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A Distance-Based Approach to Strong Target Control of Dynamical Networks

Henk J. van Waarde^{1b}, M. Kanat Camlibel^{1b}, *Member, IEEE*, and Harry L. Trentelman^{1b}, *Fellow, IEEE*

Abstract—This paper deals with controllability of dynamical networks. It is often unfeasible or unnecessary to fully control large-scale networks, which motivates the control of a prescribed subset of agents of the network. This specific form of output control is known under the name target control. We consider target control of a family of linear control systems associated with a network, and provide both a necessary and a sufficient topological condition under which the network is strongly targeted controllable. Furthermore, a leader selection algorithm is presented to compute leader sets achieving target control.

Index Terms—Dynamical networks, leader selection algorithm, output controllability, target control, zero forcing.

I. INTRODUCTION

DURING the last two decades, networks of dynamical agents have been extensively studied. It is customary to represent the infrastructure of such networks by a graph, where nodes are identified with agents and arcs correspond to the communication between agents. In the study of controllability of networks, two types of nodes are distinguished: Leaders, which are influenced by external input, and followers whose dynamics are completely determined by the behavior of their neighbors. Network controllability comprises the ability to drive the states of all nodes of the network to any desired state, by applying appropriate input to the leaders.

Motivated by model uncertainties, the notion of structural controllability of linear control systems described by the pair (A, B) was introduced by Lin [12]. Here the entries of the matrices A and B are either fixed zeros or free parameters. In this framework, weak structural controllability requires almost all realizations of (A, B) to be controllable. That is, for almost all parameter settings of the entries of A and B , the pair (A, B) is controllable. Lin provided a graph-theoretic condition under which (A, B) is weakly structurally controllable in the single-input case. Many papers followed [12], amongst others we name Glover and Silverman [9] and Shields and Pearson [24] in which

extensions to multiple leaders are given, and the article [14], that introduces strong structural controllability, which requires all realizations of (A, B) to be controllable.

In recent years, structural controllability gained much attention in the study of networks of dynamical agents [2], [3], [13], [17], [27]. With a given network graph, a family of linear control systems is associated, where the structure of the state matrix of each system depends on the network topology, and the input matrix is determined by the leader set. In this framework, a network is said to be weakly (strongly) structurally controllable if almost all (all) systems associated with the network are controllable. The graph-theoretic results obtained in classical papers [12], [14] lend themselves excellently to the study of structural controllability of networks. A topological condition for (weak) structural controllability of networks is given in terms of maximum matchings in [13], while strong structural controllability is fully characterized in terms of zero forcing sets in [17], and in terms of constrained matchings in [3].

However, in large-scale networks with high vertex degrees, a substantial amount of nodes must be chosen as leader to achieve full control in the strong sense, which is often unfeasible. Furthermore, in some applications full control over the network is unnecessary. Hence, we are interested in controlling a subset of agents, called target nodes. This specific form of output control is known under the name target control [8], [16]. Potential applications of target control within the areas of biology, chemical engineering and economic networks are identified in [8].

A network is said to be strongly targeted controllable if all systems in the family associated with the network graph are targeted controllable. In this paper, we consider strong targeted controllability for the class of state matrices called distance-information preserving matrices. The adjacency matrix and symmetric, indegree, and outdegree Laplacian matrices are examples of distance-information preserving matrices. As these matrices are often used to describe network dynamics (see, e.g., [6], [10], [22], [26], [31]), distance-information preserving matrices form an important class of matrices associated with network graphs.

Our main results are threefold. First, we provide a sufficient topological condition for strong targeted controllability of networks, that generalizes the results of [16] for the class of distance-information preserving matrices. Specifically, the results of [16] are restricted to target nodes having distance one with respect to the so-called derived set of the leaders. However, our result is applicable to target nodes that have arbitrary distance with respect to the leaders. Our sufficient

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topological condition can be understood as a “k-walk theory” [8] for strong targeted controllability. However, we remark that the k-walk theory for (weak) targeted controllability established in [8, Th. 2] is only applicable to single-input directed tree networks. On the other hand, our result is applicable to arbitrary directed networks with multiple leaders.

Second, noting that our proposed sufficient condition for target control is not a one-to-one correspondence, we establish a necessary graph-theoretic condition for strong targeted controllability.

Finally, we consider the minimum leader selection problem in the context of strong targeted controllability of networks. Recently, leader selection (and in general, actuator placement) has received much attention in the literature. Minimum actuator placement in the context of controllability is studied in [18], [19], [25], and [30]. Also, minimum actuator placement for reachability problems was studied [29]. Moreover, the selection of minimum input sets achieving (strong) structural controllability has been considered in [3], [13], [20] and [21]. It was shown [3] that determining minimum input sets for strong structural controllability is NP-hard in general, although a subclass of problems is polynomially solvable [21]. In the context of (weak) targeted controllability, a leader selection method has been given in [8]. To the best of the authors’ knowledge, this paper is the first to consider leader selection for *strong* targeted controllability of networks. We first prove that there exists no polynomial-time algorithm to determine minimum leader sets achieving strong target control (assuming $P \neq NP$). Subsequently, we provide a heuristic two-phase leader selection algorithm consisting of a binary linear programming phase and a greedy approach to obtain leader sets achieving strong target control.

The paper is organized as follows: In Section II, we introduce preliminaries and notation. Subsequently, the problem is stated in Section III. Our main results are presented in Section IV. To illustrate the main results, an example is given in Section V. Finally, Section VI contains our conclusions.

II. PRELIMINARIES

Consider a directed graph $G = (V, E)$, where V is a set of n vertices, and E is the set of directed arcs. The cardinality of a vertex set V' is denoted by $|V'|$. Throughout this paper, all graphs are assumed to be simple and without self-loops.

We define the *distance* $d(u, v)$ between two vertices $u, v \in V$ as the length of the shortest path from u to v . If there does not exist a path in the graph G from vertex u to v , the distance $d(u, v)$ is defined as infinite. Moreover, the distance from a vertex to itself is equal to zero.

For a nonempty subset $S \subseteq V$ and a vertex $j \in V$, the distance from S to j is defined as

$$d(S, j) := \min_{i \in S} d(i, j). \quad (1)$$

A directed graph $G = (V, E)$ is called *bipartite* if there exist disjoint sets of vertices V^- and V^+ such that $V = V^- \cup V^+$ and $(u, v) \in E$ only if $u \in V^-$ and $v \in V^+$. We denote bipartite

graphs by $G = (V^-, V^+, E)$, to indicate the partition of the vertex set.

A. Qualitative Class and Pattern Class

The *qualitative class* of a directed graph G is a family of matrices associated with the graph. Each of the matrices of this class contains a nonzero element in position i, j if and only if there is an arc (j, i) in G , for $i \neq j$. More explicitly, the qualitative class $\mathcal{Q}(G)$ of a graph G is given by

$$\mathcal{Q}(G) = \{X \in \mathbb{R}^{n \times n} \mid \text{for } i \neq j, X_{ij} \neq 0 \iff (j, i) \in E\}.$$

Note that the diagonal elements of a matrix $X \in \mathcal{Q}(G)$ do not depend on the structure of G , these are “free elements” in the sense that they can be either zero or nonzero.

Next, we look at a different class of matrices associated with a bipartite graph $G = (V^-, V^+, E)$, where the vertex sets V^- and V^+ are given by

$$\begin{aligned} V^- &= \{r_1, r_2, \dots, r_s\} \\ V^+ &= \{q_1, q_2, \dots, q_t\}. \end{aligned} \quad (2)$$

The *pattern class* $\mathcal{P}(G)$ of the bipartite graph G , with vertex sets V^- and V^+ given by (2), is defined as

$$\mathcal{P}(G) = \{M \in \mathbb{R}^{t \times s} \mid M_{ij} \neq 0 \iff (r_j, q_i) \in E\}. \quad (3)$$

Note that the cardinalities of V^- and V^+ can differ, hence the matrices in the pattern class $\mathcal{P}(G)$ are not necessarily square.

B. Subclass of Distance-Information Preserving Matrices

In this section, we investigate properties of the powers of matrices belonging to the qualitative class $\mathcal{Q}(G)$. The relevance of these properties will become apparent later on, when we provide a graph-theoretic condition for targeted controllability of systems defined on graphs.

We first provide the following lemma, which states that if the distance between two nodes is greater than k , the corresponding element in X^k is zero.

Lemma 1: Consider a directed graph $G = (V, E)$, two distinct vertices $i, j \in V$, a matrix $X \in \mathcal{Q}(G)$ and a positive integer k . If $d(j, i) > k$, then $(X^k)_{ij} = 0$.

Proof: The proof follows easily by induction on k , and is therefore omitted. ■

Subsequently, we consider the class of matrices for which $(X^k)_{ij}$ is nonzero if the distance $d(j, i)$ is exactly equal to k . Such matrices are called *distance-information preserving*, more precisely:

Definition 1: Consider a directed graph $G = (V, E)$. A matrix $X \in \mathcal{Q}(G)$ is called *distance-information preserving* if for any two distinct vertices $i, j \in V$ we have that $d(j, i) = k$ implies $(X^k)_{ij} \neq 0$.

Although the distance-information preserving property does not hold for all matrices $X \in \mathcal{Q}(G)$, it does hold for the adjacency and Laplacian matrices [23]. Because these matrices are often used to describe network dynamics, distance-information

Fig. 1. Graph G . Fig. 2. Force $2 \rightarrow 4$. Fig. 3. Force $4 \rightarrow 5$.

preserving matrices form an important subclass of $\mathcal{Q}(G)$, which from now on will be denoted by $\mathcal{Q}_d(G)$.

C. Zero Forcing Sets

In this section, we review the notion of zero forcing. The reason for this is the correspondence between zero forcing sets and the sets of leaders rendering a system defined on a graph controllable. More on this will follow in the next section.

For now, let $G = (V, E)$ be a directed graph with vertices colored either black or white. The *color-change rule* is defined in the following way: If $u \in V$ is a black vertex and exactly one out-neighbor $v \in V$ of u is white, then change the color of v to black [11].

When the color-change rule is applied to u to change the color of v , we say u forces v , and write $u \rightarrow v$.

Given a coloring of G , that is, given a set $C \subseteq V$ containing black vertices only, and a set $V \setminus C$ consisting of only white vertices, the *derived set* $D(C)$ is the set of black vertices obtained by applying the color-change rule until no more changes are possible [11].

A *zero forcing set* for G is a subset of vertices $Z \subseteq V$ such that if initially the vertices in Z are colored black and the remaining vertices are colored white, then $D(Z) = V$.

The *zero forcing number* $\rho(G)$ of the graph $G = (V, E)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V$. Moreover, a zero forcing set $Z \subseteq V$ is called a *minimum zero forcing set* if $|Z|$ equals $\rho(G)$.

Finally, for a given zero forcing set, we can construct the derived set, listing the forces in the order in which they were performed. This list is called a *chronological list of forces*. Note that such a list does not have to be unique.

Example 1: Consider the directed graph $G = (V, E)$ depicted in Fig. 1 and let $C = \{2\}$. Note that vertex 2 can force 4 (Fig. 2), and subsequently node 4 can force 5 (Fig. 3). No further color changes can be made, so $D(C) = \{2, 4, 5\}$. As $D(C) \neq V$, C is not a zero forcing set. However, suppose we

choose $C = \{1, 2\}$. In this case it is easy to see that we can color all vertices black, hence $C = \{1, 2\}$ is a zero forcing set.

D. Output Controllability of Linear Systems

In this section, we review the notion of output controllability for linear, time-invariant systems. This notion will become useful in the next section, where we will discuss targeted controllability of systems defined on graphs.

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (4)$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the input and $t \in \mathbb{R}_{\geq 0}$ denotes time. The real matrices A and B are of appropriate dimensions. For a given initial condition $x_0 \in \mathbb{R}^n$ and input function u , we denote the state of (4) at time t by $x_u(t, x_0)$. It is well known that system (4) is called *controllable* if for all $x_0, x_1 \in \mathbb{R}^n$, there exists an input function u and a finite time T , such that $x_u(T, x_0) = x_1$. In the case (4) is controllable, we say the pair (A, B) is controllable. If in addition to (4) we specify an output equation $y(t) = Cx(t)$, we obtain the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (5)$$

where $y \in \mathbb{R}^p$ is the output of the system and $C \in \mathbb{R}^{p \times n}$. For a given initial condition x_0 and input function u , we denote the output of (5) at time t by $y_u(t, x_0)$. We are now ready to introduce the notion of output controllability.

Definition 2: [28] System (5) is called *output controllable* if for all $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$, there exists an input function u and finite time T such that $y_u(T, x_0) = y_1$.

In the case system (5) is output controllable, we say (A, B, C) is output controllable.

E. Targeted Controllability of Systems Defined on Graphs

Consider a directed graph $G = (V, E)$, where the vertex set is given by $V = \{1, 2, \dots, n\}$. Furthermore, let $V' = \{v_1, v_2, \dots, v_r\} \subseteq V$ be a subset. The $n \times r$ matrix $P(V; V')$ is defined by

$$P_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

We now introduce the subset $V_L \subseteq V$ consisting of so-called *leader nodes*, i.e., agents of the network to which an external control input is applied. The remaining nodes $V \setminus V_L$ are called *followers*. We consider finite-dimensional linear time-invariant systems of the form

$$\dot{x}(t) = Xx(t) + Uu(t) \quad (7)$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input of the system. Here $X \in \mathcal{Q}(G)$ and $U = P(V; V_L)$, for some leader set $V_L \subseteq V$. An important notion regarding systems of the form (7) is the notion of strong structural controllability.

Definition 3: [17] A system of the form (7) is called *strongly structurally controllable* if the pair (X, U) is controllable for all $X \in \mathcal{Q}(G)$.

In the case that (7) is strongly structurally controllable, we say $(G; V_L)$ is controllable, with a slight abuse of terminology. There is a one-to-one correspondence between strong structural controllability and zero forcing sets, as stated in the following theorem.

Theorem 2: [17] Let $G = (V, E)$ be a directed graph and let $V_L \subseteq V$ be a leader set. Then $(G; V_L)$ is controllable if and only if V_L is a zero forcing set.

Remark 1: In the context of minimum leader selection, it is particularly interesting to compute *minimum* zero forcing sets. Unfortunately, the problem of computing a minimum zero forcing set has been shown to be NP-hard [1]. In fact, the inapproximability of a related problem suggests that it is even hard to *approximate* the minimum *number* of leaders (i.e., the zero forcing number $\rho(G)$). Following Trefois *et al.* [27], it can be shown that in a directed bipartite graph G with n vertices, we have that $n - \rho(G)$ is equal to the so-called maximum size *constrained matching* in G . It has been shown in [15] that it is hard to approximate the maximum size constrained matching in bipartite graphs within a factor $\mathcal{O}(n^{\frac{1}{3}-\epsilon})$ for any $\epsilon > 0$ (unless $\text{NP} = \text{ZPP}$). This suggests that also the zero forcing number (and thereby, the minimum number of leaders) cannot be approximated in polynomial time within a large factor. The above considerations are for bipartite graphs. However, note that if the zero forcing number is hard to approximate in bipartite graphs, it is certainly hard to approximate in general directed graphs.

In this paper, we are primarily interested in controlling the states of a subset $V_T \subseteq V$ of nodes, called *target nodes*. We specify an output equation $y(t) = Hx(t)$, which together with (7) yields the system

$$\begin{aligned}\dot{x}(t) &= Xx(t) + Uu(t) \\ y(t) &= Hx(t)\end{aligned}\quad (8)$$

where $y \in \mathbb{R}^p$ is the output of the system consisting of the states of the target nodes, and $H = P^T(V; V_T)$. Note that the ability to control the states of all target nodes in V_T is equivalent with the output controllability of system (8) [16]. As the output of system (8) specifically consists of the states of the target nodes, we say (8) is targeted controllable if it is output controllable.

Furthermore, system (8) is called *strongly targeted controllable* if (X, U, H) is targeted controllable for all $X \in \mathcal{Q}(G)$ [16]. In case (8) is strongly targeted controllable, we say $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}(G)$. The term “with respect to $\mathcal{Q}(G)$ ” clarifies the class of state matrices under consideration. This paper mainly considers strong targeted controllability with respect to $\mathcal{Q}_d(G)$. We conclude this section with well-known conditions for strong targeted controllability. Let $U = P(V; V_L)$ and $H = P^T(V; V_T)$ be the input and output matrices respectively, and define the reachable subspace $\langle X \mid \text{im } U \rangle = \text{im } U + X \text{im } U + \dots + X^{n-1} \text{im } U$.

Proposition 3: [16] The following statements are equivalent:

- 1) $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}(G)$,
- 2) $\text{rank} \begin{pmatrix} HU & HXU & \dots & HX^{n-1}U \end{pmatrix} = p \quad \forall X \in \mathcal{Q}(G)$,

- 3) $H \langle X \mid \text{im } U \rangle = \mathbb{R}^p \quad \forall X \in \mathcal{Q}(G)$, and
- 4) $\ker H + \langle X \mid \text{im } U \rangle = \mathbb{R}^n \quad \forall X \in \mathcal{Q}(G)$.

III. PROBLEM STATEMENT

Strong targeted controllability with respect to $\mathcal{Q}(G)$ was studied in [16], and a sufficient graph-theoretic condition was provided. Motivated by the fact that $\mathcal{Q}_d(G)$ contains important network-related matrices like the adjacency and Laplacian matrices, we are interested in extending the results of [16] to the class of distance-information preserving matrices $\mathcal{Q}_d(G)$. More explicitly, the problem that we will investigate in this paper is given as follows.

Problem 1: Given a directed graph $G = (V, E)$, a leader set $V_L \subseteq V$ and target set $V_T \subseteq V$, provide necessary and sufficient graph-theoretic conditions under which $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

The study of such graph-theoretic conditions is motivated by the fact that known rank conditions for strong targeted controllability (Proposition 3, condition 2) are computationally infeasible. Indeed, verifying condition 2 of Proposition 3 would require the rank computation of an infinite number of output controllability matrices (one for each $X \in \mathcal{Q}_d(G)$). Furthermore, graph-theoretic conditions for targeted controllability may aid in finding leader selection procedures.

In addition to Problem 1, we are interested in a method to compute leader sets achieving targeted controllability. More precisely:

Problem 2: Given a directed graph $G = (V, E)$ and target set $V_T \subseteq V$, compute a leader set $V_L \subseteq V$ of minimum cardinality such that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

IV. MAIN RESULTS

Our main results are presented in this section. First, in Section IV-A, we provide a sufficient graph-theoretic condition for strong targeted controllability with respect to $\mathcal{Q}_d(G)$. Subsequently, in Section IV-B, we review the notion of sufficient richness of subclasses, and prove that the subclass $\mathcal{Q}_d(G)$ is sufficiently rich. This result allows us to establish a necessary condition for strong targeted controllability, which is presented in Section IV-C. Finally, in Section IV-D, we show there is no polynomial-time algorithm solving Problem 2 (assuming $\text{P} \neq \text{NP}$). Therefore, we provide a heuristic leader selection algorithm to determine leader sets achieving targeted controllability.

A. Sufficient Condition for Targeted Controllability

This section discusses a sufficient graph-theoretic condition for strong targeted controllability. We first introduce some notions that will become useful later on.

Consider a directed graph $G = (V, E)$ with leader set V_L and target set V_T . In this section, we assume all target nodes have finite distance with respect to V_L . This assumption is without loss of generality. Indeed, it is easy to see that $(G; V_L; V_T)$ is not targeted controllable if a target node $v \in V_T$ cannot be reached from any leader.

Fig. 4. Graph G with $V_L = \{1, 2\}$. Fig. 5. $D(V_L) = \{1, 2, 3\}$. Fig. 6. Graph G_1 .

The derived set of V_L is given by $D(V_L)$. Furthermore, let $V_S \subseteq V \setminus D(V_L)$ be a subset. We partition the set V_S according to the distance of its nodes with respect to $D(V_L)$, that is

$$V_S = V_1 \cup V_2 \cup \dots \cup V_d \quad (9)$$

where for $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$ for $i = 1, 2, \dots, d$. Moreover, we define $\tilde{V}_i$ and $\hat{V}_i$ to be the sets of vertices in V_S of distance respectively less than i and greater than i with respect to $D(V_L)$. More precisely:

$$\begin{aligned} \tilde{V}_i &:= V_1 \cup \dots \cup V_{i-1} \text{ for } i = 2, \dots, d \\ \hat{V}_i &:= V_{i+1} \cup \dots \cup V_d \text{ for } i = 1, \dots, d-1. \end{aligned} \quad (10)$$

By convention $\tilde{V}_1 = \emptyset$ and $\hat{V}_d = \emptyset$. With each of the sets $V_1, V_2, \dots, V_d$ we associate a bipartite graph $G_i = (D(V_L), V_i, E_i)$, where for $j \in D(V_L)$ and $k \in V_i$ we have $(j, k) \in E_i$ if and only if $d(j, k) = i$ in the network graph G .

Example 2: We consider the network graph $G = (V, E)$ as depicted in Fig. 4. The set of leaders is $V_L = \{1, 2\}$, which implies that $D(V_L) = \{1, 2, 3\}$ (see Fig. 5).

In this example, we define the subset $V_S \subseteq V \setminus D(V_L)$ as $V_S := \{4, 5, 6, 7, 8\}$. Note that V_S can be partitioned according to the distance of its nodes with respect to $D(V_L)$ as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4, 5\}$, $V_2 = \{6, 7\}$ and $V_3 = \{8\}$.

Fig. 7. Graph G_2 . Fig. 8. Graph G_3 .

The bipartite graphs G_1 , G_2 , and G_3 are given in Figs. 6, 7, and 8 respectively.

The main result presented in this section is given in Theorem 4. This statement provides a sufficient graph-theoretic condition for targeted controllability of $(G; V_L; V_T)$ with respect to $\mathcal{Q}_d(G)$.

Theorem 4: Consider a directed graph $G = (V, E)$, with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Let $V_T \setminus D(V_L)$ be partitioned as in (9), and assume $D(V_L)$ is a zero forcing set in $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, \dots, d$. Then $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

In the special case of a single leader, i.e., $|V_L| = 1$, the condition of Theorem 4 can be simplified. In this case, $(G; V_L; V_T)$ is targeted controllable if no pair of target nodes has the same distance with respect to the leader. This is formulated in the following corollary.

Corollary 5: Consider a directed graph $G = (V, E)$, with singleton leader set $V_L = \{v\} \subseteq V$ and target set $V_T \subseteq V$. $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$ if $d(v, i) \neq d(v, j)$ for all distinct $i, j \in V_T$.

Note that the condition of Corollary 5 is similar to the “k-walk theory” for (weak) targeted controllability established in [8, Th. 2]. However, it is worth mentioning that k-walk theory [8] was only proven for directed tree networks with a single leader. On the other hand, Theorem 4 establishes a condition for strong targeted controllability that is applicable to general directed networks with multiple leaders.

Furthermore, note that Theorem 4 significantly improves the known condition for strong targeted controllability given in [16] for the class $\mathcal{Q}_d(G)$. In Theorem 4, target nodes with arbitrary distance with respect to the derived set are allowed, while the main result of [16, Th. VI.6] is restricted to target nodes of distance one with respect to $D(V_L)$. Before proving Theorem 4, we provide an illustrative example and two auxiliary lemmas.

Example 3: Once again, consider the network graph depicted in Fig. 4, with leader set $V_L = \{1, 2\}$ and assume the target set is given by $V_T = \{1, 2, \dots, 8\}$. The goal of this example is to prove that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

Note that $V_S := V_T \setminus D(V_L)$ is given by $V_S = \{4, 5, 6, 7, 8\}$, which is partitioned according to (9) as $V_S = V_1 \cup V_2 \cup V_3$, where $V_1 = \{4, 5\}$, $V_2 = \{6, 7\}$ and $V_3 = \{8\}$. The graphs G_1 ,

G_2 , and G_3 have been computed in Example 2. Note that $D(V_L) = \{1, 2, 3\}$ is a zero forcing set in all three graphs (see Figs. 6–8). We conclude by Theorem 4 that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

Lemma 6: Consider a directed graph $G = (V, E)$ with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. Let $\mathcal{Q}_s(G) \subseteq \mathcal{Q}(G)$ be any subclass. Then $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_s(G)$ if and only if $(G; D(V_L); V_T)$ is targeted controllable with respect to $\mathcal{Q}_s(G)$.

Proof: Let $U = P(V; V_L)$ index the leader set V_L and $W = P(V; D(V_L))$ index the derived set of V_L . Furthermore, let the matrix H be given by $H = P^T(V; V_T)$. We have that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_s(G)$ if and only if

$$H\langle X \mid \text{im } U \rangle = \mathbb{R}^p \text{ for all } X \in \mathcal{Q}_s(G). \quad (11)$$

However, as $\langle X \mid \text{im } U \rangle = \langle X \mid \text{im } W \rangle$ for any $X \in \mathcal{Q}(G)$ (see [16, Lemma VI.2]), (11) holds if and only if

$$H\langle X \mid \text{im } W \rangle = \mathbb{R}^p \text{ for all } X \in \mathcal{Q}_s(G). \quad (12)$$

We conclude that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_s(G)$ if and only if $(G; D(V_L); V_T)$ is targeted controllable with respect to $\mathcal{Q}_s(G)$. ■

Lemma 7: Let $G = (V^-, V^+, E)$ be a bipartite graph and assume V^- is a zero forcing set in G . Then all matrices $M \in \mathcal{P}(G)$ in the pattern class of G have full row rank.

Proof: Note that forces of the form $u \rightarrow v$, where $u, v \in V^+$ are not possible, as G is a bipartite graph. Relabel the nodes of V^- and V^+ such that a chronological list of forces is given by $u_i \rightarrow v_i$, where $u_i \in V^-$ and $v_i \in V^+$ for $i = 1, 2, \dots, |V^+|$. Let $M \in \mathcal{P}(G)$ be a matrix in the pattern class of G . Note that the element M_{ii} is nonzero, as $u_i \rightarrow v_i$. Furthermore, M_{ji} is zero for all $j > i$. The latter follows from the fact that u_i would not be able to force v_i if there was an arc $(u_i, v_j) \in E$. We conclude that the columns $1, 2, \dots, |V^+|$ of M are linearly independent, hence M has full row rank. ■

Proof of Theorem 4: Let $D(V_L) = \{1, 2, \dots, m\}$, and assume without loss of generality that the matrix U has the form (see Lemma 6):

$$U = \begin{pmatrix} I_{m \times m} & 0_{m \times (n-m)} \end{pmatrix}^T. \quad (13)$$

Furthermore, we let $V_S := V_T \setminus D(V_L)$ be given by $\{m+1, m+2, \dots, p\}$, where the vertices are ordered in nondecreasing distance with respect to $D(V_L)$. Partition V_S according to the distance of its nodes with respect to $D(V_L)$ as

$$V_S = V_1 \cup V_2 \cup \dots \cup V_d \quad (14)$$

where for $j \in V_S$ we have $j \in V_i$ if and only if $d(D(V_L), j) = i$ for $i = 1, 2, \dots, d$. Finally, assume the target set V_T contains all nodes in the derived set $D(V_L)$. This implies that the matrix H is of the form

$$H = \begin{pmatrix} I_{p \times p} & 0_{p \times (n-p)} \end{pmatrix}. \quad (15)$$

Note that by the structure of H and U , the matrix HX^iU is simply the $p \times m$ upper left corner submatrix of X^i . We now

claim that HX^iU can be written as follows:

$$HX^iU = \begin{pmatrix} \Lambda_i \\ M_i \\ 0_i \end{pmatrix} \quad (16)$$

where $M_i \in \mathcal{P}(G_i)$ is a $|V_i| \times m$ matrix in the pattern class of G_i , Λ_i is an $(m + |\hat{V}_i|) \times m$ matrix containing elements of lesser interest, and 0_i is a zero matrix of dimension $|\hat{V}_i| \times m$.

We proceed as follows: First we prove that the bottom submatrix of (16) contains zeros only, second, we prove that $M_i \in \mathcal{P}(G_i)$. From this, we conclude that (16) holds.

Note that for $k \in D(V_L)$ and $j \in \hat{V}_i$, we have $d(k, j) > i$ and by Lemma 1 it follows that $(X^i)_{jk} = 0$. As $D(V_L) = \{1, 2, \dots, m\}$, this means that the bottom $|\hat{V}_i| \times m$ submatrix of HX^iU is a zero matrix.

Subsequently, we want to prove that M_i , the middle block of (16), is an element of the pattern class $\mathcal{P}(G_i)$. Note that the j th row of M_i corresponds to the element $l := m + |\hat{V}_i| + j \in V_i$.

Suppose $(M_i)_{jk} \neq 0$ for a $k \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, |V_i|\}$. As M_i is a submatrix of HX^iU , this implies $(HX^iU)_{lk} \neq 0$. Recall that HX^iU is the $p \times m$ upper left corner submatrix of X^i , therefore, it holds that $(X^i)_{lk} \neq 0$. Note that for the vertices $k \in D(V_L)$ and $l \in V_i$ we have $d(k, l) \geq i$ by the partition of V_S . However, as $(X^i)_{lk} \neq 0$ it follows from Lemma 1 that $d(k, l) = i$. Therefore, by the definition of G_i , there is an arc $(k, l) \in E_i$.

Conversely, suppose there is an arc $(k, l) \in E_i$ for $l \in V_i$ and $k \in D(V_L)$. This implies $d(k, l) = i$ in the network graph G . By the distance-information preserving property of X we consequently have $(X^i)_{lk} \neq 0$. We conclude that $(M_i)_{jk} \neq 0$ and hence $M_i \in \mathcal{P}(G_i)$. This implies that (16) holds. We compute the first dm columns of the output controllability matrix $(HU \ HXU \ HX^2U \ \dots \ HX^dU)$ as follows:

$$\begin{pmatrix} I & * & * & \dots & * & * \\ 0 & M_1 & * & \dots & * & * \\ 0 & 0 & M_2 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & * & * \\ \vdots & \vdots & \vdots & \ddots & M_{d-1} & * \\ 0 & 0 & 0 & \dots & 0 & M_d \end{pmatrix} \quad (17)$$

where zeros denote zero matrices and asterisks denote matrices of less interest. As $D(V_L)$ is a zero forcing set in G_i for $i = 1, 2, \dots, d$, the matrices $M_1, M_2, \dots, M_d$ have full row rank by Lemma 7. We conclude that the matrix (17) has full row rank, and consequently $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. ■

Note that the condition given in Theorem 4 is sufficient, but not necessary. One can verify that the graph $G = (V, E)$ with leader set $V_L = \{1\}$ and target set $V_T = \{2, 3\}$ depicted in Fig. 9 is an example of a graph for which $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. However, this graph does not satisfy the conditions of Theorem 4.

Fig. 9. Theorem 4 not necessary.

B. Sufficient Richness of $\mathcal{Q}_d(G)$

The notion of sufficient richness of a qualitative subclass was introduced in [17]. We provide an equivalent definition as follows.

Definition 4: Let $G = (V, E)$ be a directed graph with leader set $V_L \subseteq V$. A subclass $\mathcal{Q}_s(G) \subseteq \mathcal{Q}(G)$ is called *sufficiently rich* if $(G; V_L)$ is controllable with respect to $\mathcal{Q}_s(G)$ implies $(G; V_L)$ is controllable with respect to $\mathcal{Q}(G)$.

The following geometric characterization of sufficient richness is proven in [17].

Proposition 8: A qualitative subclass $\mathcal{Q}_s(G) \subseteq \mathcal{Q}(G)$ is sufficiently rich if for all $z \in \mathbb{R}^n$ and $X \in \mathcal{Q}(G)$ satisfying $z^T X = 0$, there exists an $X' \in \mathcal{Q}_s(G)$ such that $z^T X' = 0$.

The goal of this section is to prove that the qualitative subclass of distance-information preserving matrices is sufficiently rich. This result will be used later on, when we provide a necessary condition for targeted controllability with respect to $\mathcal{Q}_d(G)$. First, however, we state two auxiliary lemmas which will be the building blocks to prove the sufficient richness of $\mathcal{Q}_d(G)$.

Lemma 9: Consider q nonzero multivariate polynomials $p_i(x)$, where $i = 1, 2, \dots, q$ and $x \in \mathbb{R}^n$. There exists an $\bar{x} \in \mathbb{R}^n$ such that $p_i(\bar{x}) \neq 0$ for $i = 1, 2, \dots, q$.

Proof: The proof follows immediately from continuity of polynomials and is omitted. ■

Remark 2: Without loss of generality, we can assume that the point $\bar{x} \in \mathbb{R}^n$ has only nonzero coordinates. Indeed, if $p_i(\bar{x}) \neq 0$ for $i = 1, 2, \dots, q$, there exists an open ball $B(\bar{x})$ around $\bar{x}$ in which $p_i(x) \neq 0$ for $i = 1, 2, \dots, q$. Obviously, this open ball contains a point with the aforementioned property.

Lemma 10: Let $X \in \mathcal{Q}(G)$ and $D = \text{diag}(d_1, d_2, \dots, d_n)$ be a matrix with variable diagonal entries. If $d(i, j) = k$ for distinct vertices i and j , then $((XD)^k)_{ji}$ is a nonzero polynomial in the variables $d_1, d_2, \dots, d_n$.

Proof: Note that $((XD)^k)_{ji}$ is given by

$$\sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_{k-1}=1}^n (XD)_{i_1, i} (XD)_{i_2, i_1} \dots (XD)_{j, i_{k-1}}$$

which equals

$$\sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_{k-1}=1}^n d_i X_{i_1, i} d_{i_1} X_{i_2, i_1} \dots d_{i_{k-1}} X_{j, i_{k-1}}. \quad (18)$$

Since the distance $d(i, j)$ is equal to k , there exists at least one path of length k from i to j , which we denote by $(i, i_1), (i_1, i_2), \dots, (i_{k-1}, j)$. It follows that the corresponding elements of the matrix X , i.e., the elements $X_{i_1, i}, X_{i_2, i_1}, X_{j, i_{k-1}}$

are nonzero. Therefore, the term

$$d_i X_{i_1, i} d_{i_1} X_{i_2, i_1} \dots d_{i_{k-1}} X_{j, i_{k-1}} \quad (19)$$

is nonzero (as a function of $d_i, d_{i_1}, d_{i_2}, \dots, d_{i_{k-1}}$). Furthermore, this combination of k diagonal elements is unique in the sense that there does not exist another summand on the right-hand side of (18) with exactly the same elements. This implies that the term (19) does not vanish (as a polynomial). We conclude that $((XD)^k)_{ji}$ is a nonzero polynomial function in the variables $d_1, d_2, \dots, d_n$. ■

Theorem 11: The subclass $\mathcal{Q}_d(G)$ is sufficiently rich.

Proof: Given a matrix $X \in \mathcal{Q}(G)$, using Lemmas 9 and 10, we first prove there exists a diagonal matrix $\bar{D}$ with nonzero diagonal components such that $X\bar{D} \in \mathcal{Q}_d(G)$. From this we will conclude $\mathcal{Q}_d(G)$ is sufficiently rich.

Let $D = \text{diag}(d_1, d_2, \dots, d_n)$ be a matrix with variable diagonal entries. We define $p_{ij} := ((XD)^{d(i, j)})_{ji}$ for distinct $i, j = 1, 2, \dots, n$. By Lemma 10 we have that $p_{ij}(d_1, d_2, \dots, d_n)$ is a nonzero polynomial in the variables $d_1, d_2, \dots, d_n$. Moreover, Lemma 9 states the existence of nonzero real constants $\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n$ such that

$$p_{ij}(\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n) \neq 0 \text{ for distinct } i, j = 1, 2, \dots, n. \quad (20)$$

Therefore, the choice $\bar{D} = \text{diag}(\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n)$ implies $X\bar{D} \in \mathcal{Q}_d(G)$. Let $z \in \mathbb{R}^n$ be a vector such that $z^T X = 0$ for an $X \in \mathcal{Q}(G)$. The choice of $X' = X\bar{D}$ yields a matrix $X' \in \mathcal{Q}_d(G)$ for which $z^T X' = 0$. By Proposition 8 it follows that $\mathcal{Q}_d(G)$ is sufficiently rich. ■

C. Necessary Condition for Targeted Controllability

In addition to the previously established sufficient condition for targeted controllability, we give a necessary graph-theoretic condition for targeted controllability in Theorem 12.

Theorem 12: Let $G = (V, E)$ be a directed graph with leader set $V_L \subseteq V$ and target set $V_T \subseteq V$. If $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$ then $V_L \cup (V \setminus V_T)$ is a zero forcing set in G .

Proof: Assume without loss of generality that $V_L \cap V_T = \emptyset$. Hence, $V_L \cup (V \setminus V_T) = V \setminus V_T$. We partition the vertex set V into V_L , $V \setminus (V_L \cup V_T)$, and V_T . Without loss, we write $V_L = \{1, 2, \dots, m\}$ and $V_T = \{n - p + 1, n - p + 2, \dots, n\}$. Moreover, we use the shorthand notation $\bar{n} = n - p - m$ to denote the number of nodes that are neither a target node nor a leader node. Accordingly, the input and output matrices $U = P(V; V_L)$ and $H = P^T(V; V_T)$ satisfy

$$U = \begin{pmatrix} I_{m \times m} & 0_{m \times \bar{n}} & 0_{m \times p} \end{pmatrix}^T \quad (21)$$

and

$$H = \begin{pmatrix} 0_{p \times m} & 0_{p \times \bar{n}} & I_{p \times p} \end{pmatrix}. \quad (22)$$

Note that $\ker H = \text{im } R$, where $R := P(V; (V \setminus V_T))$ is given by

$$R = \begin{pmatrix} I_{m \times m} & 0_{m \times \bar{n}} & 0_{m \times p} \\ 0_{\bar{n} \times m} & I_{\bar{n} \times \bar{n}} & 0_{\bar{n} \times p} \end{pmatrix}^T. \quad (23)$$

Fig. 10. Theorem 12 not sufficient.

Fig. 11. Zero forcing set.

Since for all $X \in \mathcal{Q}_d(G)$ we have

$$\ker H + \langle X \mid \text{im } U \rangle = \mathbb{R}^n, \quad (24)$$

or, equivalently,

$$\text{im } R + \langle X \mid \text{im } U \rangle = \mathbb{R}^n \quad (25)$$

we obtain

$$\langle X \mid \text{im } (U R) \rangle = \mathbb{R}^n. \quad (26)$$

As $\text{im } U \subseteq \text{im } R$, (26) implies $\langle X \mid \text{im } R \rangle = \mathbb{R}^n$ for all $X \in \mathcal{Q}_d(G)$, or, equivalently, the pair (X, R) is controllable for all $X \in \mathcal{Q}_d(G)$. Furthermore, by sufficient richness of $\mathcal{Q}_d(G)$, it follows that (X, R) is controllable for all $X \in \mathcal{Q}(G)$. We conclude from Theorem 2 that $V \setminus V_T$ is a zero forcing set. ■

Example 4: Consider the directed graph $G = (V, E)$ with leader set $V_L = \{1, 2\}$ and target set $V_T = \{1, 2, \dots, 8\}$ as depicted in Fig. 4. We know from Example 3 that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. The set $V_L \cup (V \setminus V_T) = \{1, 2, 9, 10\}$ is colored black in Fig. 11. Indeed, $V_L \cup (V \setminus V_T)$ is a zero forcing set in G . A possible chronological list of forces is: $1 \rightarrow 3$, $3 \rightarrow 4$, $2 \rightarrow 5$, $4 \rightarrow 6$, $6 \rightarrow 8$, and $9 \rightarrow 7$.

The condition provided in Theorem 12 is necessary for targeted controllability, but not sufficient. To prove this fact, consider the directed graph with leader set $V_L = \{1\}$ and target set $V_T = \{4, 5\}$ given in Fig. 10. It can be shown that $(G; V_L; V_T)$ is not targeted controllable with respect to $\mathcal{Q}_d(G)$, even though $V_L \cup (V \setminus V_T) = \{1, 2, 3\}$ is a zero forcing set.

So far, we have provided a necessary and a sufficient topological condition for targeted controllability. However, given a network graph with target set, it is not clear how to choose leaders achieving target control. Hence, in the following section we focus on a leader selection algorithm.

D. Leader Selection Algorithm

We now address Problem 2, as introduced in Section III. That is, given a directed graph $G = (V, E)$ with target set $V_T \subseteq V$, we want to find a leader set $V_L \subseteq V$ of minimum cardinality such that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. Such a leader set is called a *minimum leader set*. In general, a graph G with target set V_T can have multiple minimum leader sets. In this section, we first prove that there is no polynomial-time algorithm that solves Problem 2 (assuming $P \neq NP$). Subsequently, we provide a heuristic algorithm to determine leader sets achieving targeted controllability.

Theorem 13: Assuming $P \neq NP$, there is no polynomial-time algorithm that solves Problem 2.

Proof: Assume that $P \neq NP$. The problem of finding a minimum zero forcing set was proven to be NP-hard in [1], by a reduction from the directed Hamiltonian cycle problem. Consequently, by Theorem 2 there is no polynomial-time algorithm to determine a minimum leader set V_L that achieves controllability of $(G; V_L)$ with respect to $\mathcal{Q}(G)$. Recall from Theorem 11 that the subclass $\mathcal{Q}_d(G)$ is sufficiently rich. Hence, controllability of $(G; V_L)$ with respect to $\mathcal{Q}(G)$ is equivalent with controllability of $(G; V_L)$ with respect to $\mathcal{Q}_d(G)$. Therefore, there is no polynomial-time algorithm to determine a minimum leader set V_L such that $(G; V_L)$ is controllable with respect to $\mathcal{Q}_d(G)$. Note that “ordinary” controllability of $(G; V_L)$ can be regarded as a special case of targeted controllability of $(G; V_L; V_T)$, where $V_T = V$. We conclude that there is no polynomial-time algorithm solving Problem 2 (assuming $P \neq NP$). ■

Next, we propose a heuristic approach to compute a (minimum) leader set that achieves targeted controllability. The algorithm consists of two phases. First, we identify a set of nodes in the graph G from which all target nodes can be reached. These nodes are taken as leaders. Second, this set of leaders is extended to achieve targeted controllability.

To explain the first phase of the algorithm, we introduce some notation. First of all, we define the notion of *root set*.

Definition 5: Consider a directed graph $G = (V, E)$ and a target set $V_T \subseteq V$. A subset $V_R \subseteq V$ is called a *root set* of V_T if for any $v \in V_T$ there exists a vertex $u \in V_R$ such that $d(u, v) < \infty$.

A root set of V_T of minimum cardinality is called a *minimum root set* of V_T . Note that the cardinality of a minimum root set of V_T is a lower bound on the minimum number of leaders rendering $(G; V_L; V_T)$ targeted controllable. Indeed, it is easy to see that if there are no paths from any of the leader nodes to a target node, the graph is not targeted controllable. The first step of the proposed algorithm is to compute the minimum root set of V_T . Let the vertex and target sets be given by $V = \{1, 2, \dots, n\}$ and $V_T = \{v_1, v_2, \dots, v_p\} \subseteq V$ respectively. Furthermore, define a matrix $A \in \mathbb{R}^{p \times n}$ in the following way. For $j \in V$ and $v_i \in V_T$ let

$$A_{ij} := \begin{cases} 1 & \text{if } d(j, v_i) < \infty, \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

That is, the matrix A contains zeros and ones only, where coefficients with value one indicate the existence of a path between the corresponding vertices.

Remark 3: The matrix A can be found using p runs of Dijkstra's algorithm [5], with total computational complexity $\mathcal{O}(pn^2)$. This can be done by transposing the graph $G = (V, E)$, i.e., computing $\bar{G} = (V, \bar{E})$, where $(j, i) \in \bar{E}$ if and only if $(i, j) \in E$. Dijkstra's algorithm can be applied to find the distance from a (target) node to all other nodes. We apply Dijkstra's algorithm in $\bar{G}$ to all target nodes, to find a $p \times n$ distance matrix D , with elements in $\mathbb{N} \cup \{\infty\}$. Here, each element D_{ij} is equal to the distance from node $v_i \in V_T$ to $j \in V$ in the graph $\bar{G}$. As the graph $\bar{G}$ is the transposed of G , we have that D_{ij} equals the distance from $j \in V$ to $v_i \in V_T$ in the original graph G . Consequently, the matrix A is easily obtained from D by changing " ∞ "-elements in D to 0, and all other elements in D to 1. Dijkstra's algorithm and graph transposition have computational complexity $\mathcal{O}(n^2)$. As we execute Dijkstra's algorithm p times, the total procedure has computational complexity $\mathcal{O}(pn^2)$. The distance matrix D will also become useful in the second phase of the leader selection algorithm.

Now, finding a minimum root set of V_T boils down to finding a binary vector $x \in \mathbb{R}^n$ with minimum number of ones such that $Ax \geq \mathbb{1}_p$, where the inequality is defined element-wise and $\mathbb{1}_p$ denotes the p -dimensional vector of all ones. In the vector x , coefficients with value one correspond to elements in the root set of V_T . It is for this reason we can formulate the minimum root set problem as a binary integer linear program:

$$\begin{aligned} & \text{minimize } \mathbb{1}_n^T x \\ & \text{subject to } Ax \geq \mathbb{1}_p \\ & \text{and } x \in \{0, 1\}^n. \end{aligned} \quad (28)$$

Linear programs of this form can be solved using software like CPLEX or MATLAB.

Remark 4: Note that the minimum root set problem (and in general binary integer programming) is NP-hard. This can be shown by constructing an NP-reduction from the well-known NP-hard set-covering problem [4] to the minimum root set problem. Therefore, for large-scale problems it is advisable to apply an approximation algorithm to compute an approximate minimum root set. The greedy algorithm for the set-covering problem [4] can be directly applied to the minimum root set problem. This algorithm has computational complexity $\mathcal{O}(p^2n)$, and has an approximation ratio of $\mathcal{O}(\ln(n))$. That is, the approximation algorithm returns a root set containing at most $\mathcal{O}(\ln(n))$ times the optimal number of vertices in the minimum root set. Furthermore, it has been shown in [7] that no polynomial-time algorithm for the set-covering problem can have a better approximation ratio than $\mathcal{O}(\ln(n))$.

In the following example, we illustrate how the minimum root set problem can be regarded as a binary integer linear program.

Example 5: Consider the directed graph $G = (V, E)$ with target set $V_T = \{1, 4, 5, 6, 7\}$ depicted in Fig. 12. The goal of this example is to find a minimum root set for V_T . The matrix

Fig. 12. Directed graph $G = (V, E)$ with target set $V_T = \{1, 4, 5, 6, 7\}$.

A , as defined in (27), is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}. \quad (29)$$

Note that $x = (1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0)^T$ satisfies the inequality $Ax \geq \mathbb{1}_p$ and the constraint $x \in \{0, 1\}^7$. Furthermore, the vector x minimizes $\mathbb{1}_n^T x$ under these constraints. This can be seen in the following way: There is no column of A in which all elements equal 1, hence there is no vector x with a single one such that $Ax \geq \mathbb{1}_p$ is satisfied. Therefore, x solves the binary integer linear program (28), from which we conclude that the choice $V_R = \{1, 4\}$ yields a minimum root set for V_T . Indeed, observe in Fig. 12 that we can reach all nodes in the target set starting from the nodes 1 and 4. It is worth mentioning that the choice of minimum root set is not unique: the set $\{1, 2\}$ is also a minimum root set for V_T .

In general, the minimum root set V_R of V_T does not guarantee targeted controllability of $(G; V_R; V_T)$ with respect to $\mathcal{Q}_d(G)$. For instance, it can be shown for the graph G and target set V_T of Example 5 that the leader set $V_L = \{1, 4\}$ does not render $(G; V_L; V_T)$ targeted controllable with respect to $\mathcal{Q}_d(G)$. Hence, we propose a greedy approach to extend the minimum root set of V_T to a leader set that does achieve targeted controllability.

Recall from Theorem 4 that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$ if $D(V_L)$ is a zero forcing set in the bipartite graphs $G_i = (D(V_L), V_i, E_i)$ for $i = 1, 2, \dots, d$, where $V_i \subseteq V_T$ is the set of target nodes having distance i from $D(V_L)$. Given an initial set of leaders V_L , we compute its derived set $D(V_L)$ and verify whether we can force all nodes in the bipartite graphs G_i for $i = 1, 2, \dots, d$. Suppose that in the bipartite graph G_k the set V_k cannot be forced by $D(V_L)$ for a $k \in \{1, 2, \dots, d\}$. In this case, we choose an additional leader as follows. Let $V_u \subseteq V_k$ be the set of vertices in V_k that can't be forced. Suppose $v_i \in V_u$ is the vertex in V_u from which most target nodes can be reached. Then we choose v_i as additional leader. Consequently, we have extended our leader set V_L to $V_L \cup \{v_i\}$. Note that v_i can be easily found by computing the column sums of the columns in A corresponding to the nodes in V_u .

With the extended leader set we can repeat the procedure, until the leaders render the graph targeted controllable. This idea is captured more formally in the following leader selection

Algorithm 1: Leader Selection Procedure.**Input:** Directed graph $G = (V, E)$, target set $V_T \subseteq V$;**Output:** Leader set $V_L \subseteq V$ achieving target control;

```

1: Let  $V_L = \emptyset$ ;
2: Compute matrix  $A$ , given in (27);
3: Compute a solution  $x$  to the linear program (28);
4: for  $i = 1$  to  $n$  do
5:   if  $x_i = 1$  then
6:      $V_L \leftarrow V_L \cup \{i\}$ ;
7:   end if
8: end for
9: Compute  $D(V_L)$ ;
10:  $i \leftarrow 1$ ;
11: repeat
12:   Compute  $V_i$  and  $G_i = (D(V_L), V_i, E_i)$ ;
13:   if  $D(V_L)$  forces  $V_i$  in  $G_i$  then
14:      $i \leftarrow i + 1$ ;
15:   else
16:     Find unforced  $v \in V_i$  reaching the most targets;
17:      $V_L \leftarrow V_L \cup \{v\}$ ;
18:     Compute  $D(V_L)$ ;
19:      $i \leftarrow 1$ ;
20:   end if
21: until  $d(D(V_L), w) < i$  for all  $w \in V_T$ ;
22: return  $V_L$ .

```

algorithm. One should recognize the two phases of leader selection: first, a minimum root set is computed. Subsequently, the minimum root set is greedily extended to a leader set achieving targeted controllability.

Remark 5: Algorithm 1 can be implemented with computational complexity $\mathcal{O}(p^2 n^2)$ if the heuristic set-covering procedure [4] is used to compute an approximate minimum root set (step 3). This can be seen as follows. First, note that the repeat (step 11) runs at most p^2 times. Moreover, every step within the repeat runs in time at most $\mathcal{O}(n^2)$. Indeed, using the distance matrix D (see Remark 3), we can compute V_i and $G_i = (D(V_L), V_i, E_i)$ in time $\mathcal{O}(pn)$. Furthermore, the derived sets in steps 13 and 18 can be computed in $\mathcal{O}(n^2)$ time [27]. Finally, step 16 compares the column sums of at most p columns of length p , and can hence be implemented in $\mathcal{O}(p^2)$. As $p \leq n$, we find that all steps within the repeat run with computational complexity at most $\mathcal{O}(n^2)$. Consequently, steps 11–21 have computational complexity $\mathcal{O}(p^2 n^2)$. Steps 1–10 of the algorithm run in time complexity less than $\mathcal{O}(p^2 n^2)$ (see Remarks 3 and 4). We conclude that Algorithm 1 can be implemented with computational complexity $\mathcal{O}(p^2 n^2)$ if the heuristic set-covering procedure is applied to step 3.

Note that Algorithm 1 is a heuristic algorithm, and the quality of its solution with respect to the actual minimum leader set is not known. The problem of finding a minimum leader set achieving target control is more general than the problem of finding a minimum leader set achieving (full) strong structural controllability. Indeed, by the choice $V_T = V$, one can solve the latter problem using the former. Consequently, Remark 1

Fig. 13. Directed graph $G = (V, E)$ with encircled target nodes $V_T = \{2, 3, 6, 8, 10, 13, 15, 16, 17, 20\}$.

suggests that the minimum number of leaders achieving target control cannot be approximated within a large factor.

However, it is worth mentioning that Algorithm 1 does return optimal results for some specific types of graphs. In the case of a cycle or a complete graph (with target set $V_T = V$), Algorithm 1 returns leader sets of respectively 2 and $n - 1$ leaders. This is in agreement with the optimal results found in [17] for cycle and complete graphs.

V. ILLUSTRATIVE EXAMPLE

In this section, we illustrate the heuristic algorithm using an example. Consider the directed graph given in Fig. 13, with target set $V_T = \{2, 3, 6, 8, 10, 13, 15, 16, 17, 20\}$. The goal of this example is to compute a leader set V_L such that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$.

The first step of Algorithm 1 is to compute the matrix A , defined in (27). For this example, A is given as follows.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Using the MATLAB function `intlinprog`, we find the optimal solution

$$x = (0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T$$

Fig. 14. Graph G_1 for derived set $D(V_L) = \{4, 13\}$. Fig. 15. Graph G_1 for derived set $D(V_L) = \{2, 4, 13\}$. Fig. 16. Graph G_2 for derived set $D(V_L) = \{2, 4, 13\}$. Fig. 17. Graph G_3 for derived set $D(V_L) = \{2, 4, 13\}$.

to the binary linear program (28). Hence, a minimum root set for V_T is given by $\{4, 13\}$. Following Algorithm 1, we define our initial leader set $V_L = \{4, 13\}$. As nodes 4 and 13 both have three out-neighbors, the derived set of V_L is simply given by $D(V_L) = \{4, 13\}$. The next step of the algorithm is to compute the first bipartite graph $G_1 = (D(V_L), V_1, E_1)$, which we display in Fig. 14. Observe that the nodes 2 and 3 cannot be forced. As both nodes can reach the same number of target nodes, we simply choose node 2 as additional leader. The process now repeats itself, we redefine $V_L = \{2, 4, 13\}$ and compute $D(V_L) = \{2, 4, 13\}$. Furthermore, for this leader set, the graph $G_1 = (D(V_L), V_1, E_1)$ is given in Fig. 15. In this case, the set $V_1 = \{3, 17\}$ of nodes having distance one with respect to $D(V_L)$ is forced. Therefore, we continue with the second bipartite graph $G_2 = (D(V_L), V_2, E_2)$, given in Fig. 16. The set V_2 is forced by $D(V_L)$ in the graph G_2 , hence we continue to investigate the third bipartite graph consisting of nodes having distance three with respect to $D(V_L)$. This graph is displayed in Fig. 17. As neither node 8 nor 10 can be forced, we have to add another leader. Node 8 can reach 4 target nodes, while node 10 can reach 7 target nodes. Hence, we choose node 10 as additional leader. In other words, we redefine $V_L = \{2, 4, 10, 13\}$. Furthermore, the derived set of V_L is given by $D(V_L) = \{2, 4, 10, 13\}$. As we adapted the derived set, we have to recalculate the bipartite graphs G_1 , G_2 , and G_3 (see Fig. 18). Note that in this case $D(V_L)$ is a zero forcing set in all three bipartite graphs. Furthermore, since $d(D(V_L), v) < 4$ for all $v \in V_T$, Algorithm 1

Fig. 18. Graphs G_1 , G_2 and G_3 for $D(V_L) = \{2, 4, 10, 13\}$. Fig. 19. Network graph $G = (V, E)$ with target nodes $V_T = \{2, 3, 6, 8, 10, 13, 15, 16, 17, 20\}$, and leader set $V_L = \{2, 4, 10, 13\}$.

returns the leader set $V_L = \{2, 4, 10, 13\}$. This choice of leader set guarantees that $(G; V_L; V_T)$ is targeted controllable with respect to $\mathcal{Q}_d(G)$. For the sake of clarity, we display the network graph in Fig. 19, where the leader nodes are colored black, and the target nodes are encircled.

VI. CONCLUSION

In this paper, strong targeted controllability for the class of distance-information preserving matrices has been discussed. We have provided a sufficient graph-theoretic condition for strong targeted controllability, expressed in terms of zero-forcing sets of particular distance-related bipartite graphs. We have shown that this result significantly improves the known sufficient topological condition [16] for strong targeted controllability of the class of distance-information preserving matrices.

Motivated by the observation that the aforementioned sufficient condition is not a one-to-one correspondence, we provided a necessary topological condition for strong targeted controllability. This condition was proved using the fact that the subclass of distance-information preserving matrices is sufficiently rich. Finally, we showed that there is no polynomial-time algorithm to compute minimum leader sets achieving targeted controllability (assuming $P \neq NP$). Therefore, a heuristic leader selection algorithm was given to compute approximate minimum leader sets achieving target control. The algorithm comprises two phases: First, it computes a minimum root set of the target set, i.e., a set of vertices from which all target nodes can be reached.

Second, this minimum root set is greedily extended to a leader set achieving target control.

Both graph-theoretic conditions for strong targeted controllability provided in this paper are not one-to-one correspondences. Hence, finding a necessary and sufficient topological condition for strong targeted controllability is still an open problem. Furthermore, investigating other system-theoretic concepts like disturbance decoupling and fault detection for the class of distance-information preserving matrices is among the possibilities for future research.

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