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Prescribing Transient and Asymptotic Behaviour of Non-linear Systems with Stochastic Initial Conditions

M. Dresscher & B. Jayawardhana

Abstract—We study a control design problem for nonlinear affine systems whose initial condition is a random variable with known distribution. In this control problem, the control objectives are two-folds: (i). the closed-loop system attains a minimum cumulative distribution over a prescribed containment set at the end of transient time; and (ii) it converges exponentially to a desired trajectory. A solution to the control problem is obtained by imposing desired contraction properties on the closed-loop system such that both control objectives can be met. The efficacy of the proposed controller design is shown on the control of robotic manipulator.

I. INTRODUCTION

Most of modern control techniques are originated from deterministic and linear assumption where important insights and key methods were readily obtained using existing mathematical methods. Although the class of systems that satisfy these assumptions is very small (and is famously claimed in [1] that it has a measure of zero), most of physical systems can be treated or approximated by models satisfying these two key assumptions. When there are any components in the system that induce stochasticity, we tend to simplify them or take them into account explicitly as noise or disturbances as is commonly done in the field of stochastic control [2], [3]. The latter complicates the control design, but this can be a necessity if these disturbances or noise have a significant influence on the closed-loop system dynamics. In stochastic control literature, the state can sometimes be expressed as a probability density function (pdf) and one of the main control objectives is to shape this pdf such that some performance criteria are satisfied [4], [5]. In general, the generalization of the results from the deterministic setting to the stochastic one is not trivial. First of all, we may have to deal with infinite dimensional systems corresponding to the evolution of the pdf. Secondly, dealing with stochastic disturbance that cannot be measured/predicted means that we cannot anticipate the detrimental influence of random noise through real-time control input; we can only suppress it.

With recent advances in sensor and actuator systems, we can now realize advanced mechatronics systems / smart machines where external disturbance or noise becomes negligible so that most deterministic control design tools can directly be applicable. Such improvement in the measurement and actuation precision has enabled us now to deal with

another source of stochasticity that can influence the system's final performance and has not been addressed in literature. This corresponds to the case of deterministic systems with stochastic initial conditions.

Generally speaking, when we apply existing control methods for deterministic systems, we assume that the initial conditions belong to a known set / intervals and the control design has to guarantee that any transient performance (such as, rise time or settling time) will be met for all initial conditions. In practice, this leads to a conservative control design, in particular, if the set of admissible initial conditions are very large.

In [6], we consider the deterministic control problem with stochastic initial conditions where we have apriori knowledge on the pdf of the initial states. This has allowed us to complement the standard asymptotic convergence criterion by combining it with transient performance criteria, based on characteristics of the pdf of the state at a transient time of interest. This facilitates *ab-initio* controller design where we can guarantee that the evolution of the states' pdf satisfy a prescribed transient performance (for example, the cumulative distribution with respect to a set exceeds a minimum value at a given transient time). The results in [6] are for linear time-invariant (LTI) systems.

In this paper, we extend the main result of [6] to the non-linear time-invariant affine systems where we rely on existing results in contraction control methods.

The results obtained in [6] and this paper are relevant for many systems where the dynamics are relatively well-understood and precise, so that stochasticity in the initial condition becomes important in the performance criteria. Such systems can be found in, for example; micro-electronics manufacturing, where precursor materials can have fluctuating purity; or robotics, where positions of joints can be uncertain upon initialization. Furthermore, the description of the system in this manner can be seen as an extension to the normal system description since it includes the case where the initial condition satisfies a Dirac-delta function as pdf, resulting in a single value for the state, e.g. the completely deterministic case.

For background reading on stochastic processes, we refer the reader to [7], [8], for background reading on non-linear systems control and analysis to [9], [10], [11], [12] and for background reading on contraction (and the closely related notion of incremental stability) to [13], [14], [15], [16].

The remainder of this paper is structured as follows. Section II introduces the system dynamics, the performance criteria and the control problem that we will consider. Section

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III presents existing results on contraction and partial contraction that we will implement further in order to solve our control problem. In Section IV we present our main result, giving explicit controller characteristics that will guarantee the desired performance. In Section V, we implement our result for a nontrivial exemplary system and we discuss the obtained results. Lastly, Section VI rounds up with the conclusions.

II. CONTROL PROBLEM DESCRIPTION

Consider non-linear time-invariant affine systems described by

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0, \quad (1)$$

where the state $x \in \mathcal{X} \subseteq \mathbb{R}^n$, the drift vector field $f : \mathcal{X} \mapsto \mathbb{R}^n$, the control vector field $g : \mathcal{X} \mapsto \mathbb{R}^{n \times m}$, the input $u \in U \subseteq \mathbb{R}^m$ and the initial condition x_0 is a random variable. We denote the probability density function (pdf) of x_0 by ϕ_{x_0} . We note that this system description includes the case where ϕ_{x_0} is given by Dirac function, in which case we are dealing with the standard deterministic non-linear affine system. Since we will study the evolution of ϕ_{x_0} subject to the system dynamics as in (1) and a control law $u(t) = k(x(t), t)$, we will denote the time evolution of ϕ_{x_0} by $\phi_{x_0, t}$. For any point $x_1, x_2 \in \mathcal{X}$, we denote the distance between x_1 and x_2 by $d(x_1, x_2)$ which will be defined later in Section III.

Given a compact subset $\Xi \subseteq \mathcal{X}$ and pdf evolution $\phi_{x_0, t}$, the *cumulative density* $\Phi_{\Xi, t}$ is defined by

$$\Phi_{\Xi, t} := \int_{\xi \in \Xi} \phi_{x_0, t}(\xi) d\xi. \quad (2)$$

Equivalently, $\Phi_{\Xi, t} = P(x(t) \in \Xi)$ where $P(x(t) \in \Xi)$ is the probability of $x(t)$ being in Ξ .

Containment Control Problem (CCP): Given a desired containment set Ξ , a desired containment level $p^* \in (0, 1)$, a transient time $T > 0$ and a target trajectory x_d , design a control law $u(t) = k(x(t), t)$ such that

$$\text{CCPa: } \Phi_{\Xi, T} \geq p^*$$

$$\text{CCPb: } \lim_{t \rightarrow \infty} d(x(t), x_d(t)) = 0. \quad \triangle$$

Let us shortly elaborate on the two control objectives **CCPa** and **CCPb** above. The control objective **CCPa** implies that at the transient time T , the probability that the state $x(T)$ is in the set Ξ is at least p^* . On the other hand, the control objective **CCPb** requires that the distance between the state trajectory and the target trajectory converges to zero, i.e., the trajectory x converges to the desired trajectory x_d .

III. PRELIMINARIES ON THE CONTRACTION-BASED CONTROL METHOD

In this section, we will recall relevant results on contraction-based control method that is relevant to our system and our containment control problem. Contraction theory refers typically to analysis tool for studying the contraction behaviour among trajectories of autonomous system. For

instance, it allows us to characterize the incremental stability property of autonomous system. The use of contraction theory in control design is fairly recent and we refer interested reader to [13], [14], [15], [17].

In the contraction-based control design method, the main basic principle is to design a control law $u(t) = k(\bar{x}(t), t)$, where $\bar{x}$ can be the desired state trajectory, such that the closed-loop system

$$\dot{x} = f(x) + g(x)k(\bar{x}, t) = h(x, \bar{x}, t) \quad (3)$$

has the contraction property. Furthermore, we consider h as a $\mathbb{C}^1$ vector field which maps each $(x, t) \in \mathcal{X} \times \mathbb{R}_{\geq 0}$ to a tangent vector belonging to the tangent space of $\mathcal{X}$ at $x \in \mathcal{X}$, which we denote by $T_x \mathcal{X}$. Accordingly, we consider $T\mathcal{X} = \bigcup_{x \in \mathcal{X}} \{x\} \times T_x \mathcal{X}$ as the tangent bundle of $\mathcal{X}$.

For contraction analysis, system (3) can be written as the prolonged system [18] when we combine it with its corresponding variational system as

$$\begin{aligned} \dot{x} &= h(x, \bar{x}, t), \\ \delta x &= \frac{\partial h}{\partial x}(x, \bar{x}, t) \delta x. \end{aligned} \quad (4)$$

Here, $(x, \delta x, t) \in T\mathcal{X} \times \mathbb{R}_{\geq 0}$, e.g. δx is a tangent vector, which has length with respect to a distance.

A. Finsler-Lyapunov function and Finsler distance

In the literature on contraction, a commonly used distance is the *Finsler distance*. This distance has a relation to a *Finsler-Lyapunov function* as defined below. We adopt the following definitions from [19].

Definition 1: (Finsler-Lyapunov function) A $\mathbb{C}^1$ function $V : T\mathcal{X} \mapsto \mathbb{R}_{\geq 0}$, that maps every $(x, \delta x) \in T\mathcal{X}$ to $V(x, \delta x) \in \mathbb{R}_{\geq 0}$, is a candidate Finsler-Lyapunov function for (4), if there exist $c_1, c_2 \in \mathbb{R}_{\geq 0}$, $p \in \mathbb{R}_{\geq 1}$, and a Finsler structure $F : T\mathcal{X} \mapsto \mathbb{R}_{\geq 0}$ such that, $\forall (x, \delta x) \in T\mathcal{X}$,

$$c_1 F(x, \delta x)^p \leq V(x, \delta x, t) \leq c_2 F(x, \delta x)^p. \quad (5)$$

The Finsler structure F satisfies the following conditions:

- 1) F is a $\mathbb{C}^1$ function for each $(x, \delta x) \in T\mathcal{X}$ such that $\delta x \neq 0$;
- 2) $F(x, \delta x) > 0$ for each $(x, \delta x) \in T\mathcal{X}$ such that $\delta x \neq 0$;
- 3) $F(x, \lambda \delta x) = \lambda F(x, \delta x)$ for each $\lambda \geq 0$ and each $(x, \delta x) \in T\mathcal{X}$;
- 4) $F(x, \delta x_1 + \delta x_2) < F(x, \delta x_1) + F(x, \delta x_2)$ for each $(x, \delta x_1), (x, \delta x_2) \in T\mathcal{X}$ such that $\delta x_1 \neq \lambda \delta x_2$ for any given $\lambda \in \mathbb{R}$.

The *Finsler-Lyapunov function* is then a measure of length of the tangent vector. This measure corresponds to the *Finsler distance* as induced by F on $\mathcal{X}$ via integration.

Definition 2: (Finsler distance) Consider a candidate *Finsler-Lyapunov function* V on $\mathcal{X}$ and the associated Finsler structure F as in *Definition 1*. For any two points $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$, let $\Gamma(x_1, x_2)$ be the collection of piecewise $\mathbb{C}^1$ curves $\gamma : \mathcal{I} \mapsto \mathcal{X}$, $\mathcal{I} := \{s \in \mathbb{R} \mid 0 \leq s \leq 1\}$, $\gamma(0) = x_1$ and $\gamma(1) = x_2$. The distance (or metric) $d : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}_{\geq 0}$ induced by F satisfies

$$d(x_1, x_2) := \inf_{\Gamma(x_1, x_2)} \int_{\mathcal{I}} F(\gamma(s), \dot{\gamma}(s)) ds. \quad (6)$$

B. Contraction and partial contraction

Lemma 1: Consider the system (4) on the smooth manifold $\mathcal{X}$ with F a $\mathbb{C}^2$ function, a connected and forward invariant set $\mathcal{C} \subseteq \mathcal{X}$ and a function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$. Let V be a candidate *Finsler-Lyapunov function* such that,

$$\frac{\partial V(x, \delta x)}{\partial x} h(x, \bar{x}, t) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial h(x, \bar{x}, t)}{\partial x} \delta x \leq \alpha(V(x, \delta x)) \quad (7)$$

for each $t \in \mathbb{R}$, $x \in \mathcal{C} \subseteq \mathcal{X}$, and $\delta x \in T_x \mathcal{X}$. Then, (4) is

- incrementally stable (IS) on $\mathcal{C}$ if $\alpha(s) = 0$, for each $s \geq 0$;
- incrementally asymptotically stable (IAS) on $\mathcal{C}$ if α is a class $\mathcal{K}$ function;
- incrementally exponentially stable (IES) on $\mathcal{C}$ if $\alpha(s) = \lambda s$.

We refer to [19] for the proof of this lemma.

For a given Finsler distance d , we say that the system (4) is *contracting* if there exists V as in Lemma 1 such that (7) holds with $\alpha \in \mathcal{K}$. Moreover, it is said to be *exponentially contracting* if $\alpha(s) = \lambda s$. The function V will be called the contraction measure and $\mathcal{C}$ will be called the contraction region.

For our control design, we are interested in the contraction property with respect to a specific target trajectory. To this end, we can use results on partial contraction as follows [20], [21].

Lemma 2: (Partial contraction) Consider the nonlinear system as in (3) with an admissible target trajectory $x_r(t)$, i.e., x_r satisfies $\dot{x}_r(t) = h(x_r(t), x_r(t), t)$ for all $t \geq 0$. Consider a virtual system

$$\dot{x} = h(x, x_r, t). \quad (8)$$

If the virtual system (8) is exponentially contracting w.r.t. x then x converges to $x_r(t)$ exponentially.

The proof of the lemma follows the result from [21] and [19].

Correspondingly, we say that the system (4) is *partially contracting* if (8) is exponentially contracting w.r.t. x .

IV. CONTRACTION-BASED CONTAINMENT CONTROL DESIGN

In this section, we will use the contraction-based control design approach as discussed in the preceding section to solve our **CCP**. The main idea of the method is that we design control law for the system (1) with an appropriate design of x_r such that the closed-loop system will contract to a reference trajectory $x_r(t)$ with a desired rate of convergence and moreover, $\lim_{t \rightarrow \infty} d(x_r(t), x_d(t)) = 0$.

To design the control law, we can use an adaption of *Lemma 2* as presented in [14], [22] as follows. For the system (1), assume that for a given Finsler distance d , we have a control law $u(t) = k(x_r(t), t)$ such that the closed-loop system given by (8) is partially contracting. Using this control, the rate of convergence λ and the reference trajectory x_r become control design parameters for achieving the two control objectives in **CCP**.

The following proposition presents particular design of λ and x_r that solves our **CCP**. Prior to presenting our main result, we require the following definition on an open ball induced by the Finsler distance d . For a given $\kappa > 0$, we define the ball (induced by the Finsler distance d) with radius κ centred at x_1 by

$$\mathbb{D}_\kappa(x_1) = \{x_2 \in \mathcal{X} \mid d(x_1, x_2) < \kappa\}. \quad (9)$$

Proposition 1: Consider the system (1) with the control law $u(t) = k(x, x_r, t)$ such that the closed-loop system is partially contracting w.r.t. x , with the contraction rate λ . Suppose that there exist $\mu \in \mathcal{C}$, $\xi \in \Xi$, reference trajectory x_r and $\kappa_1, \kappa_2 > 0$ such that the following two conditions hold.

- 1) The reference signal x_r satisfies

$$\dot{x}_r = f(x_r) + g(x_r)k(x_r, x_r, t) \quad (10)$$

for all $t \geq 0$, $x_r(0) = \mu$, $x_r(T) = \xi$ and $\lim_{t \rightarrow \infty} d(x_r(t), x_d(t)) = 0$.

- 2)

$$\int_{\mathbb{D}_{\kappa_1}(\mu)} \phi_{x_0}(\xi) d\xi \geq p^*, \quad (11)$$

$\mathbb{D}_{\kappa_2}(\xi) \subseteq \Xi$, and

$$\lambda \geq -\frac{\ln\left(\frac{\kappa_2}{\kappa_1}\right)}{T}, \quad (12)$$

Then the control law $u(t) = k(x, x_r, t)$ solves **CCP**.

Proof: We prove the proposition by showing that the initial ball $\mathbb{D}_{\kappa_1}(x_r(0))$, which has the desired minimum cumulative distribution, will contract to $\mathbb{D}_{\kappa_2}(x_r(T))$, which is contained in the desired containment set Ξ , at the transient time T .

By the hypothesis of the proposition, the closed-loop system is partially contracting w.r.t. x which implies that all trajectories starting in $\mathbb{D}_{\kappa_1}(\mu)$ converge to x_r which is also an admissible solution of the virtual system, with exponential rate λ . Hence,

$$d(x_r(t), x(t)) \leq d(x_r(0), x_0) e^{-\lambda t}, \quad (13)$$

for all $x_0 \in \mathbb{D}_{\kappa_1}(\mu)$. For all initial conditions $x_0 \in \mathbb{D}_{\kappa_1}(\mu)$, we have that $d(x_r(0), x_0) \leq \kappa_1$. Hence

$$d(x_r(t), x(t)) \leq \kappa_1 e^{-\lambda t}. \quad (14)$$

Thus at time T , by the hypothesis on λ as in (12),

$$d(x_r(T), x(T)) \leq \frac{\kappa_2}{\kappa_1} \kappa_1 \Rightarrow d(x_r(T), x(T)) \leq \kappa_2. \quad (15)$$

Hence, for this λ we have

$$x_0 \in \mathbb{D}_{\kappa_1}(\mu) \Rightarrow x(T) \in \mathbb{D}_{\kappa_2}(x_r(T)) \subset \Xi \quad (16)$$

It follows then that

$$\int_{\Xi} \phi_{x_0, T}(\xi) d\xi \geq \int_{\mathbb{D}_{\kappa_2}(x_r(T))} \phi_{x_0, T}(\xi) d\xi \geq p^*, \quad (17)$$

which implies that the first control objective **CCPa** is satisfied.

It remains to show that all trajectories with initial conditions $x_0 \in \mathcal{X}$ converge to x_d as $t \rightarrow \infty$. Since the reference signal x_r is such that $\lim_{t \rightarrow \infty} d(x_r(t), x_d(t)) = 0$, it is sufficient to show that all trajectories converge to x_r . From the partial contraction property, we have

$$\lim_{t \rightarrow \infty} d(x(t), x_r(t)) = 0 \Rightarrow \lim_{t \rightarrow \infty} d(x(t), x_d(t)) = 0, \quad (18)$$

for all $x_0 \in \mathcal{X}$. Hence, **CCPb** is satisfied.

This concludes the proof. $\blacksquare$

V. EXAMPLE ON THE CONTROL OF ROBOTIC MANIPULATOR

In this section, we will evaluate the application of our control design to a standard 2-nd order mechanical system operating with 3-DOF, which is a SCARA robot as presented in [20].

A. Dynamics and control design

The robot operates on the manifold $\mathcal{X} = \mathcal{Q} \times \mathbb{R}^3$, with states $q \in \mathcal{Q} = \mathcal{S}^1 \times \mathcal{S}^1 \times \mathbb{R}$, where $\mathcal{S}^1$ the unitary circumference, and $p \in \mathbb{R}^3$. Here, $q^T = [\theta_1, \theta_2, z]$ is the generalized position, and $p^T = [p_{\theta_1}, p_{\theta_2}, p_z] = M(q)\dot{q}$ is the generalized momentum, with $M(q) = M^T(q)$ the inertia matrix, and $u^T = [\tau_1, \tau_2, f]$ the generalized force. The system satisfies the port-Hamiltonian form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_3 & I_3 \\ -I_3 & -D(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} + \begin{bmatrix} 0_3 \\ G(q) \end{bmatrix} u, \quad (19)$$

where $H(q, p)$ is the Hamiltonian function, $D(q) = D^T(q) : \mathcal{Q} \mapsto \mathbb{R}_{\geq 0}^{3 \times 3}$ is the damping matrix and $G(q) : \mathcal{Q} \mapsto \mathbb{R}^{3 \times 3}$ is the input matrix. For the Hamiltonian function we have the total energy as

$$H(q, p) = \frac{1}{2} p^T M^{-1}(q) p + \bar{V}(q) \quad (20)$$

with $\bar{V}(q) = (m_1 + m_2 + m_3)gz$ the potential energy, where m_1, m_2 and m_3 are the masses of the robot manipulator links. For the mass matrix we have

$$\mathbf{M}(q) = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{12} & m_3 l_2^2 & 0 \\ 0 & 0 & (m_1, m_2, m_3)g \end{bmatrix}, \quad (21)$$

where

$$M_{11} = (m_2 + m_3)l_1^2 + m_3 l_2^2 + 2m_3 l_1 l_2 \cos \theta_2,$$

$$M_{12} = m_3 l_2^2 + m_3 l_1 l_2 \cos \theta_2.$$

We assume stochastic initial conditions for the two rotational joints, satisfying a normal distribution. Hence, we have $q_0 \sim \mathcal{N}(\mu_q, \Sigma_q)$, where

$$\mu_q = \begin{bmatrix} \mu_{q,1} \\ \mu_{q,2} \\ \mu_{q,3} \end{bmatrix}, \Sigma_q = \begin{bmatrix} \sigma_{q,1}^2 & 0 & 0 \\ 0 & \sigma_{q,2}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (22)$$

It can be checked that this corresponds to

$$q_0 = \begin{bmatrix} \mu_{q,1} + \sigma_{q,1} s_1 \\ \mu_{q,2} + \sigma_{q,2} s_2 \\ \mu_{q,3} \end{bmatrix} = \begin{bmatrix} \theta_{0,1} \\ \theta_{0,2} \\ z_0 \end{bmatrix}, \quad (23)$$

where s_1 and s_2 are two independent standard normal random variables. We take the system to be idle upon initialization, e.g. $p_0^T = [0, 0, 0]$. The initial conditions $x_0^T = [q_0, p_0]^T \sim \mathcal{N}(\mu, \Sigma)$ then satisfy

$$\mu = \begin{bmatrix} \mu_q \\ 0_{3 \times 1} \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_q & 0_3 \\ 0_3 & 0_3 \end{bmatrix}. \quad (24)$$

In this example, we will use $D = \text{diag}([0.2, 0.2, 0.2])$, $G = \mathbf{I}_3$, $[m_2, m_3] = [1.5, 1, 0.5]$, $[l_1, l_2] = [2, 1]$, $g = 9.81$, $\mu = [1, 0, 0]^T$ and $[\sigma_{q,1}, \sigma_{q,2}] = [1, 1]$. We consider for the desired trajectory $q_d = [\sin(t) + 1, \sin(t), \sin(t)]^T$ and $p_d(t) = M(q_d(t))\dot{q}_d(t)$. For the reference signal, we design q_r s.t. $q_r(0) = \mu_q$, $q_r(T) = \delta \in \Xi$ and $\lim_{t \rightarrow \infty} q_r(t) = q_d(t)$. We will discuss characteristics of Ξ shortly. Assume Ξ is such that we can take $\delta = [\sin(T) + 1, \sin(T), \sin(T)]^T$ and since we also have that $q_d(0) = \mu$ we can conveniently let $q_r(t) = q_d(t)$.

Subsequently, the error system is given by

$$\zeta := \begin{bmatrix} \tilde{q} \\ \omega \end{bmatrix} = \begin{bmatrix} q - q_d \\ p - p_r \end{bmatrix}, \quad (25)$$

where p_r is a momentum reference signal, to be defined. The dynamics of $\tilde{q}$ are given by

$$\dot{\tilde{q}} = M^{-1}(\omega + q_d)p - M^{-1}(q_d)p_d. \quad (26)$$

We define $p_r = p_{d\omega} - \Lambda \tilde{q}$, with $p_{d\omega} = M(\tilde{q} + q_d)\dot{\tilde{q}}$ and $-\Lambda = \Lambda$ Hurwitz. Hence, we obtain the properties $\lim_{t \rightarrow \infty} q(t) = q_d(t)$ and $\lim_{t \rightarrow \infty} p_r(t) = p_d(t)$. The full error dynamics are given by

$$\begin{aligned} \dot{\tilde{q}} &= M^{-1}(\tilde{q} + q_d)(\omega - \Lambda \tilde{q}) \\ \dot{\omega} &= - \left[\frac{\partial H}{\partial q}(q, p) + D \frac{\partial H}{\partial p}(q, p) - u + \dot{p}_r \right] \end{aligned} \quad (27)$$

Accordingly, we obtain the closed-loop error system by applying the control law

$$\begin{aligned} u &= u_{eq} + u_{at}, \\ u_{eq} &= \dot{p}_r + \frac{\partial H}{\partial q}(q, p_r) + D \frac{\partial H}{\partial p}(q, p_r), \\ u_{at} &= -K_d \frac{\partial H}{\partial p}(q, \omega) - M^{-1}(q)\Lambda \tilde{q} + \frac{\partial}{\partial q}(p_r^T M^{-1}(q)\omega), \end{aligned} \quad (28)$$

where K_d is such that

$$D + K_d + \frac{1}{2}I_3 - \frac{1}{4}(M^{-1} + M) > 0. \quad (29)$$

Our system then specifies the contraction properties proven in, and for a virtual system as provided in, [20]. This virtual system admits x and $[q_d^T, p_r^T]^T$ as solutions. The candidate *Finsler-Lyapunov function* for this virtual system, as in *Definition 1* is given by

$$V(x_v, \delta x_v) = \frac{1}{2} \delta x_v^T \Theta^T P(\zeta) \Theta \delta x_v, \quad (30)$$

with $\delta x_v = [\delta q_v, \delta p_v]^T$ as in (4),

$$\Theta = \begin{bmatrix} I_3 & 0_3 \\ \Lambda & I_3 \end{bmatrix}, \quad (31)$$

$$P(\zeta) = \begin{bmatrix} \Lambda & 0_3 \\ 0_3 & M^{-1}(\tilde{q} + q_d) \end{bmatrix}. \quad (32)$$

It follows that the distance as in *Definition 2* satisfies

$$d(x, x_r) = \inf_{\Gamma(x, x_r)} \int_I \sqrt{\left(V(\gamma(s)) \frac{\partial \gamma(s)}{\partial s} \right)} ds. \quad (33)$$

Lastly, we obtain the property

$$d(x(t), x_r(t)) < d(x(0), x_r(0))e^{-\lambda t}, \quad (34)$$

related to a rate λ as

$$\lambda(\zeta) = \min \text{eig}(P^{1/2}(\zeta)\Upsilon(\zeta)P^{1/2}(\zeta)), \quad (35)$$

with

$$\Upsilon(\zeta) = \begin{bmatrix} 2M^{-1}(\tilde{q} + q_d) & (M^{-1}(\tilde{q} + q_d) - I_3) \\ (M^{-1}(\tilde{q} + q_d) - I_3) & 2(D + K_d) \end{bmatrix}. \quad (36)$$

B. Control design parameters

In this section, we will discuss the relevant control design parameters T , p^* , Ξ , Λ and K_d . Firstly, let us apply our notion of the distance as in (6), with respect to the reference signal $x_r = [q_d^T, p_r^T]^T$ as $d(x, x_r)$, with $\gamma(s) = [q_d^T, p_r^T]^T + \zeta s$, e.g. a line, which is a valid infimum for our coordinates. Hence, we also have $\frac{\partial \gamma(s)}{\partial s} = \zeta$. Notice that the distance is furthermore dependent on Λ , which simultaneously influences the contraction rate. Λ should be chosen such that the minimal contraction rate λ is achievable, for our purpose, we can use $\Lambda = \text{diag}\{2, 2, 2\}$. Furthermore, we take $T = 10$, $p^* = 0.7$ and Ξ such that

$$\Xi = \{x \mid d(x, x_r(T)) \leq \kappa_2\} = \mathbb{D}_{\kappa_2}(x_r(T)), \quad (37)$$

for $\kappa_2 = 6$.

Remark 1: For a higher order, multidimensional system such as this, it is not trivial to relate the containment set to the original coordinates. The main reason is that the map $d^{-1} : \mathbb{R}_{\geq 0} \mapsto \mathcal{X} \times \mathcal{X}$ is not unique. However, desired restrictions on the distances in the transient can still be obtained by conducting a study of the distances for different coordinates or empirically. Both options being often time-consuming. Here, we will suffice with the choice of Ξ as given above.

For the initial conditions, we can specify a distance set as follows. For the pdf of our initial condition around μ , we can use the bivariate standard normal pdf for θ_1 and θ_2 , since the other 4 states are deterministic. This is expressed in polar coordinates as

$$\phi_{x_0}(r, \theta) = \frac{r}{2\pi} e^{-0.5r^2}. \quad (38)$$

We then find that

$$\int_0^{1.5517} \frac{r}{2\pi} e^{-0.5r^2} dr = 0.7. \quad (39)$$

Fig. 1. In this figure, we show the circumference of the two distance sets that are relevant for our example; $\mathbb{D}_{\kappa_1}(x_r(0))$ and $\mathbb{D}_{\kappa_2}(x_r(T))$, and the contour of a set of initial conditions whose cumulative density is p^* . The angle (ρ) of this polar plot is interpretable with respect to θ_1 and θ_2 , where $\rho = -\pi \mathbf{1}(-\theta_1 + \mu_1) + \tan^{-1} \left(\frac{\theta_1 - \mu_1}{\theta_2 - \mu_2} \right)$, $\mathbf{1}(\cdot)$ being the step function.

By mapping the contour of this radius through d , we obtain the shape in Fig. 1. These points can hence be captured by a distance set as in (9), $\mathbb{D}_{\kappa_1}(\mu)$, with $\kappa_1 = 15.73$. Accordingly, we have

$$\int_{\mathbb{D}_{\kappa_1}(\mu)} \phi_{x_0}(\xi) d\xi \geq p^*. \quad (40)$$

Remark 2: In contrast with $\Xi = \mathbb{D}_{\kappa_2}(x_r(T))$, we are now able to easily relate the distance set $\mathbb{D}_{\kappa_1}(\mu)$ to the original coordinates. This is due to the knowledge that we assume on the initial conditions, and specifically due to the variations being only present in two dimensions upon initialization.

Let us now determine the minimal contraction rate λ and the corresponding K_d such that we achieve this rate. We find λ as

$$\lambda \geq -\frac{\ln \left(\frac{\kappa_2}{\kappa_1} \right)}{T} = 0.0964. \quad (41)$$

Accordingly, we can choose $K_d = \text{diag}\{3, 1, 25\}$ which is such that this minimal rate is always satisfied in accordance with (35). In fact, we find that that (35) yields $\lambda = 0.0965$ for all trajectories in our example.

C. Simulation results

The simulation results are shown in Fig. 2. It is easy to see that we have achieved

$$\Phi_{\Xi, T} = 1 \geq p^* \quad (42)$$

and thus satisfy **CCPa**. The performance of our system is strong, due to the following:

- 1) The distance set $\mathbb{D}_{\kappa_1}(\mu)$ is greater than a marginal set covering p^* fraction of initial conditions.
- 2) The rate λ is a minimal rate, but the other eigenvalues from (35) generally cause the convergence to be faster than this minimum.

In Figures 3, 4 and 5, we show the convergence of an initial condition that satisfies $d(x_0, x_r) = \kappa_1$. The asymptotic convergence, satisfying **CCPb**, to the reference signal can

Fig. 2. In this figure, we show the distances $d(x(t), x_r(t))$ of both the initial distribution at time $t = 0$ (left) and the distribution at time $t = T$ (right). At time T , all trajectories are in the set $\Xi = \mathbb{D}_{\kappa_2}(x_r(T))$, hence we have achieved our desired performance $\Phi_{\Xi, T} \geq p^*$. The interpretation of the plot angle ρ is as in Fig. 1.

Fig. 3. This figure depicts the time evolution of $\tilde{q}(t) = q(t) - q_d(t)$, for a trajectory satisfying $d(x(0), x_r(0)) = \kappa_1$.

clearly be seen, as well as the fast decay of the initial distance $d(x_0, x_r)$.

VI. CONCLUSIONS

In this paper, we propose control design method for solving containment control problem where we prescribe the transient behaviour of nonlinear systems with stochastic initial conditions, in addition to the standard asymptotic convergence requirement. The method is based on recent results on the contraction-based control design and extends our previous work for LTI systems with stochastic initial conditions. We have shown the applicability and efficacy of our method to the control of robotic manipulator systems.

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Fig. 4. This figure depicts the time evolution of $\tilde{p}(t) = p(t) - p_d(t)$, for the same initial condition as used in Fig. 3.

Fig. 5. This figure depicts the time evolution of the distance, for the same initial condition as used in Fig. 3. Furthermore, we show the difference with the nominal decay, which is an effective upper bound. Lastly, the plot shows convergence to a distance $d = 0$.

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