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Price competition on graphs

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Abstract

This paper extends Hotelling’s model of price competition with quadratic transportation costs from a line to graphs. We derive an algorithm to calculate firm-level demand for any given graph, conditional on prices and firm locations. These graph models of price competition may lead to spatial discontinuities in firm-level demand. We show that the existence result of d’Aspremont et al. (1979) does not extend to graphs that cannot be reduced to a line or circle.

JEL classification: D43, L10, R12

Keywords: spatial competition, Hotelling, graphs.

1 Introduction

Firms face two opposing incentives in the decision where to locate relative to competitors. A location close to one’s competitors maximizes the opportunities to capture one’s competitors’ consumers, but at the same time, little spatial differentiation increases price competition among firms. Hotelling (1929) introduced a stylized linear model of spatial (or product) differentiation to analyze which of these is the dominant force.

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The current paper generalizes Hotelling’s line model to graphs. The prime motivation for this extension is that in reality, firms do not locate on a line but they also cannot locate just anywhere on a plane due to constraints caused by zoning, geography and roads. As a result, observations of clustering by firms in physical space are not the exclusive result from firm conduct but may as well reflect the limits imposed by the given spatial structure. Recent empirical studies acknowledge this and use techniques from spatial statistics to develop measures of spatial clustering that correct for this (Picone et al., 2009).

For markets with two competing firms, we outline an algorithm that calculates firm-specific demand as a function of the firms’ prices, conditional on their position in the graph. In other words, for any structure of lines and intersections that one can draw on a piece of paper and the position of the firms on this structure, the algorithm will give firm-level demand. The constructed graphs may be as arbitrary as the patterns of released sticks in the game of Pick-up sticks, after the isolated sticks have been removed.¹ Consumers are assumed to be uniformly distributed on the graph’s edges. The line model arises as a special case.

Importantly, a number of standard results do not carry over from the unit interval to graph models of price competition. First, when transportation costs are quadratic – as we will assume throughout, following the seminal contribution by d’Aspremont et al. (1979)² – spatial discontinuities in firm-level demand may occur. That is, for given prices, consumers with a preference for a firm’s product may be surrounded by consumers with a preference for its competitor’s product. Second, in contrast to d’Aspremont et al. (1979), the assumption of quadratic transportation cost no longer is a sufficient condition for the existence of a Nash equilibrium in pure strategies.³

¹The presented graphs are best interpreted as models of differentiation in physical space but interpretations as models of differentiation in product space may be possible.

²d’Aspremont et al. (1979) show the invalidity of Hotelling’s original claim that with transportation cost linear with respect to distance, firms tend to minimally differentiate. They demonstrate that, in a model with transportation cost quadratic in distance, a price equilibrium solution exists for any pair of locations and firms maximally differentiate. Irmen and Thisse (1998) conclude that, despite differences in modeling assumptions, the outcome of most theoretical models is that firms seek to differentiate in order to avoid price competition. They however show that when the analysis is extended from one-dimensional to multi-dimensional characteristics space while upholding the quadratic transportation costs, firms only maximally differentiate in a single dimension and thus Hotelling was “almost right”.

³The assumption of quadratic transportation cost, where disutility rises more than proportional with distance is often thought to be more appropriate in models where “distance” is not interpreted as a physical distance but proxies for the difference between the characteristics of the product bought and the most preferred variety. Within the current model, non-linear transportation costs may however reflect increased search cost: the greater the distance between the consumer and the firm, the more crossroads the

The algorithm to compute demand allows us to establish the properties of the demand function and to derive sufficient conditions for the non-existence of pure-strategy Nash equilibria in prices in any given graph with a given firm location configuration. With some examples, we illustrate for which graphs and firm locations one is less likely to find pure-strategy price equilibria.

We show that for arguably even the simplest extension of the line model, the “Hotelling line with a junction” – more formally called a $K_{1,3}$ graph or S_3 star,⁴ see Figure 2 – one can always come up with firm location configurations for which the price competition game does not possess a noncooperative equilibrium in pure strategies. We prove this non-existence result in general (Theorem 8): For every graph with at least one node with degree 3, firms can always be located such that no equilibrium price solution exists.

Finally, we prove in a straightforward extension of Dasgupta and Maskin (1986b, Th. 3) that equilibria in mixed strategies always exist and we derive the mixed-strategy equilibrium price distribution for the $K_{1,3}$ graph. To the best of our knowledge, no theoretical models exist that evaluate what price profit-maximizing firms would choose on a graph, conditional on their own location and those of competitors.

The paper is related to studies on pricing on networks that have appeared, like Bloch and Quérou (2013). These studies however locate firms and consumers at nodes, following the modeling methodology common in social network analysis. In particular, the edges in these models are “void”: they are not inhabited by a density of consumers but only serve the purpose of connecting two nodes.⁵ The graphs presented here are fundamentally different because consumers are assumed uniformly distributed along the edges of the graph.⁶ This is in the spirit of Hotelling’s line model, Salop (1979)’s circular model and von Ungern-Sternberg (1991)’s pyramid model. In a recent contribution that complements this paper, Fournier and Scarsini (2016) also assume uniformly distributed consumers. In their set up, firms compete on location but take prices as given. For that setting, they prove the existence of a pure Nash equilibrium when the number of firms is sufficiently

consumer has to take the right turn to reach the firm.

⁴A graph $\mathcal{G}$ is bipartite if its nodes can be divided into two classes N_1 and N_2 such that $N_1 \cap N_2 = \emptyset$ and $N_1 \cup N_2 = N$ and every edge joins a node of N_1 to a node of N_2 . A complete bipartite graph $K_{n,m}$ is a bipartite graph that contains all possible edges joining the n nodes in N_1 with the m nodes in N_2 . The Hotelling line can be described as a $K_{1,2}$ graph: two outer nodes each with an edge to a central node. For any integer $k \geq 1$, $K_{1,k}$ or S_k graphs are called stars because k edges connect the outer nodes to one central node. The $K_{1,3}$ star is also called a claw. See e.g. Bollobás (1998) for a formal treatment of graph theory.

⁵Buechel and Röhl (2015) characterize the set of equilibria in a firm location game on such a network. They establish that, in robust equilibria, firms cluster, thereby vindicating Hotelling’s intuition.

⁶Commuting behavior of consumers is not considered, see Claycombe and Mahan (1991); Raith (1996) for theoretical contributions and Houde (2012) for a state-of-the-art empirical study.

large. In contrast to their approach, firms set prices in our model but take locations as given. That is, we focus on the second stage of the two-stage game with firms competing in prices in the second stage after having chosen their location in the first stage.

A number of papers (Mills and Lav, 1964; Eaton and Lipsey, 1976; Greenhut et al., 1976; Holahan and Schuler, 1981) have studied location choice and price competition on two-dimensional spatial markets with constant transport cost per unit distance and free entry. The starting point of this literature is the well-known result first asserted by Lösch (1954) and formally proven by Bollobás and Stern (1972) that, conditional on every consumer in the plane being served and constant transport cost per unit distance, a division of market demand into hexagons is socially optimal. Subsequent contributions have questioned whether the hexagonal configuration is the unique equilibrium when the number of firms is given and the extent to which this configuration results under free entry (Eaton and Lipsey, 1976). These studies have in common that consumers are assumed uniformly distributed over a plane. Instead, the current paper is concerned with studying the generic properties of spatial models of product competition with consumers uniformly distributed along the edges of a given graph. We do not study which firm configurations result under entry for any particular (class of) graphs.

We limit attention to the situation with two firms and quadratic transportation cost but conceptually, the analytical approach can be extended in a straightforward manner to cover situations with non-quadratic cost and multi-firm competition. In fact, as in the pyramid model in von Ungern-Sternberg (1991), graphs (or network structures) easily allow for multi-firm competition. The difference with the model by von Ungern-Sternberg is that no analytical solutions are available for less stylized graphs.

1.1 Examples

Before we embark on the formal treatment, we introduce a number of motivating examples to sharpen our intuition which firm location configurations on given graphs lead to a price equilibrium in pure strategies and which configurations only allow for equilibria in mixed strategies. The examples also illustrate how elusive the intuition of equilibrium existence is for this class of games.

1.1.1 From line to kite

Different from the familiar Hotelling line case, firms face a tradeoff in selecting a location on a graph. On the one hand, moving away as far possible from the competitor softens competition by increasing product differentiation. On the other hand, such a move creates a group of consumers in

Figure 1: From line to kite. Note that the mass of consumers is normalized to one by taking $\alpha = 1/(4 + \sqrt{2})$.

between firms for whom the difference in distance to firm A and firm B is identical. Competition for these consumers can be fierce because they all make a similar tradeoff in their decision from whom to buy.

The kite-shaped graph in Figure 1 illustrates how this tradeoff causes the Hotelling $K_{1,2}$ line but not the $K_{1,3}$ graph to have an equilibrium in pure strategies. For $\delta = 0$, firm A is located at node 1 and firm B at node 3, a situation akin to Hotelling's line with the firms competing simultaneously on three main streets. For $\delta = 1$, firm A is located at node 2 and firm B at node 4. This case is similar to a $K_{1,3}$ -graph since both have a junction located midway between firm A and firm B .

As δ increases, two things happen. The distance between the firms increases. This effect will lead to less competition and higher equilibrium prices. However, at the same time, the consumers located on the “diagonal” (i.e. the edge connecting node 1 and node 3) are increasingly price-sensitive since the difference in transportation cost between the two firms is decreasing for this group. This effect will eventually destabilize the pure strategy equilibrium and increase competition between the firms. Table 1 illustrates this: profits peak at $\delta = 0.7$. For higher values, only mixed strategy equilibria exist with expected prices and profits lower than for $\delta = 0.7$.⁷

⁷More detailed calculations for the mixed strategy equilibria in this paper are available from the authors upon request.

δ	$p_A = p_B$	p_{min}	p_{avg}	p_{max}	$\pi_1 = \pi_2$
0	0.108				0.054
0.1	0.114				0.057
0.2	0.119				0.060
0.3	0.123				0.062
0.4	0.127				0.064
0.5	0.131				0.065
0.6	0.134				0.067
0.7	0.136				0.068
0.8	–	0.0879	0.1259	0.1326	0.063
0.9	–	0.0664	0.0832	0.0979	0.041
1	–	0.0352	0.0491	0.0869	0.023

Table 1: Equilibrium price (range) and profits for the example in Figure 1.

1.1.2 Shifting hinterlands

The example in Figure 2(a) illustrates the stabilizing effect of firms having a hinterland ($\delta = 0, 1, \dots, 8$, because of symmetry we focus on $\delta \geq 4$). We define the hinterland of firm A (B) as the set of those consumers who can only travel to firm B (A) by passing firm A (B). In all cases 1/11 of all consumers (those at the vertical part of the graph) are at equidistance from firm A and B . For $\delta = 4$, the game is symmetric with both firms having a hinterland of equal size. As δ increases, the hinterland of firm A grows while the hinterland of firm B shrinks. In Section 4.2.1, we will formally prove that this game has a pure-strategy equilibrium for $\delta = 8$. For smaller values, there only is a mixed-strategy equilibrium. The plots of the equilibrium price strategies in Figure 3(a) and 3(b) show that the redistribution of hinterland from firm B to firm A softens price competition: both firms increase their average prices as δ increases. Interestingly, for firm B initially this increase in equilibrium prices is sufficient to make up for the loss of hinterland: both firms' profits increase (Table 2). When $\delta = 8$, firm A 's hinterland has grown such that it is no longer profit-increasing for firm A to compete for the consumers at equidistance. In the pure strategy equilibrium that results, firm A sets a high price ($p_A = 0.220$) and only serves consumers in its hinterland.

Figure 2(b) depicts the slightly different situation where the hinterland lost by firm A is not added to the other firm's hinterland but shifted to the perpendicular edge. Table 2 indicates that the effect of such a shift to “no man's land” is as expected and qualitatively the same: competition intensifies when δ decreases and, because no hinterland is redistributed to firm B , both firm's are left worse off. The equilibrium price strategies (not shown) are similar to the ones depicted in Figure 3.

Figure 2: Shifting hinterlands.

2 Notation and auxiliary results

This section introduces the notation to describe the position of firms and consumers on graphs, the shortest paths and shortest path distances between them. This notational framework will be put to work in Section 3 when we derive firm-level demand for given graphs with given firm location configurations.

A graph $\mathcal{G}$ is described by a finite set of nodes $N = \{1, \dots, n\}$ and a set of edges $E \subset N \times N$. Generic elements of N are denoted by i and j . If i and j share an edge, then $(i, j) \in E$. We abbreviate (i, j) by ij . Furthermore, node i has a physical location $x_i \in \mathbb{R}^2$. Consequently, we think of an edge

	I		II	
δ	π_A	π_B	π_A	π_B
0	0.056	0.133	0.001	0.001
1	0.062	0.117	0.004	0.004
2	0.061	0.093	0.010	0.008
3	0.055	0.068	0.019	0.014
4	0.046	0.046	0.032	0.022
5	0.068	0.055	0.051	0.032
6	0.093	0.061	0.077	0.044
7	0.117	0.062	0.107	0.054
8	0.133	0.056	0.133	0.056

Table 2: Equilibrium profits for Shifting Hinterlands I & II.

(a) Situation I: Firm A

(b) Situation I: Firm B

Figure 3: Equilibrium price strategies shifting hinterlands Figure 2(a).

ij as a straight line connecting x_i and x_j . The length of edge ij is then $\ell_{ij} = \|x_i - x_j\|$. We assume that $\mathcal{G}$ is connected.

Let there be two firms, A and B , who have zero cost of production and who simultaneously set prices; p_A and p_B , respectively. Consumers are uniformly distributed along a graph (with a total mass of one) on which the firms have a fixed location. Suppose that firm A is located at node 1 and firm B at node 2. This is without loss of generality in the sense that we can always create an extra node on the edge where the firm is located and relabel the nodes accordingly. This does not change the underlying graph. Mostly it will generate redundant nodes that lie on a straight line between two other nodes. If the firm happens to be located on a node, then it will only lead to relabeling of the nodes. The analysis however becomes considerably easier if we can exclude the possibility that firms are located in the interior of an edge.

A location on an edge is denoted by $z \in ij$. Note that $z = tx_i + (1-t)x_j$ for some $t \in (0, 1)$. It is useful to be able to talk about small perturbations along edges: let $z_\varepsilon = z + \varepsilon(x_j - x_i)$, where ε is sufficiently close to zero such that $z_\varepsilon \in ij$.

We assume that the market is covered and that consumers minimize transportation cost plus price. Transportation cost is the square of the distance of the consumer to the firm. On a graph, finding the shortest path from one point to another is less trivial than in Euclidean spaces. For the moment we postpone the discussion of how to calculate these shortest paths between points and assume that for each node $i > 2$, the distances to nodes 1 and 2 are known.⁸ We denote the corresponding distance to firm A and B by d_{Ai} and d_{Bi} , respectively. This information is sufficient to calculate the minimal distance for each point $z \in ij$ on the graph to firm A :

$$d_A(z) = \min\{d_{Ai} + \|z - x_i\|, d_{Aj} + \|z - x_j\|\}, \quad (1)$$

and $d_B(z)$ is defined analogously. From this formula, we see that the shortest path goes either via node i or via node j . Shortest paths are usually unique, but at some locations, it does not matter whether one travels to firm A via node i or j because the paths are equally long. At those points, the direction of the shortest path changes. On each edge, there can be at most one point where the shortest path is non-unique because the distance via i or

⁸Formally (Bollobás, 1998; Jackson, 2008), a path P between nodes i to j is a sequence of edges $i_1i_2, i_2i_3, \dots, i_{K-1}i_K$ such that $i_ki_{k+1} \in E$ for each $k \in \{1, \dots, K-1\}$ with $i_1 = i$ and $i_K = j$. Let $e(P)$ denote the length of path P , where $e(P) = \sum_{ij \in P} \ell_{ij}$. Let $\mathcal{P}_{ij}$ denote the set of all paths between nodes i and j . Then the shortest path P_{ij}^{SP} between the nodes i and j is defined as:

$$P_{ij}^{SP} = \{P \in \mathcal{P}_{ij} \mid e(P) \leq e(\tilde{P}), \quad \forall \tilde{P} \in \mathcal{P}_{ij}\}.$$

Dijkstra's algorithm (Bertsekas, 1991, pp. 68–75), for example, calculates the shortest path from a node to all other nodes.

j changes linearly along the edge, increasing in one direction and decreasing in the other. We refer to these points as *change points* (for firm A or B) as these are the points where the direction of the shortest path (to firm A or B) changes. The distance of a perturbation z_ε of z to firm A is related to the distance of z to firm A , provided ε is sufficiently small:

$$d_A(z_\varepsilon) = d_A(z) + \varepsilon \ell_{ij} \text{ (if the shortest path goes via } i), \quad (2)$$

$$d_A(z_\varepsilon) = d_A(z) - \varepsilon \ell_{ij} \text{ (if the shortest path goes via } j), \quad (3)$$

$$d_A(z_\varepsilon) = d_A(z) - |\varepsilon| \ell_{ij} \text{ (if } z \text{ is a change point)}. \quad (4)$$

Note that the last expression states that change points for firm A are located further from firm A than points in the change point's neighborhood.⁹

3 Demand

A consumer located at z buys from firm A if $d_A(z)^2 + p_A < d_B(z)^2 + p_B$ or if

$$f(z) \equiv d_A(z)^2 - d_B(z)^2 < \mu \equiv p_B - p_A,$$

where μ is the price difference. Hence the consumer buys from A if $f(z) < \mu$, buys from B if $f(z) > \mu$ and is indifferent when $f(z) = \mu$. Assume that demand from indifferent consumers is split equally between the firms. As in Hotelling's model, the location of the indifferent consumers allow us to calculate demand. On Hotelling's line, all consumers on the left of the indifferent consumer will buy from one firm, those on the right from the other. On more complicated graphs, the picture is less straightforward. To illustrate all different possibilities, we first focus on a single edge and determine demand on this edge.

3.1 Demand on a single edge

Consider edge ij and suppose that the price difference is μ . In most of the proofs below, we compare the value of $f(z_\varepsilon)$ to the value of $f(z)$. By combining (2–4) with the definition of $f(\cdot)$, we get the following cases:

$$f(z_\varepsilon) = f(z) - 2\ell_{ij}d_A(z)\square_A - 2\ell_{ij}d_B(z)\square_B, \quad (5)$$

where $\square_A \in \{-\varepsilon, \varepsilon, |\varepsilon|\}$ and $\square_B \in \{\varepsilon, -\varepsilon, -|\varepsilon|\}$, depending on whether the shortest path to firm A (B) goes via i , via j or whether z is a change point, respectively.

Proposition 1. *An edge ij contains an interval of indifferent consumers if and only if $\mu = 0$ and there exists $z \in ij$ such that $d_A(z) = d_B(z)$ and both shortest paths have the same direction.*

⁹This is a direct consequence of the fact that the minimum of two affine functions is a concave function (cf. (1)).

This proposition (whose proof is relegated to A, as are the proofs of the subsequent propositions and theorems) establishes that intervals of indifferent consumers can only appear when both firms set the same price (i.e. $\mu = 0$) and gives necessary and sufficient conditions for such intervals of indifference to occur in terms of the properties of the graph and the position of the firms therein.¹⁰

In Figure 4, the simplest example is illustrated. In this case, $d_A(z) = d_B(z)$ for all $z \in ij$: the entire edge is indifferent. Figure 5 adds one edge to this simplest example, which leads to a change point for firm A on the edge ij , labeled z_A . The conditions of Proposition 1 no longer apply between z_A and x_j since the direction of the shortest path to firm A now points in another direction as the shortest path to B . Only consumers between x_i and z_A are indifferent.

In both examples, if one of the firms lowers its price, then all these consumers will buy from the firm with the lowest price. Therefore, demand for the firm may not be continuous at $\mu = 0$. The next proposition gives a sharper characterization of edges where an interval of the consumers is indifferent.

Proposition 2. *Suppose $\mu = 0$ and there exists $z \in ij$ such that $d_A(z) = d_B(z)$ and both shortest paths have the same direction. Then either the entire edge is indifferent between A and B or all indifferent consumers are located between one of the nodes and a change point.*

Note that this proposition implies that if, for a given edge, one of the end nodes is equidistant from both firms and the shortest paths have the same direction at this node, then part of the edge will be indifferent when $\mu = 0$. If this is true for both nodes (such as in Figure 4), then the entire edge is indifferent.

For the remainder of this paragraph, we exclude the possibility of intervals of indifferent consumers and focus on isolated indifference points by assuming that either $\mu \neq 0$, or there does not exist a $z \in ij$ such that $d_A(z) = d_B(z)$ and both shortest paths have the same direction. We start with an investigation of the location of indifferent consumers. It is useful to distinguish between weak and strong indifference points.

Definition 1 (Weak and strong indifference points). *A location z is an indifference point for a given value of μ if $f(z) = \mu$. A weak indifference point is an indifference point with the property that consumers on both sides of z strictly prefer to buy from the same firm. A strong indifference point separates consumers who buy from firm A from those who buy from firm B .*

¹⁰For given prices p_A and p_B (and thus μ), the set of indifferent consumers of a graph is defined as $S^{ind}(\mu) \equiv \{z \mid f(z) = \mu\}$. Only if $\mu = 0$, part of this set may consist of intervals of indifferent consumers. Such an interval I_{ij}^{ind} at edge ij is then defined as: $I_{ij}^{ind} \equiv \{z \in S^{ind}(0) \cap ij \mid \exists \varepsilon > 0 \text{ s.t. } z_\varepsilon \in S^{ind}(0)\}$.

Figure 4: At $\mu = 0$, all consumers on edge ij are indifferent (numbers indicate distances).

Figure 5: If $\mu = 0$, all consumers between x_i and z_A are indifferent (numbers indicate distances).

The importance of weak indifference points is that we expect a bifurcation to occur at weak indifference points, i.e. a structural change from an edge where everybody buys A (or alternatively B) to an edge where in the middle of the edge a group of consumers prefer B over A (with two strong indifference points on either side). Weak indifference points are linked to change points, as the following proposition establishes.

Proposition 3. *A weak indifference point is a change point either for firm A or for firm B (or both). Moreover, if z is a change point for firm A and $f(z) > 0$ (or for firm B and $f(z) < 0$), then it is a weak indifference point for $\mu = f(z)$.*

With the aid of this proposition, we can immediately establish whether a change point is a weak indifference point. Figure 6(a) shows an example of a graph where one of the edges has both a change point for firm A , located at $z_A = \frac{1}{3}x_i + \frac{2}{3}x_j$, and a change point for firm B , located at $z_B = \frac{2}{3}x_i + \frac{1}{3}x_j$. Figure 6(b) shows how the value of f changes along the edge. When $\mu < -11$ (that is, $f(z) > \mu$ for all z and all strictly prefer to buy from B) or when $\mu > 11$ (that is, $f(z) < \mu$ for all z and all strictly prefer to buy from A) there is no indifference point and therefore no weak indifference point. In particular, neither z_A and z_B are weak or strong indifference points. If $\mu \in (-9, 9)$, then the indifference point lies between z_A and z_B and so no indifference point is a weak indifference point since no indifference point is a change point of either firm. If $\mu \in (-11, -9]$, then there are two indifference points: one to the left of z_B and one between z_B and z_A : everyone buys from firm B with the exception of the consumers located around z_B who buy firm A 's product. Since neither of these indifference points is a change point, both are strong indifference points and there are no weak indifference points. Analogously, both of the two indifference points are strong when $\mu \in [9, 11)$. Thus, there is a weak indifference point if and only if $p_B - p_A = \mu = -11$ (when z_B is a weak indifference point) or $\mu = 11$ (when z_A is a weak indifference point). For any $\mu \in (-11, 11)$ all indifference points are strong. For any μ such that $|\mu| > 11$, there are no indifference points.¹¹

There is nothing special about strong indifference points, in the sense that any location on the graph will be a strong indifference point for some value of μ . At a strong indifference point, the difference in transportation cost is either increasing or decreasing. At weak indifference points, the difference in transportation cost instead reaches a minimum or a maximum.

¹¹At this point, it is important to recall that not all change points necessarily are weak indifference points for some value of μ . Suppose for example that we change the graph in Figure 6(a) such that the length of the edges x_{jB} and x_{iB} becomes 8 and 9, respectively, then $f(z_A) = d_A(z_A)^2 - d_B(z_A)^2 = 36 - 100 = -64 < 0$ and there does not exist any price difference $\mu \equiv p_A - p_B$ that would make the consumer at point z_A indifferent while all consumers in her direct neighborhood would have a strict preference to buy from the same firm.

Figure 6: An example with multiple weak indifference points

Therefore, in the absence of change points for which $f(z) > 0$ (or < 0), we know that the entire edge buys from firm A (or B) if the consumers located at node i and j buy from A (or B). When the consumers located at these nodes buy from different firms, there has to be a strong indifference point, separating those who buy from A from those who buy from B .

By combining these observations, one can determine the demand for firm A on any given edge as follows.

Proposition 4. *Suppose that for a given edge ij we know the following: the distance to firm A and B from node i and j , the location of the change points (and the distance to firm A and B from these points). Then demand for firm A , $\nu_{ij}(\mu)$, along the edge can be determined by the following algorithm.*

1. IF there are no change points:

- (a) IF $f(x_i) = f(x_j) = 0$ and $\mu = 0$, then $\nu_{ij}(\mu) = \frac{1}{2}\ell_{ij}$
- (b) ELSE:
 - If $\min\{f(x_i), f(x_j)\} \geq \mu$, then $\nu_{ij}(\mu) = 0$
 - If $\max\{f(x_i), f(x_j)\} \leq \mu$, then $\nu_{ij}(\mu) = \ell_{ij}$
 - Else find the value of t such that $f(tx_i + (1-t)x_j) = \mu$
 - If $f(x_i) \leq \mu$, then $\nu_{ij}(\mu) = t\ell_{ij}$.
 - Else $\nu_{ij}(\mu) = (1-t)\ell_{ij}$

2. ELSE: Suppose the change points are located at $z_0, z_1, \dots, z_m$. Then create auxiliary edges $ik_0, k_0k_1, \dots, k_mj$ from x_i to z_0 , z_0 to z_1 , $\dots$ and z_k to x_j . Demand on edge ij is given by $\nu_{ij}(\mu) = \nu_{ik_0}(\mu) + \nu_{k_0k_1}(\mu) + \dots + \nu_{k_mj}(\mu)$.

The reason why this algorithm will properly calculate the demand on a single edge is that it breaks up edges at change points if these change points are located in their interior. In doing so, we circumvent all special cases discussed above.

Figures 7(a) and 7(b) show the demand for firm A from consumers along the designated edge ij in Figures 5 and 6(a). Note that demand is piecewise linear. This follows from the fact that if z is a strong indifference point but not a change point, and $\mu \neq 0$, then

$$f(z_\varepsilon) = \mu - 2\ell_{ij}d_A(z)\square_A - 2\ell_{ij}d_B(z)\square_B, \quad (6)$$

where $\square_A \in \{-\varepsilon, \varepsilon\}$ and $\square_B \in \{\varepsilon, -\varepsilon\}$. Therefore a change in μ is offset by a constant change in ε , i.e. the location of any of the indifferent consumers. This translates into a constant slope of the demand curve. The points in Figure 7, where the slope changes, are at values of μ for which either one of the change points on the edge is a weak indifference point or one of the end nodes of the edge is an indifference point.

(a) Example in Figure 5: demand jumps at $\mu = 0$.

(b) Example in Figure 6(a): kinks because of the weak indifference points.

Figure 7: Demand on a single edge

3.2 Firm-level demand and profit

Demand for firm A 's product is simply the sum over demand on each single edge:

$$D(\mu) = \frac{\sum_{ij \in E} \nu_{ij}(\mu)}{\sum_{ij \in E} \ell_{ij}}$$

where we have normalized demand to 1. Demand for firm B is $1 - D(\mu)$. The profits are given by (remember that $\mu = p_B - p_A$):

$$\begin{aligned}\pi_A(p_A, p_B) &= D(p_B - p_A)p_A \\ \pi_B(p_A, p_B) &= [1 - D(p_B - p_A)]p_B\end{aligned}$$

Firms simultaneously set prices. The properties of $D(\cdot)$ will determine whether equilibria in pure strategies exist. We close this section by observing that $D(\mu)$ inherits most of its properties from ν_{ij} , namely:

Proposition 5. *Demand for firm A as a function of the price differential μ is a non-decreasing, piecewise linear function that is continuous everywhere, except possibly at $\mu = 0$. Furthermore*

1. $\check{\mu} = \sup\{\mu \mid D(\mu) = 0\} \in (-\infty, 0]$
2. $\hat{\mu} = \inf\{\mu \mid D(\mu) = 1\} \in [0, \infty)$
3. *The cardinality of the set of points where $D(\mu)$ is non-differentiable is at most four times the number of edges.*

Observe that $\check{\mu} = \hat{\mu} = 0$ corresponds to the pathological case where firm A and firm B are located at the same node. Since the equilibrium is trivial in this point ($p_A = p_B = 0$), we will exclude this possibility.

At this point, a natural question to ask is whether it is possible to abstract from the graph-theoretic foundation and to reduce the consumers' characteristics that are relevant for firm-level demand to a single dimension. That is, would we obtain similar results if we would just assume the two firms to be located at the two ends of a unit interval with the distribution F of consumers on this line not being uniform but displaying atoms. This indeed is possible. Suppose consumers are indexed by $\theta \in \mathbb{R}$ and that the distribution of θ is given by the cumulative distribution function F . Consumer θ receives utility $-p_A$ if he buys from firm A and utility $\theta - p_B$ if he buys from B . Therefore, he buys from A if $-p_A > \theta - p_B \Rightarrow \theta < \mu$. Firm-level demand for firm A is thus given by $F(\mu)$. However, without the graph-theoretic foundation, there is no justification for the result in Proposition 5 that this demand function F is piecewise linear and continuous, except possibly at $\mu = 0$. It is this foundation that allows us to relate differences in demand functions to differences in the underlying graph and firm location therein, such as between Figure 5 and 6(a).

4 Pure Equilibrium Existence

We want to establish conditions under which pure strategy Nash-equilibria exist. As Vives (1999, Ch. 6) shows, in games with differentiated products and simultaneous price-setting, the existence of pure-strategy equilibria is not guaranteed.¹² As a first step, let us examine the properties of the reaction function.¹³ The reaction function for firm A is

$$R_A(p_B) = \arg \max_{p_A \geq 0} D(p_B - p_A)p_A.$$

We have:

1. The best-response to $p_B = 0$ is to set $p_A > p_B$: Suppose $p_B = 0$: Note that the profit of firm A is zero if it sets $p_A = 0$. However demand, and therefore profit, is strictly positive for any $p_A \in (0, -\check{\mu})$. It follows that $R_A(0) > 0$.
2. The best-response to $p_B = \frac{\hat{\mu} - \check{\mu}D(0)}{1 - D(0)}$ is to set $p_A < p_B$. Note that $p_A = p_B - \hat{\mu} = \frac{D(0)[\hat{\mu} - \check{\mu}]}{1 - D(0)} > 0$. At this price, firm A captures the entire market and profit is therefore equal to $p_B - \hat{\mu}$. This is a lower bound on profit when firm A undercuts firm B . To show that profit for any $p_A \geq p_B$ is beneath this lower bound, note that

$$\max_{p_A \geq p_B} D(p_B - p_A)p_A = \max_{p_A \in [p_B, p_B - \check{\mu}]} D(p_B - p_A)p_A < D(0)(p_B - \check{\mu}) = p_B - \hat{\mu},$$

where the first equality follows from the fact that demand is zero for prices above $p_B - \check{\mu}$ and the inequality follows from the fact that demand is strictly decreasing. We conclude that for p_B sufficiently high, $R_A(p_B) < p_B$.

3. The best-response curve is upward-sloping whenever the maximizer is at a differentiable point of the profit function: The first-order condition for profit-maximization is

$$-D'(p_B - p_A)p_A + D(p_B - p_A) = 0.$$

Using the implicit function theorem, we get

$$\frac{dp_A}{dp_B} = \frac{D''(p_B - p_A)p_A - D'(p_B - p_A)}{D''(p_B - p_A)p_A - 2D'(p_B - p_A)} = \frac{1}{2}$$

since demand is (piece-wise) linear.

¹²Note that the existence-result by Anderson et al. (1997) does not apply in our case because the graph-theoretic foundation does not imply log-concavity of $D(\mu)$.

¹³We present the proofs for firm A , the proofs for firm B are similar.

Together with continuity of the reaction function this implies the existence of pure-strategy Nash-equilibria. However, to obtain a continuous reaction function, we need to make strong assumptions on the demand function such as quasiconcavity or convexity of the inverse demand function (Caplin and Nalebuff, 1991). This is in general not met for the type of games discussed in this paper.

It is well known that even when the demand function is continuous, pure-strategy equilibria can fail to exist because profits are not quasi-concave, see e.g. Gabszewicz and Thisse (1986).¹⁴ Section 4.1 will give sufficient conditions for the non-existence of pure-strategy equilibria for the case when demand is discontinuous at $\mu = 0$, that is, when firms set equal prices. Section 4.2 points out the implications of this non-existence result for graphs: it will turn out that for the simplest graph that is not a Hotelling line one can always find a pair of firm locations that satisfy the conditions of the non-existence result. This holds more generally for every graph with at least one node with degree 3 or higher. We also show how the presence of a sufficiently large hinterland for one of the firms generates a pure-strategy Nash equilibrium. Finally, in Section 4.3 we will state the rather obvious result that any two-firm graph model will have an equilibrium in mixed strategies and compute such an equilibrium for one particular graph with a given firm location configuration.

4.1 Non-existence of a pure-strategy Nash-equilibrium when demand is discontinuous at $\mu = 0$

This section sets out to prove the following proposition, which provides sufficient conditions for the non-existence of pure-strategy Nash-equilibria in cases where firms are located on a graph $\mathcal{G}$ such that the demand function $D(\cdot)$ has a discontinuity at $\mu = 0$, that is, when firms A and B charge the same price.

Theorem 6. *Suppose firms are positioned such that the demand function $D(\cdot)$ has a discontinuity at $\mu = 0$. Sufficient conditions for the non-existence of pure-strategy Nash-equilibria are that*

$$D(0^-) \equiv \lim_{\mu \uparrow 0} D(\mu) < \frac{1}{2} \text{ and}$$

$$D(0^+) \equiv \lim_{\mu \downarrow 0} D(\mu) > \frac{1}{2}.$$

Note that this condition for non-existence is weak. The only requirement is that there is a discontinuity and, at the discontinuity, demand jumps from below one half to above one half. In the next Section, we will use these

¹⁴B shows for one specific graph how continuity of demand does not guarantee the existence of a pure strategy equilibrium.

criteria to find firm locations for which pure-strategy Nash-equilibria do and do not exist.

4.2 Pure-strategy price equilibria and types of graphs

In graph-theoretic terms, the Hotelling line is a complete bipartite $K_{1,2}$ graph, that is, a tree with one internal node and 2 leaves (edges). The most straightforward extension of the Hotelling line is the $K_{1,3}$ graph (or star S_3) of which Figure 4 and Figure 8 depict an example. In this section, we consider $K_{1,3}$ graphs and show that for every such graph, one can position firm A and B such that pure-strategy Nash-equilibria do not exist.

Theorem 7. *For every $K_{1,3}$ graph, there exists a continuum of firm locations for which the price competition game does not possess a pure-strategy Nash equilibrium.*

Proof. Consider the graph in Figure 8. Without loss of generality, $\delta \geq \max\{\xi + \varepsilon, \varphi + \varepsilon\}$. Note that we have $D(0^-) = \varphi + \varepsilon$ and $D(0^+) = \varphi + \delta + \varepsilon$. Since $\delta \geq \max\{\xi + \varepsilon, \varphi + \varepsilon\}$, we see that $D(0^-) < \frac{1}{2}$ and $D(0^+) > \frac{1}{2}$. From Theorem 6, we see that a pure-strategy Nash equilibrium does not exist. $\square$

Figure 8: Location of firms; numbers indicate the length of the edges, the total mass of consumers is $2\varepsilon + \varphi + \delta + \xi = 1$.

Thus, the most minor graph-theoretic extension of the d'Aspremont et al. (1979) two-firm line model with quadratic transportation costs is sufficient to lead to a non-existence result. The proof shows that no pure-strategy equilibrium exists if both firms are at equidistance from the junction and the longest edge is not inhabited by any of the firms.

In light of the sufficient conditions formulated in the literature for non-uniform consumer distributions, notably Caplin and Nalebuff (1991) and Anderson et al. (1997), this non-existence result is not so surprising. Anderson et al. state that pure-strategy do exist “if the density is not ‘too asymmetric’ and not ‘too concave.’” This condition indeed is not met here, just as the ρ -concavity sufficient condition formulated in Caplin and Nalebuff (1991). A brief comparison of our result with Varian’s non-existence result (Varian, 1980, Proposition 2) is also appropriate. In Varian (1980),

the absence of symmetric price equilibria is due to the assumption of declining average cost curves and the fact that a slight price cut by one of the stores leads this store to capture all informed consumers. The behavior of these informed customers is akin to the flocking of consumers along the junction of the $K_{1,3}$ -graph to the firm charging the lowest price including transportation cost. Other than in Varian's model, a noncooperative equilibrium in pure strategies may therefore exist unless the fraction of informed customers – i.e. the length of the junction – is sufficiently large for firms to start a price war. The proof essentially shows that in every $K_{1,3}$ graph one can position the firms such that this condition holds.¹⁵

4.2.1 Example: Hinterlands

In order to provide further intuition on how the non-existence result of Theorem 7 relates to actual graphs, we give the next example that shows the importance of hinterlands. Figure 9(a) shows a graph where firm A has a hinterland, i.e. the consumers on the most left edge can only reach firm B by traveling via firm A . The parameter δ is a measure of the size of the hinterland. Firm A has an incentive to exploit these customers: especially if they are located deep in the hinterland, firm A can charge a substantial premium before these consumers switch to firm B . One can check that for $\delta < 1$ all conditions in Theorem 6 are met, and therefore there will not be a pure strategy equilibrium. In Figure 9(b), we see the reaction functions when $\delta = 8$. For firm A , the best-response is to set a higher price than firm B if firm B 's price is below a certain threshold. However firm A will just undercut firm B when p_B is high. Firm B 's reaction function is similar, but in addition it will set a higher price than firm A if firm A sets a really low price. The equilibrium prices are $p_A^* = \frac{80}{363} \approx 0.220$ and $p_B^* = \frac{52}{363} \approx 0.143$. Note that the location of the jump in firm A 's reaction function is crucial in determining whether a pure-strategy equilibrium exists or not. If the jump had occurred for a lower value of p_B , then it would have jumped over the reaction function of firm B .

4.2.2 Beyond $K_{1,3}$ -graphs

The next theorem states that the non-existence result holds for all graph models of price competition involving two or more firms:

¹⁵In Economides (1986), consumers are evenly distributed on a surface. He shows that demand and profit functions are continuous for fairly general distance functions including the Euclidean metric but not for the block metric. In our application, the distance between two points x and y is determined by the length of the shortest path between these two points. That is, we cannot use a different distance function to remove the “thickness” of consumers at the boundary to restore existence.

(a) Graph with a hinterland for firm A

(b) Reaction curves for firm A and B ($\delta = 8$)

Figure 9: Example with hinterland

Theorem 8. *For every graph $\mathcal{G}$ with at least one node having degree 3 or higher, there exist firm locations for which the price competition game does not possess a pure-strategy Nash equilibrium.*

The condition that at least one of the graph's nodes has to be of degree 3 or higher rules out the structures for which we know that they do have a pure-strategy equilibrium, such as the line and circle.¹⁶ The sufficient condition in Theorem 6 allows us to find locations for which the price competition game does not possess a pure-strategy Nash equilibrium. The online appendix illustrates for a more complicated graph how to find firm configurations for which the price competition games does not possess a pure-strategy Nash equilibrium.

4.3 Mixed-strategy price equilibria

Demand discontinuities lead to the non-existence of equilibria in pure strategies. However, for the model with two firms, mixed-strategy price equilibria do exist because the profit functions $\pi_i(p_A, p_B)$ ($i = A, B$) are bounded and weakly lower semi-continuous in p_i and $\sum_{i=1}^2 \pi_i(\mathbf{p})$ is upper-semicontinuous (Dasgupta and Maskin, 1986a, Th. 5). For the model with two firms, we therefore have the following positive result, which essentially extends Theorem 3 in Dasgupta and Maskin (1986b).

Theorem 9. *The two-firm graph model of price competition has a mixed-strategy equilibrium.*

This theorem does not use the assumption of quadratic transportation cost. This implies that the result extends to graph models where consumers face other forms of nonlinear or linear transportation cost.

4.3.1 Example of a mixed-strategy equilibrium

The online appendix shows how to compute a symmetric mixed-strategy equilibrium for the $K_{1,3}$ graph of Figure 8. For the specific case with $\phi = \varepsilon = \xi = \frac{3}{16}$ and $\delta = \frac{1}{4}$, Figure 10a shows the equilibrium price distribution. Note the concave shape of the distribution which implies that most of the prices are at the lower end of the distribution with an occasional higher price. Figure 10b shows the expected profit as a function of price: the area in between the vertical red bars denotes the equilibrium support of the price distribution.

¹⁶Note that the line and circle both have the following property: For any given locations of the firms A and B , at any point either the direction of the shortest path to firm A is opposite to the direction of the shortest path to firm B , or the point belongs to the hinterland of one of the firms. This rules out the possibility of having a positive mass of indifferent consumers.

(a) Price distribution.

(b) Expected profit as a function of price.

Figure 10: The symmetric mixed-strategy equilibrium of Figure 8 ($\phi = \varepsilon = \xi = \frac{3}{16}$ and $\delta = \frac{1}{4}$).

5 Summary and discussion

This paper is a first contribution to the analysis of graph models of price competition. The algorithm introduced allows one to numerically evaluate firm-level demand and profits for all graphs where consumers are uniformly distributed along the edges and face quadratic transportation cost and where two firms compete in prices conditional on their location. One important phenomenon for these type of models is that spatial discontinuities in demand may occur. The most important result is that the existence result by d'Aspremont et al. (1979) for the $K_{1,2}$ graph does not extend to the $K_{1,3}$ graph, arguably the most straightforward extension of the original model.

We believe that the framework presented in this paper offers ample scope for future research. Natural directions for further investigation include the analysis of markets with three or more firms, issues related to

endogenous entry and markets where consumers face non-linear, but not necessarily quadratic transportation costs. Furthermore, whereas the present paper presents numerical evaluations for a number of specific graphs, it is worthwhile to investigate more systematically the relationship between graph characteristics, firm locations within the graph and pricing equilibria. One of the results in d’Aspremont et al. (1979) is that for the line model with linear transportation cost, pure-strategy equilibria exist if firms are far enough apart. Are there classes of graphs for which a similar result can be obtained?

Another avenue for research is the study of the relationship between graph characteristics, firm location and the occurrence and characteristics (length, amplitude, symmetry) of price cycles. Theoretical Edgeworth cycles, first described by Edgeworth (1925) and given a solid game-theoretic foundations by Maskin and Tirole (1988), are characterized by strongly asymmetric periods of price cuts followed by a rapid price increase. Theoretically, Edgeworth price cycles are most likely to occur in markets characterized by homogenous goods and extremely price-sensitive consumers. Consistent with this, one particular market in which asymmetric price cycles have been consistently found is the market for retail gasoline. Typically, these studies start with the observation of price cycles in a certain market, verify whether or not the cycles are asymmetric, and, conditional on finding asymmetries, look for the possible causes.¹⁷ Noel (2009) for example decomposes asymmetric price cycles into a part that can be explained by Edgeworth cycles and a part driven by other unknown sources. Less attention has been paid to why some firms are cycling and other are not. Exceptions are Noel (2007a) and de Roos and Katayama (2013) who use a Markov switching-regression model and allow for differences in the price cycles of major firms and independents. The location of a firm on a given road network relative the location of its competitors might be an important additional variable explaining the occurrence and shape of these price cycles.

A Proofs

A.1 Proof of Proposition 1

The “only if”-part is proved as follows. Suppose there is an interval of indifferent consumers and let z be a location on this interval. Then for ε

¹⁷These empirical studies give evidence for price cycles in the US (Castanias and Johnson, 1993; Lewis, 2011; Lewis and Noel, 2011; Noel, 2007a,b; Eckert, 2003; Wang, 2009; de Roos and Katayama, 2013). Bachmeier and Griffin (2003) do not uncover asymmetric cycles.

sufficiently close to zero, we have

$$\begin{aligned} f(z) &= f(z_\varepsilon), \\ 0 &= 2\ell_{ij}d_A(z)\square_A + 2\ell_{ij}d_B(z)\square_B. \end{aligned}$$

Note that the RHS can only be equal to zero when $d_A(z) = d_B(z)$ and the shortest paths have the same direction. Since $d_A(z) = d_B(z)$ and z is the location of an indifferent consumer, it follows that $\mu = 0$.

The proof of the if-part is as follows. Suppose there exists $z \in ij$ such that $d_A(z) = d_B(z)$, then $f(z) = 0$. Since $\mu = 0$, the consumer located at z is indifferent. Since the shortest paths have the same direction, from (5) we get,

$$f(z_\varepsilon) = f(z) \pm 2\ell_{ij}\varepsilon[d_A(z) - d_B(z)] = 0.$$

Hence all consumers in the neighborhood of z are also indifferent. $\square$

A.2 Proof of Proposition 2

Since change points are precisely those points where the direction of the shortest path changes, it is clear that if there exists $z \in ij$ such that $d_A(z) = d_B(z)$ in between a node and a change point, then all consumers on that part of the edge are indifferent. Moreover, on an edge without change points, the entire edge will be indifferent. We have to show that it is not possible for consumers located between two change points to be indifferent. Let z_A denote the change point for the shortest path to A , and z_B the change point for the shortest path to B . Assume without loss of generality that z_A is closer to x_i than z_B . Points in between z_A and z_B are closer to x_j than z_A and closer to x_i than z_B . Since the direction of the shortest path changes at z_A and z_B , we conclude that for points in between z_A and z_B , the direction of the shortest path to A goes via x_j and the direction of the shortest path to B goes via x_i . Hence, the shortest paths have different direction and consumers located between z_A and z_B are not indifferent. $\square$

A.3 Proof of Proposition 3

The proof of the first part is by contradiction. Suppose z is a weak indifference point (for some value of μ) but not a change point. There are four different cases (depending on the direction of the shortest paths). We focus on the case where, at z , the shortest path to both firm A and B leaves the edge via node i , the proof for the other three cases is similar. Then, since z is assumed not to be a change point,

$$\begin{aligned} d_A(z_\varepsilon) &= d_A(z) + \varepsilon\ell_{ij} \\ d_B(z_\varepsilon) &= d_B(z) + \varepsilon\ell_{ij} \end{aligned}$$

and, consequently,

$$f(z_\varepsilon) = f(z) + 2\ell_{ij}\varepsilon[d_A(z) - d_B(z)] = \mu + 2\ell_{ij}\varepsilon[d_A(z) - d_B(z)].$$

Note that $d_A(z) \neq d_B(z)$ by assumption. If $d_A(z) > d_B(z)$, then $f(z_\varepsilon) < \mu$ for $\varepsilon < 0$ and $f(z_\varepsilon) > \mu$ for $\varepsilon > 0$. If $d_A(z) < d_B(z)$, then $f(z_\varepsilon) < \mu$ for $\varepsilon > 0$ and $f(z_\varepsilon) > \mu$ for $\varepsilon < 0$. In both cases, z separates consumers who buy from firm A from those who buy from firm B . It follows that z is a strong indifference point and therefore not weak.

Suppose z is a change point for firm A and $f(z) > 0$. In that case (1) $d_A(z_\varepsilon) = d_A(z) - |\varepsilon|\ell_{ij}$ and (2) $d_A(z) > d_B(z)$. We focus on the case where $d_B(z_\varepsilon) = d_B(z) + \varepsilon\ell_{ij}$. The other case is similar. Then

$$f(z_\varepsilon) = f(z) - 2\ell_{ij}[d_A(z)|\varepsilon| + d_B(z)\varepsilon]$$

It follows that $f(z_\varepsilon) < f(z)$ for both $\varepsilon > 0$ and $\varepsilon < 0$. Hence for $\mu = f(z)$, z is a weak indifference point. $\square$

A.4 Proof of Proposition 5

The first properties follow directly from the properties of demand on a single edge.

Ad (1) and (2): Note that there exists a location z^* which minimizes f on $\mathcal{G}$. Set $\check{\mu} = f(z^*)$. Consequently $f(z) \geq \check{\mu}$ for all $z \in \mathcal{G}$. Hence $D(\check{\mu}) = 0$. Note that $\check{\mu}$ is finite. To show that $\check{\mu}$ is non-positive, observe that $f(z^*) \leq f(x_1) \leq 0$ since x_1 is the location of firm A . The proof for $\hat{\mu}$ is analogous.

Ad (3): From (6) we derive that the slope changes only if μ is such that the position of the indifferent consumer coincides with a change point, or when a small change in μ leads the indifferent consumer to move to another edge (with possibly a different length). Since there are at most two change points per edge and at most two points where the indifferent consumer can drift off the edge, the number of non-differentiable points is at most 4 times the number of edges. $\square$

A.5 Proof of Theorem 6

Denote the equilibrium prices by p_A^* and p_B^* and the price difference by $\mu^* = p_B^* - p_A^*$. First, we show that the Nash-equilibrium is at a point where the demand function is differentiable.

Suppose, by contradiction, that the Nash-equilibrium is at a point where the demand function is not differentiable. Let $D'_L(\mu^*)$ denote the value of left derivate at this point and $D'_R(\mu^*)$ denote the value of the right derivative. By the properties of the demand function, both derivatives are well-defined. A necessary condition for p_A^* to be a best-response to p_B^* is that left derivative

of the profit function is non-decreasing and the right derivative of the profit function is non-increasing:

$$\begin{aligned} -D'_R(\mu^*)p_A^* + D(\mu^*) &\geq 0 \\ -D'_L(\mu^*)p_A^* + D(\mu^*) &\leq 0 \end{aligned}$$

Observe that the expression for the left derivative of the profit function contains the right derivative of the demand function (and vice versa for the other expression). The reason for this is that the left derivative is the increase in profit if firm A charges a slightly lower price than p_A^* , which means that the slope of the demand function is evaluated at a slightly higher value than μ^* , which is the right derivative of the demand function (idem for the other expression). Note that both inequalities can only hold when $D'_R(\mu^*) \leq D'_L(\mu^*)$.

Analogously, a necessary condition for p_B^* to be a best-response to p_A^* is:

$$\begin{aligned} -D'_L(\mu^*)p_B^* + 1 - D(\mu^*) &\geq 0 \\ -D'_R(\mu^*)p_B^* + 1 - D(\mu^*) &\leq 0 \end{aligned}$$

Observe that now the left derivative of the profit function contains the left derivative of the demand function (idem for the other expression). The reason for this is that the left derivative is the increase in profit if firm B charges a slightly lower price than p_B^* , which means that the slope of the demand function is evaluated at a slightly lower value than μ^* , which is the left derivative of the demand function (idem for the other expression). Note that both inequalities can only hold when $D'_R(\mu^*) \geq D'_L(\mu^*)$. Hence $D'_R(\mu^*) = D'_L(\mu^*)$, i.e. $D(\cdot)$ is differentiable at μ^* , and the Nash-equilibrium can only be at a point where the demand function is differentiable.

The necessary conditions for profit maximization then simplify to:

$$D'(\mu^*)p_A^* = D(\mu^*) \text{ and } D'(\mu^*)p_B^* = 1 - D(\mu^*).$$

By taking the difference of these conditions, we obtain:

$$D'(\mu^*)\mu^* = 1 - 2D(\mu^*). \tag{7}$$

If $\mu^* < 0$, then $D(\mu^*) \leq D(0^-) < \frac{1}{2}$ by assumption. This implies that the LHS of (7) is strictly negative, while the RHS is strictly positive. This implies that there is no equilibrium where $\mu^* < 0$. If $\mu^* > 0$, then $D(\mu^*) \geq D(0^+) > \frac{1}{2}$ by assumption. This implies that the LHS of (7) is strictly positive, while the RHS is strictly negative. This implies that there is no equilibrium where $\mu^* > 0$.

Finally, for sake of completeness, if $\mu^* = 0$, then $\lim_{\varepsilon \downarrow 0} D(\varepsilon)(p_A^* - \varepsilon) = D(0^+)p_A^* > D(0)p_A^*$ and firm A profits from this deviation if ε is sufficiently small. Hence there cannot be a Nash-equilibrium for this value of μ^* as well.

□

A.6 Proof of Theorem 8

First, we give the proof for a graph which contains a node with degree 3. Then we generalize the result to graphs which only contain nodes of degree 4 or higher.

Take any node with degree 3, denoted by N . There are three edges connected to this node, which we denote by E_1 , E_2 and E_3 . The consumers are divided into several categories. There are consumers for which every shortest path to N goes over E_i (where $i = 1, 2, 3$). Let $\delta_i > 0$ denote the fraction of consumers for which this is true. There may be consumers who have shortest paths to N via E_1 and shortest paths via E_2 (mass $\delta_{12} \geq 0$). We define $\delta_{13} \geq 0$ and $\delta_{23} \geq 0$ in a similar way. Finally, there may exist a group of consumers who have shortest paths to N via E_1 , via E_2 and via E_3 (mass δ_{123}). For the moment, we assume that $\delta_{123} = 0$. Obviously, $\delta_1 + \delta_2 + \delta_3 + \delta_{12} + \delta_{13} + \delta_{23} = 1$.

The last thing we need to keep track of are change points. Suppose there are k_{12} change points on $\mathcal{G}$ which separate the part of the graph where the shortest paths to N go only via E_1 and the part of the graph where the shortest paths to N go only via E_2 .¹⁸ If $k_{12} > 0$, then we refer to these change points as C_ℓ^{12} (where $\ell = 1, \dots, k_{12}$). For $\ell = 1, \dots, k_{12}$, we define $m_{12}^{\ell,1}$ as the number of edges originating from change point C_ℓ^{12} where the direction of the shortest path to N is away from C_ℓ^{12} and goes via E_1 . Note that if the change point is not located at a node, then $m_{12}^{\ell,1} = 1$. Similarly, $m_{12}^{\ell,2}$ are the number of edges originating from change point C_ℓ^{12} where the direction of the shortest path to N is away from C_ℓ^{12} and goes via E_2 . We define k_{13} and k_{23} etc. in a similar fashion.

Suppose that we consider three potential locations for the two firms, all located a distance ε away from N along the three separate nodes, where ε is positive but sufficiently small. In order to use Theorem 6, we need to calculate $D(0^-)$ and $D(0^+)$ given the locations of the firms. Note that $D(0^-)$ corresponds to the demand from consumers who buy from A despite the fact that A charges an infinitesimally higher price than B , i.e. all consumers who are located (strictly) closer to A than to B . When firm A charges an infinitesimally lower price, the demand for its product increases because the consumers who are equidistant from A and B now also buy from A : this is $D(0^+)$.

To give an example, suppose that firm A is located on E_1 and firm B is located on E_2 (both a distance ε from N). For the consumers for which every shortest path to N goes over E_1 , it is optimal to buy from firm 1 even if it charges a slightly lower price (mass δ_1). The same is true for consumers who have shortest paths to N via E_1 and shortest paths via E_3

¹⁸Note that this is a slight abuse of the terminology since change points were defined with reference to a firm.

(mass δ_{13}). Consumers for whom every shortest path to N leads via E_3 are located equidistant from A and B , namely the distance to N plus ε . Finally, the consumers with shortest paths to N leading via E_1 and shortest paths leading via E_2 are also equidistant from both firms (mass δ_{12}). It follows that if firm A is located on E_1 and firm B is located on E_2 , then $D(0^-) = \delta_1 + \delta_{13}$ and $D(0^+) = \delta_1 + \delta_{13} + \delta_3 + \delta_{12}$.

Similarly, if firm A is located on E_1 and firm B is located on E_3 , then $D(0^-) = \delta_1 + \delta_{12}$ and $D(0^+) = \delta_1 + \delta_{12} + \delta_2 + \delta_{13}$. And if firm A is located on E_2 and firm B is located on E_3 , then $D(0^-) = \delta_2 + \delta_{12}$ and $D(0^+) = \delta_2 + \delta_{12} + \delta_1 + \delta_{23}$.¹⁹

The presence of change points leads to some slight complications. Going back to the example, where firm A is located on E_1 and firm B is located on E_2 (both a distance ε from N), we see that the consumers located at C_{13}^ℓ (with $\ell = 1, \dots, k_{13}$) are actually a distance 2ε closer to firm A than to firm B . This means that the points where the consumer is equidistant from firm A and firm B shift a distance ε away from C_{13}^ℓ unto the edges where the shortest paths to N go via E_3 . Note that the consumers in between the equidistant consumers and the consumers located at C_{13}^ℓ are closer to firm A than to firm B and will buy from firm A if this firm charges a infinitesimally higher price. Since there are $\sum_{\ell=1}^{k_{13}} m_{13}^{\ell,3}$ of these edges, $D(0^-)$ increases by $\varepsilon \sum_{\ell=1}^{k_{13}} m_{13}^{\ell,3}$. Similarly, the consumers located at C_{23}^ℓ (with $\ell = 1, \dots, k_{23}$) are a distance 2ε closer to firm B than to firm A . This means that the points where the consumer is equidistant from firm A and firm B shift a distance ε away from C_{23}^ℓ unto the edges where the shortest paths to N go via E_3 . Note that the consumers in between the equidistant consumers and the consumers located at C_{23}^ℓ are closer to firm B than to firm A and will not buy from firm A if this firm charges a infinitesimally lower price. Since there are $\sum_{\ell=1}^{k_{23}} m_{23}^{\ell,3}$ of these edges, $D(0^+)$ decreases by $\varepsilon \sum_{\ell=1}^{k_{23}} m_{23}^{\ell,3}$. In the next paragraph, we will examine whether there exist firm locations such that $D(0^-) < 1/2$ and $D(0^+) > 1/2$. Since these inequalities are strict, as long as ε is sufficiently small, we can ignore the fact that demand is either increased by $\varepsilon \sum_{\ell=1}^{k_{13}} m_{13}^{\ell,3}$ or decreased by $\varepsilon \sum_{\ell=1}^{k_{23}} m_{23}^{\ell,3}$. To conclude, the change points present no difficulties.

We claim that the sufficient conditions in Theorem 6 for non-existence, i.e. $D(0^-) < 1/2$ and $D(0^+) > 1/2$, will apply for one of the possible firm locations. Suppose otherwise, i.e. for all possible firm locations either

¹⁹The three other combinations, where the positions of firm A and firm B are switched, are equivalent after appropriate relabeling.

$D(0^-) \geq 1/2$ or $D(0^+) \leq 1/2$:

$$\delta_1 + \delta_{13} \geq \frac{1}{2} \quad \text{or} \quad \delta_1 + \delta_{13} + \delta_3 + \delta_{12} \leq \frac{1}{2} \quad (8)$$

$$\text{and } \delta_1 + \delta_{12} \geq \frac{1}{2} \quad \text{or} \quad \delta_1 + \delta_{12} + \delta_2 + \delta_{13} \leq \frac{1}{2} \quad (9)$$

$$\text{and } \delta_2 + \delta_{12} \geq \frac{1}{2} \quad \text{or} \quad \delta_2 + \delta_{12} + \delta_1 + \delta_{23} \leq \frac{1}{2} \quad (10)$$

To complete the proof we need to check, case by case, that it indeed impossible to have values of δ_1 , δ_2 , δ_3 , δ_{12} , δ_{13} , and δ_{23} such that the inequalities all hold.

We show how to establish a contradiction in the first case, the details for the other seven cases are available upon request. In the first case,

$$\delta_1 + \delta_{13} \geq \frac{1}{2} \quad (11)$$

$$\delta_1 + \delta_{12} \geq \frac{1}{2} \quad (12)$$

$$\delta_2 + \delta_{12} \geq \frac{1}{2} \quad (13)$$

Note that adding (11) and (13) yields

$$\delta_1 + \delta_{13} + \delta_2 + \delta_{12} \geq 1.$$

Since $\delta_1 + \delta_2 + \delta_3 + \delta_{12} + \delta_{13} + \delta_{23} = 1$, this implies that

$$\delta_3 + \delta_{23} \leq 0,$$

which contradicts the fact that $\delta_3 > 0$.

The other seven cases proceed along similar lines. This establishes that close to a node with degree 3 we can find firm locations such that the conditions of Theorem 6 apply and, hence, a pure-strategy Nash-equilibrium does not exist.

Next, suppose that $\delta_{123} > 0$. Note that if $\delta_{123} > 0$ and if the firms are located a distance ε from N , then these consumers are equidistant from firm A and firm B and will only buy from firm A if it charges the lowest price. This leads to an increase in $D(0^+)$ and it will become more difficult to satisfy the constraint $D(0^+) \leq 1/2$. Given the fact that the system of inequalities in (8-10) had no solution for the case $\delta_{123} = 0$, the same will be true when $\delta_{123} > 0$.

Finally, suppose that the graph has a node with degree 4 or higher (but not a node with degree equal to 3). Take any node N , with degree 4 or higher and any 3 edges connecting to N and call them E_1 , E_2 and E_3 as before. The node will also have additional edges E_4 , E_5 etcetera. Define δ_i as the mass of consumers with either all shortest path to N going via

E_i or consumers whose shortest paths go via E_i, E_4, E_5 etcetera (where $i = 1, 2, 3$). Define δ_{12} as the mass of consumers with shortest paths that lead to N via E_1, E_2, E_4, E_5 etcetera. We define δ_{13} and δ_{23} in a similar manner. Finally, define δ_{123} as the mass of consumers whose shortest paths to N do not lead through E_1, E_2 or E_3 . Then, by an analogous argument as before, we can find firm locations such that the conditions of Theorem 6 apply and, hence, a pure-strategy Nash-equilibrium does not exist. $\square$

B Non-existence of a pure-strategy Nash-equilibrium when demand is continuous

Figure 11 shows a graph that does not strictly adhere to our model, but is useful to illustrate how continuity of demand is not sufficient to guarantee the existence of pure-strategy equilibria. Note that if the indifference point is between A and C or between D and B , it is unique. If there is an indifference point between C and D instead, there necessarily are m of these points. Consequently, the slope of the demand curve is m times greater when the price differential μ is such that the indifferent consumers are located between C and D .

Figure 11: An example where demand is continuous

In order to analyze this game, we introduce a coordinate x , such that firm A is located at $x = 0$, node C at $x = \frac{1}{3}$, node D at $x = \frac{1}{3} + \frac{1}{3m}$ and firm B at $x = \frac{2}{3} + \frac{1}{3m}$. For $\frac{1}{3} \leq x \leq \frac{1}{3} + \frac{1}{3m}$, x represents a location on each

of the m branches. It is straightforward but tedious to show that:

$$D(\mu) = \begin{cases} 0 & \text{if } \mu \leq -\left(\frac{2m+1}{3m}\right)^2 \\ \hat{x} & \text{if } -\left(\frac{2m+1}{3m}\right)^2 < \mu \leq -\frac{2m+1}{(3m)^2} \\ \frac{1}{3} + m\left(\hat{x} - \frac{1}{3}\right) & \text{if } -\frac{2m+1}{(3m)^2} < \mu \leq \frac{2m+1}{(3m)^2} \\ \frac{1}{3} + m\left(\hat{x} - \frac{1}{3} - \frac{1}{3m}\right) & \text{if } \frac{2m+1}{(3m)^2} < \mu \leq \left(\frac{2m+1}{3m}\right)^2 \\ 1 & \text{if } \mu > \left(\frac{2m+1}{3m}\right)^2 \end{cases} \quad (14)$$

where

$$\hat{x} = \frac{2m+1}{6m} + \frac{3m}{2(2m+1)}\mu.$$

Denote the equilibrium prices by p_A^* and p_B^* and the price difference by $\mu^* = p_B^* - p_A^*$. From the proof of Theorem 6, we know that the equilibrium is at a point where the demand function is differentiable and where the following condition holds:

$$D'(\mu^*)\mu^* = 1 - 2D(\mu^*). \quad (15)$$

Note that if $\mu^* < 0$, then $D(\mu^*) > \frac{1}{2}$ if (15) is supposed to hold. However, $\mu^* < 0$ implies that $p_A^* > p_B^*$ such that, due to the firms' position in the graph being symmetric, $D(\mu^*) < \frac{1}{2}$. Hence $\mu^* = 0$ is the only candidate solution. One can verify that

$$p_A^* = p_B^* = \frac{2m+1}{3m^2},$$

which tends to zero as $m \rightarrow \infty$. Since demand is equal to $\frac{1}{2}$, we see that profit tends to zero as well. Clearly if $m = 1$, this is a Nash-equilibrium (this is the Hotelling-line). However, if m is sufficiently large, then it will be profitable for firm A to deviate to $p_A = \frac{2}{9}$ (for instance). Note that as $m \rightarrow \infty$, $D\left(-\frac{2}{9}\right) = \hat{x} = \frac{1}{6}$. Hence the profit of deviating is $\frac{1}{6} \times \frac{2}{9} > 0$. So given sufficiently many branches, there is no Nash-equilibrium at a point where the demand function is differentiable.

This establishes that given sufficiently many branches, there is no pure-strategy Nash-equilibrium. The reason is that as m gets large, the demand function around $\mu = 0$ becomes very steep. This means that the equilibrium price must decrease (since demand is very elastic). But at the same time firms can keep the demand of a non-negligible part of the consumers by raising the price. This destabilizes the pure-strategy equilibrium.

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