has a similar role w.r.t. our reduction axioms of Sect. 5.

Proposition 1. $\varphi \equiv \psi$ *iff* $\models \varphi \Leftrightarrow \psi$.

The operator $\equiv$ will be widely used for the demonstrations of Sect. 5.

3 Tableaux

In this section we will describe a tableaux system for FVEL[4]. A tableau is a tree-like structure used for checking derivability and theoremhood. Each branch of the tableau is a set of restrictions that may ultimately determine a model, which is said to be the model *induced* by that branch. In the system used in this paper, each node of the tree is of the form $(\varphi, \pm i)$ or $(ir_m j)$. The first type of node tells if the formula φ is designated $(+)$ or non-designated $(-)$ in state $f(i)$, where f is a function from $\mathbb{N}$ to states (of the induced model). The second type of node says that $f(i)R_m f(j)$ for $i, j \in \mathbb{N}$, where R_m is the accessibility relation of agent m (in the induced model).

The root of the tableau is of the form $(\varphi, -0)$, where φ is the desired conclusion. This root node asserts that the conclusion is non-designated in an arbitrary state $f(0)$. Below the root comes a series of nodes $P_1, P_2, ..., P_n$ such that P_1 is child of the root, P_2 is child of P_1, P_3 is child of P_2, and so on. Each node in this sequence is in the form $(\psi, +0)$, where ψ is a premise. A *branch* is a path from the root to a leaf of the tableau. A branch is called *closed* if it contains a contradiction, that is, a pair of nodes $(\chi, +k)$ and $(\chi, -k)$ for some formula χ and $k \in \mathbb{N}$. If the branch does not contain a contradiction and its leaf node does not fulfill the conditions for the application of any rule (that was not yet applied), then we say the branch is *open*. If the branch contains no contradictions and not all applicable rules were applied, the branch is neither closed nor open, it is *incomplete*. If no branch is incomplete we say the tableau is *complete*.

A successful proof is one where all the branches of the tableau are closed, showing that it is impossible that φ is non-designated in a state where all the premises are designated, and therefore φ is provable from the premises. In that case we say $\Sigma \vdash \varphi$, where Σ is the finite set of premises and φ is the conclusion. If φ is proven from an empty set of premises we write $\vdash \varphi$, and call φ a *theorem*.

To apply a rule to the tableau, a leaf node must be chosen. If the branch to which the leaf node belongs satisfies the conditions of the rule (which are represented in the left-hand side of the rule), certain nodes can be appended as child nodes to that leaf (according to the specification in the right-hand side). Conditions require simply the existence of a set of nodes with a particular format. Some rules allow the creation of one child node, other rules allow the creation of two child nodes, in some cases in series (denoted by a comma in the rules below), in other cases in parallel (denoted by a vertical bar: $|$).

To obtain a tableau calculus for FVEL, we started with the rules from the tableau system given in [16, p. 248], which, for the paper to be self-contained, are reproduced here (rules $R1$–$R14$). We then added four more rules for classical negation (rules $R15$–$R18$). This tableau system will be further augmented in

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[4] Compare [13], which provides a tableaux for **BK** (discussed in Sect. 7).

Sect. 4 to prove some correspondence results between the tableau system and classes of frames and in Sect. 5 to cope with public announcements.

$$(\varphi \wedge \psi, +i) \Longrightarrow (\varphi, +i), (\psi, +i) \tag{R1}$$

$$(\varphi \wedge \psi, -i) \Longrightarrow (\varphi, -i) \mid (\psi, -i) \tag{R2}$$

$$(\varphi \vee \psi, +i) \Longrightarrow (\varphi, +i) \mid (\psi, +i) \tag{R3}$$

$$(\varphi \vee \psi, -i) \Longrightarrow (\varphi, -i), (\psi, -i) \tag{R4}$$

$$(\neg(\varphi \vee \psi), \pm i) \Longrightarrow (\neg\varphi \wedge \neg\psi, \pm i) \tag{R5}$$

$$(\neg(\varphi \wedge \psi), \pm i) \Longrightarrow (\neg\varphi \vee \neg\psi, \pm i) \tag{R6}$$

$$(\neg\neg\varphi, +i) \Longrightarrow (\varphi, +i) \tag{R7}$$

$$(\neg\neg\varphi, -i) \Longrightarrow (\varphi, -i) \tag{R8}$$

$$(\Box_m \varphi, +i), (i r_m j) \Longrightarrow (\varphi, +j) \tag{R9}$$

$$(\Box_m \varphi, -i) \Longrightarrow (i r_m j), (\varphi, -j) \tag{R10}$$

$$(\Diamond_m \varphi, +i) \Longrightarrow (i r_m j), (\varphi, +j) \tag{R11}$$

$$(\Diamond_m \varphi, -i), (i r_m j) \Longrightarrow (\varphi, -j) \tag{R12}$$

$$(\neg\Box_m \varphi, \pm i) \Longrightarrow (\Diamond_m \neg\varphi, \pm i) \tag{R13}$$

$$(\neg\Diamond_m \varphi, \pm i) \Longrightarrow (\Box_m \neg\varphi, \pm i) \tag{R14}$$

$$(\neg{\sim}\varphi, +i) \Longrightarrow (\varphi, +i) \tag{R15}$$

$$(\neg{\sim}\varphi, -i) \Longrightarrow (\varphi, -i) \tag{R16}$$

$$({\sim}\varphi, +i) \Longrightarrow (\varphi, -i) \tag{R17}$$

$$({\sim}\varphi, -i) \Longrightarrow (\varphi, +i) \tag{R18}$$

Notice that rules $R17$ and $R18$ invert the sign before i. In rules $R10$ and $R11$, the number j must be fresh in the branch. Figures 4 and 5 show two examples of proofs using the tableau system. In the first proof, no rule can be applied to the leaf node (its branch does not fulfill the conditions of any rule that was not already applied), and therefore the formula ${\sim}\Box_m p \vee p$ is not a theorem. The second example proves the validity $(p \vee {\sim}p) \wedge (\neg p \vee {\sim}\neg p)$.

Now we can prove soundness and completeness of this enhanced tableau with respect to FVEL.

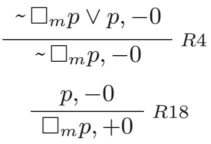

Fig. 4. An open tableau

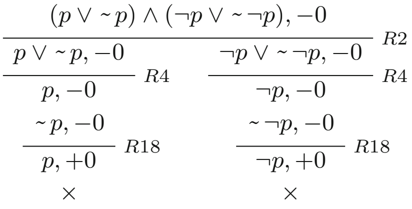

Fig. 5. A closed tableau: $p, +0$ contradicts $p, -0$, and $\neg p, +0$ contradicts $\neg p, -0$.

Theorem 1. *For any finite set of formulas* $\Sigma \cup \{\varphi\}$, $\Sigma \vdash \varphi$ *iff* $\Sigma \models \varphi$.

4 Correspondence Results

Now we will take a look at standard axioms and inference rules from modal logics. *Modus Ponens* (abbreviated as MP, $\{\varphi, \varphi \to \psi\} \vdash \psi$) is not a sound inference rule, as is the case for **FDE** (see Proposition 2). Necessitation (NEC, $\vdash \varphi \Longrightarrow \vdash \Box_m \varphi$), on the other hand, is sound. The axiom K is not a theorem of our logic. However, if K is built using classical negation instead of **FDE** negation ($\Box_m(\varphi \tilde{\to} \psi) \tilde{\to} (\Box_m \varphi \tilde{\to} \Box_m \psi)$), then it is a theorem of FVEL. Axioms T ($\Box_m \varphi \to \varphi$), 4 ($\Box_m \varphi \to \Box_m \Box_m \varphi$) and 5 ($\neg \Box_m \varphi \to \Box_m \neg \Box_m \varphi$) are not theorems (neither in their regular version, nor in their version with classical negation). Whether these or any other formulas are theorems can be easily checked using the tableau method.

Proposition 2. *MP is not a sound inference rule for $FVEL$.*

Proposition 3. *NEC is a sound inference rule for $FVEL$.*

The version of K derived from $\sim$ (let us call it $\tilde{K}$) is valid in all frames. Not surprisingly, the correspondence between some properties of frames and validity of formulas still hold, as shown by the propositions below (where the versions of T, B, 4, D and 5 derived using the classical negation are named $\tilde{T}$, $\tilde{B}$, $\tilde{4}$, $\tilde{D}$ and $\tilde{5}$, respectively).

Proposition 4. *$\mathcal{F} \models \tilde{T}$ iff $\mathcal{F}$ is reflexive.*

Proposition 5. *$\mathcal{F} \models \tilde{4}$ iff $\mathcal{F}$ is transitive.*

Proposition 6. *$\mathcal{F} \models \tilde{B}$ iff $\mathcal{F}$ is symmetric.*

Proposition 7. *$\mathcal{F} \models \tilde{D}$ iff $\mathcal{F}$ is serial.*

Proposition 8. *$\mathcal{F} \models \tilde{5}$ iff $\mathcal{F}$ is Euclidian.*

Now, it can be shown that the tableau system is complete with respect to the class of models satisfying the above properties if we augment the system with the following rules:

$$\bullet \Longrightarrow (ir_m i) \qquad (\mathrm{R}\rho)$$

$$(ir_m j), (jr_m k) \Longrightarrow (ir_m k) \qquad (\mathrm{R}\tau)$$

$$(ir_m j) \Longrightarrow (jr_m i) \qquad (\mathrm{R}\sigma)$$

$$\bullet \Longrightarrow (ir_m j) \qquad (\mathrm{R}\eta)$$

$$(ir_m j), (ir_m k) \Longrightarrow (jr_m k) \qquad (\mathrm{R}\epsilon)$$

Rules (Rρ) and (Rη) can only be applied if there is a previous appearance of the label i in the branch, and (Rη) additionally requires that j be fresh in the branch. We use the symbol $\vdash_\rho$ for the provability relation of the tableau system augmented with the rule (Rρ), and similarly for the other rules. Likewise, we use $\models_\rho$ to represent satisfiability restricted only to reflexive models, $\models_\tau$ for transitive models, $\models_\sigma$ for symmetric models, $\models_\eta$ for serial models and $\models_\epsilon$ for Euclidian models.

Theorem 2. *For all finite sets of formulas $\Sigma \cup \varphi$, the following statements hold:*

- $\Sigma \vdash_\rho \varphi$ *iff* $\Sigma \models_\rho \varphi$
- $\Sigma \vdash_\tau \varphi$ *iff* $\Sigma \models_\tau \varphi$
- $\Sigma \vdash_\sigma \varphi$ *iff* $\Sigma \models_\sigma \varphi$
- $\Sigma \vdash_\eta \varphi$ *iff* $\Sigma \models_\eta \varphi$
- $\Sigma \vdash_\epsilon \varphi$ *iff* $\Sigma \models_\epsilon \varphi$

5 Public Announcements

In this section, we extend the language with public announcements. The semantics for the new operator is defined as follows (cf. [4,15]):

$$\begin{array}{lll} M, s \models [\varphi]\psi & \text{iff} & M, s \models \varphi \text{ implies } M|_\varphi, s \models \psi \\ M, s \models \neg[\varphi]\psi & \text{iff} & M, s \models \varphi \text{ and } M|_\varphi, s \models \neg\psi \end{array}$$

where $M = \langle S, R, V \rangle$ and $M|_\varphi = \langle S', R', V' \rangle$, with $S' = \{s \in S \mid M, s \models \varphi\}$, $R' = R \cap (S' \times S')$ and $V' = V|_{P \times S'}$.

The model of Fig. 1(right), upon the public announcement of $\neg p$, would be transformed according to Fig. 6.

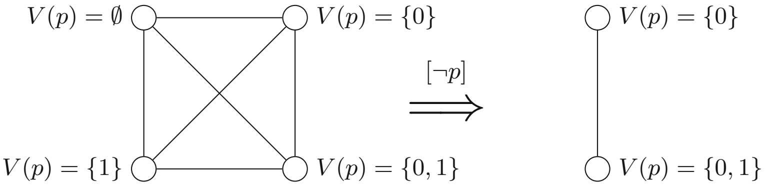

Fig. 6. The announcement of $\neg p$.

Notice that, for propositional atoms, the announcement of p does not delete worlds where $\neg p$ holds, but only worlds where p does not hold, that is, worlds where ${\sim}p$ holds. To delete worlds where $\neg p$ holds we would have to announce ${\sim}\neg p$, so that only worlds s with $M, s \models {\sim}\neg p$ (which is equivalent to $M, s \not\models \neg p$) would survive. Resorting again to our database analogy, it is possible to understand a public announcement of p (or $\neg p$) as showing to all agents the result of the query "p?" (or "$\neg p$?") to the database. More generally, the effect of the public announcement of p (or $\neg p$) is to make everybody aware that p was said true (or false) by the information source, which differs from the intuition about public announcements in standard logics.

As is the case for Public Announcement Logic [7,15], public announcements in FVEL do not increase expressivity. Any formula with public announcements can be rewritten as a standard FVEL formula, through the use of the following reduction axioms.

$$[\varphi]p \Leftrightarrow {\sim}\varphi \vee p \tag{AnAt}$$

$$[\varphi]\neg p \Leftrightarrow {\sim}\varphi \vee \neg p \tag{An$\neg$}$$

$$[\varphi](\psi \wedge \chi) \Leftrightarrow [\varphi]\psi \wedge [\varphi]\chi \tag{An$\wedge$}$$

$$[\varphi]\neg(\psi \wedge \chi) \Leftrightarrow [\varphi]\neg\psi \vee [\varphi]\neg\chi \tag{An$\neg\wedge$}$$

$$[\varphi]\Box_m\psi \Leftrightarrow {\sim}\varphi \vee \Box_m[\varphi]\psi \tag{An$\Box$}$$

$$[\varphi]\neg\Box_m\psi \Leftrightarrow {\sim}\varphi \vee \neg\Box_m[\varphi]\psi \tag{An$\neg\Box$}$$

$$[\varphi]{\sim}\psi \Leftrightarrow {\sim}\varphi \vee {\sim}[\varphi]\psi \tag{An$\sim$}$$

$$[\varphi]\neg{\sim}\psi \Leftrightarrow {\sim}\varphi \vee {\sim}{\sim}[\varphi]\psi \tag{An$\neg\sim$}$$

Proposition 9. *All above formulas for public announcements in FVEL are valid.*

Before proving that any formula with public announcements can be rewritten as an equivalent formula of FVEL where the public announcement operator does not occur, we need to prove the following lemma:

Lemma 1. *For all formulas φ, ψ, χ of FVEL with public announcements, $\varphi \equiv \psi$ implies $\chi \equiv \chi[\psi/\varphi]$. ($\chi[\psi/\varphi]$ is the formula that results from χ after uniform substitution of φ by ψ.)*

Now we can prove the following:

Proposition 10. *For any formula φ of FVEL with public announcements, a formula φ' of FVEL without public announcements can be found such that $\varphi \equiv \varphi'$.*

To account for public announcements, the tableau system can be extended with the following rule schema (which actually represents nine rules):

$$(\varphi, \pm i) \Longrightarrow (\varphi[\chi/\psi], \pm i) \quad \text{(RPA)}$$

where $\psi \Leftrightarrow \chi$ or $\chi \Leftrightarrow \psi$ is one of the public announcement axioms above[5]. Finally we can prove completeness of the extended tableau system with respect to FVEL with public announcements.

Theorem 3. *For any finite set of formulas $\Sigma \cup \{\varphi\}$ of FVEL with public announcements, $\Sigma \vdash \varphi$ iff $\Sigma \models \varphi$.*

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[5] See [1] for a different approach to tableaux for logics with public announcements, and [9] for tableaux for logics with public announcements that use translations as rules in a similar fashion.

6 A Simple Example

Now we describe the situation depicted in Fig. 7. John (j) knows that there are studies regarding health benefits of coffee consumption, for he often sees headlines about the subject. However, he never cared enough to read those articles, so he is sure that there is evidence *for* or *against* (or even *both for and against*) *coffee being beneficial for health* (p), but he does not know exactly what is the status of the evidence about p, he only knows that there is some information. Looking at Fig. 7 it is easy to see that $\Box_j((p \wedge {\sim}\neg p) \vee (\neg p \wedge {\sim} p) \vee (p \wedge \neg p))$, which is equivalent to $\Box_j(p \vee \neg p)$, holds in the actual world (s_3).

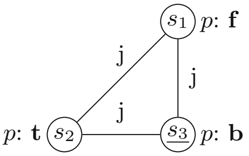

Fig. 7. Some evidence for p

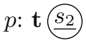

Fig. 8. No false evidence.

Kate (k), on the other hand, is a researcher on the effects of coffee on health, and for this reason she knows exactly what evidence is available (her relation R_k has only reflexive arrows, which are not represented). We can see that $M, s_3 \models \Box_k(p \wedge \neg p)$, that is, Kate actually knows that there is evidence both for and against the benefits of coffee. Moreover, John knows Kate and her job, so he also knows that she knows about p, whatever its status is (using abbreviations defined in Sect. 2.3: $\Box_j(\Box_k p^f \vee \Box_k p^t \vee \Box_k p^b)$). Likewise, Kate knows that John simply knows that there is some information about p ($\Box_k(\Box_j(p \vee \neg p) \wedge {\sim}\Box_j(p \wedge \neg p))$).

Now suppose the actual world is s_2, and so p is *true*, i.e., there is only positive evidence for p (and Kate knows that). Suppose also that Kate announces that a paper was published in a very respectable journal reassessing all the main studies that concluded that coffee was not beneficial for health, and concluded that those studies were not reliable due to sloppy methodology. Now this is equivalent to an announcement of $\Box_k {\sim}\neg p$ (Kate knows that there is no evidence for the falsity of p). This announcement results in the removal of the worlds where evidence for the falsity of p is present, namely s_1 and s_3. The resulting model is the one in Fig. 8, where John knows the status of p too. The formula ${\sim}\Box_j(p \wedge {\sim}\neg p) \wedge [\Box_k {\sim}\neg p]\Box_j(p \wedge {\sim}\neg p)$, which is satisfied in s_2 before the announcement, reflects the fact that John does not know the status of p, but after Kate's announcement he learns that p is true.

This example shows the dynamics of the agents' knowledge about available information/evidence. It might be puzzling, however, to notice that these models actually do not say much about factual knowledge. Nevertheless, it is based on information and evidence that one can form knowledge and beliefs. This observation calls for an extension of FVEL in which knowledge about evidence could be converted into factual knowledge or belief.

7 Related Work

Many authors have studied the subject of many-valued modal logics [6,11–14,17–19,21]. Of these, the most closely related to ours are Odintsov and Wansing's and Rivieccio's papers. Both papers explore some kind of four-valued epistemic logics. We will now discuss the similarities and differences between these and our approach.

In their paper [13], Odintsov and Wansing describe a logic called **BK** (a Belnapian variant of **K**), which is closely related to FVEL. They also provide a tableaux system similar to ours, but their paper does not cover public announcements, nor the correspondence results presented here. There are other small differences between the two formalisms. The logic **BK** uses two entailment symbols – support for truth ($\models^{+}$) and support for falsity ($\models^{-}$) – whereas we opted for an additional negation. While this small change still results in equi-expressive logics, we can express statements like $M, s \models \neg p \wedge \neg q$ directly, when **BK** always place the "negation" in front of the formula: $M, s \models^{-} p \vee q$. The latter has a more natural equivalent in our logic: $\neg(p \vee q)$. Moreover, this choice allows us to announce a formula like $\neg p$, which in **BK** is only expressible w.r.t. a state of a model ($M, s \models^{-} p$).

Rivieccio [17], with a very different formalism (focused on algebraic semantics), describes a logic that seems to be an extension of the one presented here, with a more expressive language and a four-valued accessibility relation. Rivieccio's logic has a symbol $\perp$ which is always evaluated to *none*, while in our language this is not expressible (no formula is evaluated to *none* if $V(p, s) \neq \emptyset$ for all p and s). His work features a Hilbert-style calculus instead of a tableau. He provides an axiomatisation which includes reduction axioms for public announcements, but it is not obvious how both axiomatisations for public announcements compare, since the languages used are slightly different.

Another work closely related to ours is being done by Majer and Sedlár [10]. They also study the logic **BK**. Their work, however, does not include public announcements nor a tableau system (as far as we know).

Finally, a unique contribution of our paper is the intuitive interpretation given to FVEL. These insights show a way in which many-valued modal logics could be used in practical applications, and open some new possibilities for research that will be discussed in the next section.

8 Conclusions and Future Work

In this paper, we presented a multi-agent four-valued logic with two distinct layers: one informational and the other epistemic. The idea of having two separate layers may be useful in the modelling of realistic scenarios where agents have access to an inconsistent or incomplete base of information. Some examples are the database scenario described in the introduction, or a robot who collects data through several sensors, which may result in inconsistent data due to sensors' inaccuracy.

First degree entailment was used as the propositional basis for the logic, with its four-valued atoms playing the role of the "informational layer", in which a proposition could be both true and false or have no value at all. A modal layer was built on top of that, introducing an epistemic aspect to the logic. The accessibility relation, then, defines the knowledge of the agents about the possibly contradictory or incomplete informational layer.

Moreover, classical negation was added to the language, increasing its expressivity. That addition allowed us to define an equivalence operator and reduction axioms for public announcements. A tableau calculus and some correspondence results were provided. While on the technical side there are similarities among our approach and others, new results have been presented and, not least, some intuition for these logics have been given.

For further work, a number of possibilities were opened. There are other possible intuitive readings for FVEL, besides the two-layered interpretation presented here (Majer and Sedlár's work offers one alternative). Furthermore, besides the public announcements studied here, a range of dynamic operators can be considered in combination with this logic. Some of these operators will act on the informational layer, and some on the epistemic layer – and perhaps some of them could act on both layers. A useful example of dynamic operation would be a method for "filtering" those inconsistent sets of beliefs in order to produce a consistent epistemic state (along the lines of belief revision, in particular [20]). It might be valuable as well to understand which actions the agents are justified to carry out on the basis of such inconsistent belief states. Other update actions (along the lines of [3]) could also be studied, for example, the actions mentioned in the introduction, which change the informational layer instead of only changing the knowledge about it. These actions, instead of removing states, could just add or remove truth (or falsity) from the value of a proposition in all worlds.

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Appendix

Proof (Lemma 1). First, let us label each atom of χ with a unique integer. In such a way, we can distinguish occurrences $\varphi_1, \varphi_2, ..., \varphi_n$ of any subformula φ that appears in χ n times, according to the labels in their atoms. The notation $\chi_\#$ will denote a labelled version of χ (that is, an occurrence of χ) in whose number we are not particularly interested. We will first prove that for any occurrence φ_k of any subformula φ of χ, if $\varphi \equiv \psi$ then $\chi \equiv \chi[\psi/\varphi_k]$, where $\chi[\psi/\varphi_k]$ denotes the formula χ after replacing the occurrence φ_k by ψ. Before starting the proof, we need to define a function sub^l from labelled formulas to a set of labelled formulas. Intuitively, $sub^l(\chi_\#)$ denotes the labelled subformulas of χ at level l, level 0 being the root, that is, $\chi_\#$ itself.

$$sub^0(\chi_\#) = \{\chi_\#\}$$

$$sub^1(\chi_\#) = \begin{cases} \emptyset, & \text{if } \chi_\# = p_\# \text{ for some atom } p. \\ \{\varphi_\#\}, & \text{if } \chi_\# \in \{{\sim}\varphi_\#, \neg\varphi_\#, \Box\varphi_\#\} \\ \{\varphi_\#, \psi_\#\}, & \text{if } \chi_\# \in \{\varphi_\# \wedge \psi_\#, [\varphi_\#]\psi_\#\} \end{cases}$$

$$sub^i(\chi_\#) = \bigcup_{\zeta_\# \in sub^{i-1}(\chi_\#)} sub^1(\zeta_\#), \qquad \text{for } i > 1$$

The proof will be by induction on the level l of φ_k. The base case is when $l = 0$, that is, $\varphi_k \in sub^0(\chi_\#)$, or simply $\varphi_k = \chi_\#$. Trivially, if $\varphi \equiv \psi$ then $\chi \equiv \chi[\psi/\varphi_k]$, since $\chi[\psi/\varphi_k] = \psi$ and $\chi \equiv \psi$ follows from $\chi = \varphi$. Induction Hypothesis (I.H.): for all $l < n$, if $\varphi_k \in sub^l(\chi_\#)$ and $\varphi \equiv \psi$, then $\chi \equiv \chi[\psi/\varphi_k]$. Given that for all occurrences $\delta_\#$ of subformulas of χ at level $n - 1$ it holds that $\delta \equiv \psi$ implies $\chi \equiv \chi[\psi/\delta_\#]$, in the induction step we need to show that for all occurrences φ_k of subformulas of χ at level n it holds that $\varphi \equiv \psi$ implies $\chi \equiv \chi[\psi/\varphi_k]$. We will divide the step in cases according to the formula $\delta_\# \in sub^{n-1}(\chi_\#)$ such that $\varphi_k \in sub^1(\delta_\#)$, and to the position of φ_k in $\delta_\#$.

$\delta_\# = \varphi_k \wedge \xi$: suppose $\varphi \equiv \psi$. Then (for all models M and states s) $M, s \models \varphi_k \wedge \xi$ iff ($M, s \models \psi$ and $M, s \models \xi$) iff $M, s \models \psi \wedge \xi$. Also, $M, s \models \neg(\varphi_k \wedge \xi)$ iff ($M, s \models \neg\varphi_k$ or $M, s \models \neg\xi$) iff ($M, s \models \neg\psi$ or $M, s \models \neg\xi$) iff $M, s \models \neg(\psi \wedge \xi)$. Therefore, $\varphi \wedge \xi \equiv \psi \wedge \xi$, and by the I.H. $\chi \equiv \chi[\psi \wedge \chi/\varphi_k \wedge \xi] = \chi[\psi/\varphi_k]$. The case for $\delta_\# = \xi \wedge \varphi_k$ is completely analogous.

$\delta_\# = \neg\varphi_k$: Suppose $\varphi \equiv \psi$. Then ($M, s \models \neg\varphi_k$ iff $M, s \models \neg\psi$) and ($M, s \models \neg\neg\varphi_k$ iff $M, s \models \neg\neg\psi$), from which it follows that $\neg\varphi \equiv \neg\psi$. By the I.H. we have that $\chi \equiv \chi[\neg\psi/\neg\varphi_k] = \chi[\psi/\varphi_k]$.

$\delta_\# = {\sim}\varphi_k$: Suppose $\varphi \equiv \psi$. Then ($M, s \models {\sim}\varphi_k$ iff $M, s \models {\sim}\psi$) and ($M, s \models \neg{\sim}\varphi_k$ iff $M, s \models \neg{\sim}\psi$), from which it follows that ${\sim}\varphi \equiv {\sim}\psi$. By the I.H. we have that $\chi \equiv \chi[{\sim}\psi/{\sim}\varphi_k] = \chi[\psi/\varphi_k]$.

$\delta_\# = \Box_i\varphi_k$. Suppose $\varphi \equiv \psi$. Then ($M, s \models \Box_i\varphi_k$ iff $\forall t$ such that sR_it $M, t \models \varphi_k$ iff $\forall t$ such that $sR_itM, t \models \psi$ iff $M, s \models \Box_i\psi$) and ($M, s \models \neg\Box_i\varphi_k$ iff $\exists t$ such that sR_it and $M, t \models \neg\varphi_k$ iff $\exists t$ such that sR_it and $M, t \models \neg\psi$ iff $M, s \models \neg\Box_i\psi$). From that it follows that $\Box_i\varphi_k \equiv \Box_i\psi$, and by the I.H. we get $\chi \equiv \chi[\Box_i\psi/\Box_i\varphi_k] = \chi[\psi/\varphi_k]$.

$\delta_\# = [\varphi_k]\xi$. Suppose $\varphi \equiv \psi$. Then $M, s \models [\varphi_k]\xi$ iff ($M, s \not\models \varphi_k$ or $M|_{\varphi_k}, s \models \xi$). But since $\varphi \equiv \psi$, $M|_{\varphi_k} = M|_\psi$. Then $M, s \models [\varphi_k]\xi$ iff ($M, s \not\models \psi$ or $M|_\psi, s \models \xi$) iff $M, s \models [\psi]\xi$. $M, s \models \neg[\varphi_k]\xi$ iff ($M, s \models \varphi_k$ and $M|_{\varphi_k}, s \models \neg\xi$) iff ($M, s \models \psi$ and $M|_\psi \models \neg\xi$) iff $M, s \models \neg[\psi]\xi$. So $[\varphi_k]\xi \equiv [\psi]\xi$, then by the I.H. $\chi \equiv \chi[[\psi]\xi/[\varphi_k]\xi] = \chi[\psi/\varphi_k]$. The case for $\delta_\# = [\xi]\varphi_k$ is similar, but even easier.

Now the induction is finished and we have proven that for any occurrence φ_k of a subformula φ of χ, if $\varphi \equiv \psi$ then $\chi \equiv \chi[\psi/\varphi_k]$. From this it is easy to see that for all φ, ψ, χ, if $\varphi \equiv \psi$ then $\chi \equiv \chi[\psi/\varphi]$. Suppose a subformula φ of χ has occurrences $\varphi_1, \varphi_2, ..., \varphi_n$ (any formula of FVEL must be finite) and $\varphi \equiv \psi$.

From the previous proof, it follows that $\chi \equiv \chi[\psi/\varphi_1] \equiv \chi[\psi/\varphi_1][\psi/\varphi_2] \equiv ... \equiv \chi[\psi/\varphi_1]...[\psi/\varphi_n] \equiv \chi[\psi/\varphi]$.

Proof (Proposition 10). First we will assume the following claims:

(Claim 1). If $[\varphi]\psi \equiv \chi$, then $\zeta([\varphi]\psi) \equiv \zeta[\chi/[\varphi]\psi]$.

(Claim 2). Given a formula $[\varphi]\psi$, where ψ contains no announcements, we can always find a formula χ without announcements such that $[\varphi]\psi \equiv \chi$.

Given any formula φ of FVEL with public announcements, we can choose a subformula $[\psi]\chi$ of it such that χ contains no announcements and, if (Claim 2) is true, find an equivalent ζ without announcements ($\zeta \equiv [\psi]\chi$). If (Claim 1) is true, we can replace $[\psi]\chi$ by ζ in φ preserving the truth value, that is, $\varphi \equiv \varphi[\zeta/[\psi]\chi]$. Since any FVEL formula with public announcements is finite, we can repeat this procedure until we reach an equivalent formula without public announcements.

Now, (Claim 1) is a corollary of Lemma 1. We now prove (Claim 2) by structural induction on ψ. Induction Hypothesis (I.H.): for all proper subformulas ψ' of ψ, $[\varphi]\psi'$ has an equivalent formula without announcements. Base: if ψ has form p, $\neg p$ or $\neg\Box\psi'$, by axioms (AnAt), (An¬) and (An¬□), respectively, we can find an equivalent formula without announcements, since ψ itself does not contain announcements. Step: for each possible connective we have a reduction axiom which reduces the original formula into another such that the formulas under the announcement φ are simpler. By the I.H., these formulas have an equivalent formula without announcements.

Proof (Theorem 3). The proof system being considered here is the tableau calculus for FVEL augmented with the public announcements' axioms and a substitution rule (if $\varphi \equiv \psi$ and $\Sigma \vdash \chi$ then $\Sigma \vdash \chi[\psi/\varphi]$). Soundness is already proven (soundness for the tableau for FVEL is proven in Theorem 1, soundness of public announcements' axioms is proven in Proposition 9 and soundness of the substitution rule follows from Lemma 1). For completeness, if φ is a formula without announcements, then $\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$ due to completeness of the FVEL proof system. If φ contains announcements, then, by Proposition 10, we can apply a finite sequence of reduction axioms $Ax_1, Ax_2, ..., Ax_n$ on φ to obtain an equivalent formula $t(\varphi)$ without announcements. Since the proof system for FVEL is complete, $t(\varphi)$ can be proven in it. Now, if we apply the substitution rule n times with the reduction axioms $Ax_n, Ax_{n-1}, ..., Ax_1$, we will obtain the original formula φ.

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