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Adaptation and disturbance rejection for output synchronization of incrementally output–feedback passive systems[‡]

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SUMMARY

This paper addresses the output synchronization problem of incrementally output–feedback passive nonlinear systems in the presence of exogenous disturbances. Two kinds of distributed controllers are proposed: one placed at the nodes and the other placed at the edges. Each of them is synthesized on the basis of the adaptive control method to cope with the shortage of passivity and on the internal model principle to deal with the disturbances. The proposed controllers synchronize the outputs of the nonlinear systems when the solution of the closed-loop system is bounded. On the basis of this, we present a class of systems for which the boundedness of the solutions is guaranteed. The analysis used in this paper is also applicable to a case where systems are coupled via links modeled by dynamical systems. Simulation results of a network of Van der Pol oscillators show the effectiveness of the proposed methods. Copyright © 2017 John Wiley & Sons, Ltd.

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KEY WORDS: output synchronization; incremental output–feedback passivity; strong coupling; disturbance rejection; solution boundedness

1. INTRODUCTION

Over the past decades, synchronization of multiple systems interconnected through a graph has received considerable attention from the control community because of its large number of applications such as rendezvous, formation control, frequency synchronization of power grids, and oscillators synchronization, to name a few. Motivated by the passive properties of physical systems such as electrical networks and oscillators, the passivity-based approach has proven its usefulness in dealing with the synchronization problem. See, for example, [2–10] (refer to [11] for other approaches based on QUAD or convergent properties). In particular, important classes of nonlinear oscillators fall into the category of systems with a shortage of passivity for which proving synchronization still relies on passivity-based arguments [3, 12].

When dealing with systems that exhibit a shortage of passivity, one of the key requirements to achieve synchronization is the so-called strong coupling condition, meaning that the algebraic connectivity of the graph should exceed a certain threshold value [3–5, 9, 11]. Another important requirement is that the solutions of the closed-loop system are bounded [7, 8, 11], which may not be guaranteed in general. This is mainly because Lyapunov functions used to show synchronization are often constructed from the differences between the states. In this case, even though the boundedness

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of the Lyapunov function is proven, it is not possible to infer the boundedness of the states but only of their differences.

Many dynamical networks are open systems that interact with the environment (e.g., an inventory system with inflow/outflow of material and a smart grid subject to unknown demand and generation). As such, they are often affected by external perturbations that can disrupt synchronization. Internal model-based controllers have been recently proposed as a mean to restore synchronization despite the effect of disturbances [9, 11, 13]. Previously, passivity-based internal model controllers were proposed to deal with formation control problem with unknown reference velocity [14]. Furthermore, passivity and internal model controllers play a role in the control of dynamical networks when the constraints and optimality considerations must be also taken into account [11, 15, 16].

Motivated by the works discussed earlier, we study the output synchronization problem of incrementally output-feedback passive (iOFP) nonlinear systems, defined on an undirected graph but affected by exogenous systems that generate external disturbances. Two different structures of distributed controllers are handled in this paper. In the first one, each local controller is placed at the corresponding node while, in the second one, a controller is associated to each edge. Either at the nodes or at the edges, each controller is a combination of an adaptive law to cope with the shortage of passivity [17, chapter 6] and internal model [18] to deal with the disturbances. It is shown in Section 3 that, despite the shortage of passivity and the presence of external disturbances, the proposed controllers enforce the output synchronization of the iOFP systems if the solution of the closed-loop system is bounded (see Remark 2 for the details of solution boundedness). Then, we show in Section 4 that such boundedness is guaranteed for nonlinear systems that are input-to-state stable (ISS) relative to a compact set $\mathcal{X}$. In Section 4, we also present a class of Lur'e-type nonlinear systems that satisfy both of the iOFP and ISS conditions and include, for example, Van der Pol oscillators and Chua's circuits as special cases.

The structure in which controllers are located at the nodes is common in the literature. It is already known that (iOFP) nonlinear systems interconnected via a *static* diffusive coupling achieve synchronization under strong coupling conditions [3]. In this paper, the difficulty descends from the fact that, as opposed to the static diffusive coupling, the systems at the nodes are now coupled via dynamical systems that aim at guaranteeing synchronization while rejecting the action of the disturbances. Meanwhile, controllers at the edges have attracted interest more recently [10, 11, 13]. In certain problems in, for example, distribution networks, the inputs to the dynamical systems at the nodes are constrained to satisfy certain physical laws (such as Kirchhoff's laws). In these cases, having controllers at the edges is more convenient because they regulate the 'flow' exchanged among the different nodes. Moreover, the analysis carried out in the case of dynamical controllers at the edges turns out to be useful to deal with the case in which the dynamics at the edges are given (Section 3.3). This feature naturally arises in those synchronization problems where the graph models physical interconnections among the nodes [12, 15, 19–21]. Summarizing, this paper contributes to a comprehensive framework in which the output synchronization of nonlinear oscillators occurs in spite of the perturbing phenomena, thanks to the newly devised dynamic controllers. The results provide new insights to the synchronization of electrical networks, complementing recent literature.

A few remarks on the proposed controllers are as follows. In contrast to [7, 8, 22], each adaptive law assigned to the corresponding node (or edge) is one-dimensional, leading in general to controllers of lower dimension. Moreover, it does not require the symmetry (required in, e.g., [7, 8, 22]) of the initial values or the update gains of the adaptive laws. Finally, the controllers are driven by the relative outputs of the nonlinear systems only, while the relative measurements of the internal models and absolute measurements of the nonlinear systems are necessary in [8] and [23], respectively. Thus, pure relative output feedback control for synchronization is achieved in our case.

Notation: We denote an undirected graph by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, A)$. The node set $\mathcal{N} = \{1, 2, \dots, N\}$ is a finite nonempty set of nodes, and the edge set $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is a set of pairs of distinct nodes satisfying $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is a symmetric nonnegative matrix defined in a way that $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. A path connecting the two nodes $i \neq j$ is a sequence of distinct nodes, $\{p_1, p_2, \dots, p_d\}$, such that $p_1 = i$, $p_d = j$, and $(p_g, p_{g+1}) \in \mathcal{E}$ with $g = 1, \dots, d - 1$. The length of the path is $d - 1$. An undirected graph is connected if, for each pair of distinct nodes, there is a path connecting them. The Laplacian

matrix of $\mathcal{G}$ is defined by $L := \Delta - A$, where Δ is the diagonal matrix whose i -th diagonal entry is $\Delta_i := \sum_{j \in \mathcal{N}} a_{ij} = \sum_{j \in \mathcal{N}} a_{ji}$ (by the symmetry of A). The incidence matrix $B = [b_{ig}] \in \mathbb{R}^{N \times E}$ of $\mathcal{G}$ with $E := |\mathcal{E}|$ is such that $b_{ig} := -\sqrt{a_{ij}}$ and $b_{jg} := \sqrt{a_{ij}}$ if (i, j) is the g -th edge of the graph ($g = 1, \dots, E$) and $b_{ig} := 0$ otherwise. By construction, $L = BB^\top$ and $B^\top 1_N = 0$, where $\top$ and $1_N \in \mathbb{R}^N$ denote the matrix transpose and the vector of ones, respectively. See, e.g., [24] for more details on graph theory.

The stacking of vectors $x_1, \dots, x_N$ is denoted by $[x_1; \dots; x_N]$. $\text{diag}(R_1, \dots, R_N)$ is the block diagonal matrix with its i -th diagonal block R_i . The Kronecker product is denoted by $\otimes$, and B^+ denotes the Moore–Penrose pseudoinverse [25] of a matrix B . Euclidean norm is denoted by $\|\cdot\|$, while the 1-norm is denoted by $\|\cdot\|_1$. For a compact set $\mathcal{X}$, we define $\|x\|_{\mathcal{X}} := \min_{y \in \mathcal{X}} \|x - y\|$. I_N is the identity matrix of dimension N and $\Pi := I_N - (1/N)1_N 1_N^\top$. $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers.

2. PROBLEM FORMULATION

Let us consider a group of N nonlinear systems, each of which is described by

$$\begin{aligned} \dot{x}_i &= f(x_i, u_i, d_i), \\ y_i &= h(x_i), \end{aligned} \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i \in \mathbb{R}^n$ and $u_i, y_i \in \mathbb{R}^q$ are the state, input, and output of the i -th system, respectively. We assume that (1) is iOFP in the sense that there exist a storage function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, a number $\sigma \in \mathbb{R}$, and two functions $\underline{\alpha}$ and $\bar{\alpha}$ of class $\mathcal{K}_\infty$ such that $\underline{\alpha}(\|x_i - x'_i\|) \leq \Phi(x_i, x'_i) \leq \bar{\alpha}(\|x_i - x'_i\|)$ and

$$\begin{aligned} &\frac{\partial \Phi}{\partial x_i}(x_i, x'_i) f(x_i, u_i, d_i) + \frac{\partial \Phi}{\partial x'_i}(x_i, x'_i) f(x'_i, u'_i, d'_i) \\ &\leq \sigma (y_i - y'_i)^\top (y_i - y'_i) + (y_i - y'_i)^\top ((u_i + d_i) - (u'_i + d'_i)) \end{aligned} \quad (2)$$

hold for all $(x_i, u_i, d_i) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q$ and $(x'_i, u'_i, d'_i) \in \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^q$. Moreover, $f(\cdot, \cdot, \cdot)$ and $h(\cdot)$ are assumed to be locally Lipschitz and continuously differentiable in their arguments, respectively. The system satisfying (2) is often referred to as σ -relaxed cocoercive system [5]. On the other hand, the system (1) is said to be incrementally passive if $\sigma \leq 0$ and incrementally output–strictly passive if $\sigma < 0$. Examples of iOFP systems include output–feedback passive linear systems, Goodwin oscillators ([9, example 1]), Chua’s circuits ([1, section IV]), and Van der Pol oscillators[‡].

The signal d_i represents the disturbance acting on the i -th system and is generated by the exogenous system

$$\begin{aligned} \dot{w}_i &= s_i(w_i), \\ d_i &= R_i w_i, \end{aligned} \quad w_i \in \mathbb{R}^{m_i}, \quad (3)$$

satisfying $s_i(0) = 0$ and, for all $(w_i, w'_i) \in \mathbb{R}^{m_i} \times \mathbb{R}^{m_i}$,

$$(w_i - w'_i)^\top (s_i(w_i) - s_i(w'_i)) \leq 0. \quad (4)$$

Note that, in this case, the solution of (3) always exists and any ball centered at the origin is forward invariant for (3). An example of systems having the property (4) is the one given by $s_i(w_i) = S_i w_i$ with S_i skew-symmetric.

[‡]It is noted that because Van der Pol oscillators, for instance, have an unstable equilibrium at the origin, they are not passive but iOFP (see Section 5). Therefore, the results based on the standard passivity such as [6] are not applicable to this case.

We assume that the N systems (1) are defined over a connected undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, A)$ with the node set $\mathcal{N} = \{1, 2, \dots, N\}$. Each node $i \in \mathcal{N}$ is associated with the i -th system in (1), and the edges in $\mathcal{E}$ define the interconnection structure. Under this setting, the objective is to design distributed controllers that enforce the asymptotic output synchronization of (1), that is,

$$\lim_{t \rightarrow \infty} \|y_i(t) - y_j(t)\| = 0, \quad i, j \in \mathcal{N}.$$

The difference of our contribution from the vast majority of the literature on nonlinear synchronization is that synchronization should be guaranteed despite the presence of exogenous time-varying signals. These exogenous signals are different from node to node and introduce a source of heterogeneity in the network. We deal with the heterogeneity by using distributed internal-model-based controllers, an approach that has received much attention for both linear and nonlinear systems (refer to, e.g., [23, 26]).

3. CONTROLLER DESIGN FOR OUTPUT SYNCHRONIZATION

In this paper, we deal with two kinds of distributed controllers: one for controllers placed at the nodes (refer to, e.g., [7–9] or Section 3.1) with their inputs being

$$\rho_i = \sum_{j \in \mathcal{N}} a_{ij}(y_j - y_i), \quad i = 1, \dots, N, \quad (5)$$

and the other for controllers placed at the edges (see, e.g., [10, 11] or Section 3.2) with their inputs constrained to be

$$\varrho_g = \sum_{j \in \mathcal{N}} b_{jg} y_j, \quad g = 1, \dots, E. \quad (6)$$

In both cases, the algebraic connectivity (denoted as) λ_2 , the second smallest eigenvalue of L of the graph $\mathcal{G}$, and the internal model principle [18] play crucial roles for the synchronizability of the systems (1) as shown in [8, 11]. On the basis of this observation, we provide internal-model-based solutions to the problem in the next sections, where the strong coupling condition $\lambda_2 > \sigma$ is no longer required. This relaxation is done by using one-dimensional adaptive controller at each node (Section 3.1) or at each edge (Section 3.2).

3.1. Controllers at the nodes

Homogeneous incrementally output-feedback systems of the form $\dot{x}_i = f(x_i) + u_i$ are known to synchronize for the static controller $u_i = \sum_{j \in \mathcal{N}} a_{ij}(x_j - x_i)$ under suitable conditions [3]. This is not guaranteed any longer if the disturbances d_i are affecting the dynamics at the nodes. The controllers we propose in this section are a natural dynamical extension of the static diffusive coupling, in which the dynamics is introduced into the controllers to deal with the heterogeneity of the disturbances.

Let us define $\bar{x} := \frac{1}{N} \sum_{j \in \mathcal{N}} x_j$ and $\bar{y} := \frac{1}{N} \sum_{j \in \mathcal{N}} y_j$. Then, we have the following.

Theorem 1

The outputs of the N systems (1) in closed-loop with the control

$$\dot{\xi}_i = s_i(\xi_i) - R_i^\top \rho_i, \quad (7a)$$

$$\dot{k}_i = \gamma_i \rho_i^\top \rho_i, \quad \gamma_i > 0, \quad (7b)$$

$$u_i = -R_i \xi_i + k_i \rho_i, \quad i \in \mathcal{N} \quad (7c)$$

synchronize asymptotically whenever the closed-loop solution is bounded. In particular, $\lim_{t \rightarrow \infty} \|y_i(t) - \bar{y}(t)\| = 0$ holds for all $i \in \mathcal{N}$.

Remark 1

A different approach to cope with the strong coupling condition is to use adaptive laws corresponding to the edges. For example, in [7, 8, 22], this is done by employing, instead of (5) and (7b), $\rho_i = \sum_{j \in \mathcal{N}} k_{ij} a_{ij} (y_j - y_i)$ and $\dot{k}_{ij} = \gamma_{ij} (y_i - y_j)^\top (y_i - y_j)$ with

$$k_{ij}(0) = k_{ij}(0), \quad \gamma_{ij} = \gamma_{ji} > 0. \quad (8)$$

On the contrary, the update law (7b) is assigned to each node without the symmetry condition (8). Thus, the number of adaptive laws in the entire system is less than that of [7, 8, 22] in general, yielding lesser dimension of controllers. In addition, only relative outputs of (1), aggregated into the variable ρ_i in (5), are required to be measurable to implement (7), while the additional communication of partial states of the internal models (7a) is necessary in the case of [8].

Proof

Let us consider the function V_1 given by

$$V_1(x) = \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} a_{ij} \Phi(x_i, x_j), \quad (9)$$

where $x := [x_1; \dots; x_N] \in \mathbb{R}^{Nn}$. We note that, by Lemma 1 in the Appendix, there are η and $\bar{\eta}$ of class $\mathcal{K}_\infty$ such that $\eta(\|\tilde{x}\|) \leq V_1(x) \leq \bar{\eta}(\|\tilde{x}\|)$, in which $\tilde{x} := (\Pi \otimes I_n)x = x - (1_N \otimes \bar{x})$. Taking its time derivative along the solution of (1) and using Lemma 2 in the Appendix, we have

$$\begin{aligned} \dot{V}_1(x) &\leq \frac{\sigma}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} a_{ij} (y_i - y_j)^\top (y_i - y_j) + \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} a_{ij} (y_i - y_j)^\top (u_i - u_j + d_i - d_j) \\ &= \sigma y^\top (L \otimes I_q) y + y^\top (L \otimes I_q) u + y^\top (L \otimes I_q) d, \end{aligned}$$

where $y := [y_1; \dots; y_N]$, $u := [u_1; \dots; u_N]$, and $d = [d_1; \dots; d_N]$. Because $u = -(KL \otimes I_q)y - R\xi$, it further becomes

$$\dot{V}_1(x) \leq y^\top \{(\sigma L - LKL) \otimes I_q\} y - y^\top (L \otimes I_q)(R\xi - d),$$

where $R := \text{diag}(R_1, \dots, R_N)$, $\xi := [\xi_1; \dots; \xi_N]$, and $K := \text{diag}(k_1, \dots, k_N)$.

Next, let us define $\tilde{\xi}_i := \xi_i - w_i$ and $\tilde{k}_i := k_i - k_\star$ with $k_\star \in \mathbb{R}$ to be determined and consider the function

$$V_2(\tilde{\xi}, \tilde{k}) = \frac{1}{2} \sum_{i \in \mathcal{N}} \tilde{\xi}_i^\top \tilde{\xi}_i + \sum_{i \in \mathcal{N}} \frac{\tilde{k}_i^2}{2\gamma_i},$$

where $\tilde{\xi} := [\tilde{\xi}_1; \dots; \tilde{\xi}_N]$ and $\tilde{k} := [\tilde{k}_1; \dots; \tilde{k}_N]$. Then, from property (4), its time derivative becomes

$$\begin{aligned} \dot{V}_2(\tilde{\xi}, \tilde{k}) &= \sum_{i \in \mathcal{N}} \left\{ \tilde{\xi}_i^\top (s_i(\xi_i) - s_i(w_i)) - \tilde{\xi}_i^\top R_i^\top \rho_i + \tilde{k}_i \rho_i^\top \rho_i \right\} \\ &\leq -\tilde{d}^\top \rho + \rho^\top (\tilde{K} \otimes I_q) \rho + \tilde{d}^\top (L \otimes I_q) y + y^\top (L \tilde{K} L \otimes I_q) y, \end{aligned}$$

where $\rho := [\rho_1; \dots; \rho_N]$, $\tilde{K} := \text{diag}(\tilde{k}_1, \dots, \tilde{k}_N) = K - k_\star I_N$, and $\tilde{d} := R\tilde{\xi}$.

We finally consider the Lyapunov function $V(x, \tilde{\xi}, \tilde{k}) := V_1(x) + V_2(\tilde{\xi}, \tilde{k})$, whose time derivative is given by

$$\dot{V}(x, \tilde{\xi}, \tilde{k}) \leq -y^\top \left((k_\star L^2 - \sigma L) \otimes I_q \right) y.$$

Let $U = [\frac{1}{\sqrt{N}} 1_N \quad Q]$ be an orthogonal matrix such that $U^\top L U = \Lambda$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, $\lambda_1 = 0$, and $\lambda_i > 0$ for $i \neq 1$. Let us assume, with no loss of generality, that

λ_2 is the smallest nonzero eigenvalue of the Laplacian. Let us choose $k_* > 0$ sufficiently large to satisfy $\epsilon := (k_* \lambda_2 - \sigma) \lambda_2 > 0$. Then, noting that $Q Q^\top = \Pi$ and $\Pi = \Pi^2$, one can obtain

$$\dot{V}(x, \tilde{\xi}, \tilde{k}) \leq -\epsilon y^\top (Q Q^\top \otimes I_q) y = -\epsilon \tilde{y}^\top \tilde{y}, \quad (10)$$

where $\tilde{y}_i := y_i - \bar{y}$ and $\tilde{y} := [\tilde{y}_1; \dots; \tilde{y}_N] = (\Pi \otimes I_q) y$. Integrating both sides of (10), we have $\tilde{y}(t) \in \mathcal{L}_2$, that is, square integrable. Because the solution is bounded, $\dot{x}_i(t)$ and hence, $\dot{\tilde{y}}_i(t)$ are bounded. The application of Barbalt lemma [17, 27] proves the result. $\square$

Remark 2

Solution boundedness is an indispensable condition for the validity of the aforementioned theorem and all other results of this section that will be given later. In Section 4, we show how a suitable ISS property of system (1) guarantees the boundedness of the solutions. We also single out an important class of systems that are both iOFP and ISS in Proposition 4 of Section 4.

In the proof of Theorem 1, the function $V_1(x)$ in (9) is used to construct a valid Lyapunov function for the closed-loop system. If the system (1) admits a quadratic storage function, that is, $\Phi(x_i, x'_i) = (x_i - x'_i)^\top P (x_i - x'_i)$ with $P = P^\top > 0$, then by using Lemma 2 in the Appendix, we have $V_1(x) = x^\top (L \otimes P) x = \tilde{x}^\top (L \otimes P) \tilde{x}$. This coincides with the one given in [28, theorem 4]. Therefore, in the absence of d_i (because [28] does not consider the disturbances), our Lyapunov function can be considered as a nonlinear extension of the one in [28, theorem 4].

As one can see from the proof of Theorem 1, the design (7) provides some flexibility. When there is no external disturbance (i.e., (1) is replaced by $\dot{x}_i = f(x_i, u_i)$, $y_i = h(x_i)$, and (2) holds with $d_i \equiv 0$ and $d'_i \equiv 0$), the controller (7b) with its output $u_i = k_i \rho_i$ solves the problem without requiring the strong coupling condition $\lambda_2 > \sigma$ imposed in [3–5]. On the other hand, if $\lambda_2 > \sigma$, then the use of the adaptive controller (7b) can be avoided. In this case, the controller is simply given by (7a) with $u_i = -R_i \xi_i + \rho_i$.

3.2. Controllers at the edges

In this section, motivated by the flow regulation problems [10, 11, 13], we tackle the same problem as before but constrain the controllers to be placed at the edges. In other words, the input to the controller placed at the g -th edge is given by (6), and the input to the i -th system (1) is constrained to be

$$u_i = - \sum_{g=1}^E b_{ig} v_g, \quad i = 1, \dots, N, \quad (11)$$

where $v_g \in \mathbb{R}^q$ is the output of the g -th controller. With this structural constraint, let us define $H := (B^+ \otimes I_q) R \in \mathbb{R}^{Eq \times m}$ and let $H_g \in \mathbb{R}^{q \times m}$ be the g -th block row of $H = [H_1; \dots; H_E]$, where $R := \text{diag}(R_1, \dots, R_N)$ and $m := \sum_{i=1}^N m_i$. This specific value of H in fact descends from the fulfillment of the regulator equations and the internal model property, shown in [10], and is necessary to solve the output agreement problem for incrementally passive nonlinear systems under the constraint (11). In particular, because $u = -(B \otimes I_q) v$ from (11) and $(B \otimes I_q) H w = (\Pi \otimes I_q) d$ from Lemma 3 in the Appendix, $v = H w$ (at steady state) implies $u_i + d_i = \frac{1}{N} \sum_{j \in \mathcal{N}} d_j =: \bar{d}$ (see (2)), where $v := [v_1; \dots; v_E]$, $d := [d_1; \dots; d_N] = R w$, and $w := [w_1; \dots; w_N]$. In other words, the net effect of the disturbances on each system (1) is the same for all $i = 1, \dots, N$ if $v = H w$. This motivates us to consider controllers at the edges, able to generate the feedforward signal $v_g = H_g w$.

Theorem 2

The outputs of (1) in closed-loop with the control

$$\dot{\zeta}_g = s(\zeta_g) + H_g^\top \varrho_g, \quad (12a)$$

$$\dot{\kappa}_g = \delta_g \varrho_g^\top \varrho_g, \quad \delta_g > 0, \quad (12b)$$

$$v_g = H_g \zeta_g + \kappa_g \varrho_g, \quad g = 1, \dots, E \quad (12c)$$

synchronize asymptotically if the corresponding closed-loop solution is bounded, where $s(\cdot)$ is defined by $s(w) := [s_1(w_1); \dots; s_N(w_N)]$. Moreover, $\lim_{t \rightarrow \infty} \|y_i(t) - \bar{y}(t)\| = 0$ holds for all $i = 1, \dots, N$.

Proof

Let us consider the function

$$W_1(x) = \frac{1}{2N} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} \Phi(x_i, x_j), \quad (13)$$

which can be shown to satisfy $\eta(\|\tilde{x}\|) \leq W_1(x) \leq \bar{\eta}(\|\tilde{x}\|)$ for some class $\mathcal{K}_\infty$ functions η and $\bar{\eta}$, again by Lemma 1 in the Appendix. Then, by using Lemma 2 with a_{ij} replaced by $1/N$, one obtains the time derivative of $W_1(x)$ as

$$\dot{W}_1(x) \leq \sigma y^\top (\Pi \otimes I_q) y + y^\top (\Pi \otimes I_q) u + y^\top (\Pi \otimes I_q) d.$$

Let $\varrho := [\varrho_1; \dots; \varrho_E]$, $\zeta := [\zeta_1; \dots; \zeta_E]$, $\bar{H} := \text{diag}(H_1, \dots, H_E)$, and $K := \text{diag}(\kappa_1, \dots, \kappa_E)$. Noting that $\Pi B = B$, $u = -(B \otimes I_q)v$, and $\varrho = (B^\top \otimes I_q)y$, one further has

$$\dot{W}_1(x) \leq \sigma y^\top (\Pi \otimes I_q) y - y^\top (BK B^\top \otimes I_q) y + y^\top (\Pi \otimes I_q) d - y^\top (B \otimes I_q) \bar{H} \zeta.$$

Let us now define $\tilde{\zeta}_g := \zeta_g - w$ and $\tilde{\kappa}_g := \kappa_g - \kappa_\star$, where $\kappa_\star \in \mathbb{R}$ is a constant to be chosen shortly. Let us consider the function

$$W_2(\tilde{\zeta}, \tilde{\kappa}) = \frac{1}{2} \sum_{g=1}^E \tilde{\zeta}_g^\top \tilde{\zeta}_g + \sum_{g=1}^E \frac{\tilde{\kappa}_g^2}{2\delta_g},$$

where $\tilde{\zeta} := [\tilde{\zeta}_1; \dots; \tilde{\zeta}_E]$ and $\tilde{\kappa} := [\tilde{\kappa}_1; \dots; \tilde{\kappa}_E]$. Then, using property (4), we obtain its time derivative as

$$\begin{aligned} \dot{W}_2(\tilde{\zeta}, \tilde{\kappa}) &= \sum_{g=1}^E \left\{ \tilde{\zeta}_g^\top (s(\zeta_g) + H_g^\top \varrho_g - s(w)) + \tilde{\kappa}_g \varrho_g^\top \varrho_g \right\} \\ &\leq \tilde{\zeta}^\top \bar{H}^\top \varrho + \varrho^\top (\tilde{K} \otimes I_q) \varrho = \tilde{\zeta}^\top \bar{H}^\top (B^\top \otimes I_q) y + y^\top (B \tilde{K} B^\top \otimes I_q) y, \end{aligned}$$

in which $\tilde{K} := \text{diag}(\tilde{\kappa}_1, \dots, \tilde{\kappa}_E) = K - \kappa_\star I_E$.

Finally, let us consider the Lyapunov function $W(x, \tilde{\zeta}, \tilde{\kappa}) = W_1(x) + W_2(\tilde{\zeta}, \tilde{\kappa})$ for the closed-loop system. Because $\tilde{\zeta} = \zeta - (1_E \otimes w)$ by construction and $(\Pi \otimes I_q) d = (BB^\top \otimes I_q) R w = (B \otimes I_q) H w = (B \otimes I_q) \bar{H} (1_E \otimes w)$ by Lemma 3 in the Appendix, the derivative of $W(x, \tilde{\zeta}, \tilde{\kappa})$ along the solutions of the system becomes

$$\dot{W}(x, \tilde{\zeta}, \tilde{\kappa}) \leq \sigma y^\top (\Pi \otimes I_q) y - \kappa_\star y^\top (L \otimes I_q) y \leq -(\kappa_\star \lambda_2 - \sigma) \tilde{y}^\top \tilde{y},$$

where λ_2 and $\tilde{y}$ are the ones in the proof of Theorem 1. If $\kappa_\star > 0$ is chosen sufficiently large so that $\varepsilon := \kappa_\star \lambda_2 - \sigma > 0$, then the result follows from similar arguments as in the proof of Theorem 1. $\square$

We note that the design (12) also possesses the flexibility discussed at the end of Section 3.1. In particular, (12b) with $v_g = \kappa_g \varrho_g$ solves the problem in the absence of disturbances and does not require the strong coupling condition in [11], while (12a) with $v_g = H_g \zeta_g + \varrho_g$ achieves the asymptotic output synchronization of (1) if $\lambda_2 > \sigma$.

Figure 1. A motivating example of edges modeled by dynamical systems. Y denotes the admittance of the electrical circuit in the dotted box, and v_L is the voltage across the load. [Colour figure can be viewed at wileyonlinelibrary.com]

3.3. Applications to electrical networks

Our scheme in the previous sections can be further applied to the case in which the dynamics at the edges are given and not free to design [12, 19], such as electrical networks [20, 21]. An example of those cases is illustrated in Figure 1(a), where effective power transfer from the two sources to the load is aimed at. Such power transfer will occur when the terminal voltages of the two sources are synchronized or, equivalently, when the two nodes coupled through the dynamic edge in Figure 1(b) are synchronized. Motivated by this example, let us consider a set of input strictly passive nonlinear systems, attached to the edges,

$$\dot{\eta}_g = \psi_g(\eta_g, \varrho_g), \quad v_g = \varphi_g(\eta_g, \varrho_g), \quad g = 1, \dots, E,$$

where ψ_g and φ_g are locally Lipschitz and continuous, respectively. By input strict passivity, we mean the system satisfies the dissipation inequality $\dot{\Psi}_g(\eta_g) \leq -\varrho_g^\top \varrho_g + v_g^\top \varrho_g$ for some positive definite Ψ_g .

We now provide the following result.

Proposition 1

Assume (1) is disturbance-free, that is, $\dot{x}_i = f(x_i, u_i)$, $y_i = h(x_i)$, and satisfies (2) with $d_i = d'_i = 0$. Then, their outputs synchronize asymptotically if u_i is given by (11), the corresponding closed-loop solution is bounded, and the strong coupling condition $\lambda_2 > \sigma$ is satisfied.

Proof

Let us consider the storage function $W(x, \eta) = W_1(x) + \sum_{g=1}^E \Psi_g(\eta_g)$, where $W_1(x)$ is in (13) and $\eta := [\eta_1; \dots; \eta_E]$. Then, noting $u = -(B \otimes I_q)v$ and $\varrho = (B^\top \otimes I_q)y$, its time derivative is given by

$$\begin{aligned} \dot{W}(x, \eta) &\leq \sigma y^\top (\Pi \otimes I_q) y + y^\top (\Pi \otimes I_q) u - \varrho^\top \varrho + v^\top \varrho \\ &= \sigma y^\top (\Pi \otimes I_q) y - y^\top (BB^\top \otimes I_q) y \leq -(\lambda_2 - \sigma) \tilde{y}^\top \tilde{y}. \end{aligned}$$

The rest can be proven by following similar arguments as in the proofs of Theorems 1 and 2. $\square$

The result of the proposition finds its application in the problems of synchronization of, for example, electrical networks without shunt elements and with dynamic heterogeneous edges. It also applies to the case of electrical networks with shunt elements in which their Kron-reduced networks[§] contain identical shunt elements with (possibly) heterogeneous dynamic edges and each node consisting of a source and its corresponding shunt element in the reduced networks is iOFP (refer to Figure 1 as an example of this case). Therefore, Proposition 1 offers a complementary result and proof technique to [20, 21]. We note however that, in all cases, the boundedness of the closed-loop solutions is indispensable for proving the synchronization. A class of systems guaranteeing such boundedness and including, for example, Van der Pol oscillators as a special case will be discussed

[§]Refer to, for example, [29] for the details on the Kron reduction process of electrical networks.

in Section 4. Note that oscillators that are not globally Lipschitz such as Van der Pol oscillators cannot be handled by the results of [20, 21].

When there are disturbances acting on the systems, the problem becomes more challenging. However, at least on a complete graph, the outputs of the systems can be synchronized as follows.

Proposition 2

Suppose that $a_{ij} = a$ for $i, j \in \mathcal{N}$ and for some $a > 0$. Then, the outputs of the N systems (1) in closed-loop with the controllers

$$\begin{aligned}\dot{\xi}_i &= s_i(\xi_i) - R_i^\top \rho_i, \\ u_i &= -R_i \xi_i - \sum_{g=1}^E b_{ig} v_g, \quad i = 1, 2, \dots, N\end{aligned}\quad (14)$$

synchronize asymptotically if $\lambda_2 > \sigma$ and the corresponding closed-loop solution is bounded.

Proof

Let us consider the storage function $W(x, \eta, \tilde{\xi}) = W_1(x) + \sum_{g=1}^E \Psi_g(\eta_g) + \sum_{i=1}^N \tilde{\xi}_i^\top \tilde{\xi}_i / (2aN)$, where $\tilde{\xi}_i := \xi_i - w_i$ and $\tilde{\xi} := [\tilde{\xi}_1; \dots; \tilde{\xi}_N]$. Taking its time derivative and bearing in mind that $\rho = -aN(\Pi \otimes I_q)y$ and $u = -R\tilde{\xi} - (B \otimes I_q)v$, we obtain

$$\begin{aligned}\dot{W}(x, \eta, \tilde{\xi}) &\leq \sigma y^\top (\Pi \otimes I_q)y + y^\top (\Pi \otimes I_q)(u + d) - \varrho^\top \varrho + v^\top \varrho - \tilde{d}^\top \rho / (aN) \\ &= -y^\top \{(L - \sigma \Pi) \otimes I_q\}y \leq -(\lambda_2 - \sigma)\tilde{y}^\top \tilde{y},\end{aligned}$$

where $\tilde{d} := R\tilde{\xi} - d$. Thus, the result again follows from similar arguments as in the previous sections. $\square$

It is also possible to achieve the output synchronization without having the strong coupling conditions imposed in the previous propositions. The idea is to consider the passive dynamical edges

$$\dot{\eta}_g = \psi_g(\eta_g, \varrho_g), \quad v_{1g} = \varphi_g(\eta_g, \varrho_g), \quad g = 1, \dots, E,$$

satisfying $\dot{\Psi}_g(\eta_g) \leq v_{1g}^\top \varrho_g$ and to assign adaptive laws to the corresponding edges. We note that such assignment may not be feasible in the case of, for example, electrical networks. However, the two corollaries given next are theoretically interesting and complement the results of the previous propositions in the sense that they provide adaptive variants of the propositions to relax strong coupling conditions.

Corollary 1

Assume the setup of Proposition 1. Then, the outputs of (1) synchronize asymptotically by the control

$$\begin{aligned}\dot{\kappa}_g &= \delta_g \varrho_g^\top \varrho_g, \quad v_{2g} = \kappa_g \varrho_g, \quad \delta_g > 0, \quad g = 1, \dots, E, \\ u_i &= -\sum_{g=1}^E b_{ig} (v_{1g} + v_{2g}), \quad i = 1, \dots, N\end{aligned}$$

if the corresponding closed-loop solution is bounded.

Proof

Let us consider the function $W(x, \eta, \tilde{\kappa}) = W_1(x) + \sum_{g=1}^E \Psi_g(\eta_g) + \sum_{g=1}^E \tilde{\kappa}_g^2 / (2\delta_g)$, where $\tilde{\kappa}_g := \kappa_g - \kappa_\star$, $\kappa_\star > 0$ and $\tilde{\kappa} := [\tilde{\kappa}_1; \dots; \tilde{\kappa}_E]$. Let us define $v_i := [v_{i1}; \dots; v_{iE}]$ with $i = 1, 2$ and $v := v_1 + v_2$. Then, noting $u = -(B \otimes I_q)v$ and $v_2 = (K \otimes I_q)\varrho = (KB^\top \otimes I_q)y$ with $K := \text{diag}(\kappa_1, \dots, \kappa_E)$, one obtains the time derivative of W as

$$\begin{aligned}\dot{W}(x, \eta, \tilde{\kappa}) &\leq \sigma y^\top (\Pi \otimes I_q)y + y^\top (\Pi \otimes I_q)u + v_1^\top \varrho + \varrho^\top (\tilde{K} \otimes I_q)\varrho \\ &= \sigma y^\top (\Pi \otimes I_q)y - y^\top (B \otimes I_q)v_2 + y^\top (B\tilde{K}B^\top \otimes I_q)y \\ &= \sigma y^\top (\Pi \otimes I_q)y - \kappa_\star y^\top (BB^\top \otimes I_q)y \leq -(\kappa_\star \lambda_2 - \sigma)\tilde{y}^\top \tilde{y}.\end{aligned}$$

The rest can be proven by following similar arguments as in the proofs of Theorems 1 and 2. $\square$

Corollary 2

Assume the setup of Proposition 2. Then, the outputs of the systems (1) in closed-loop with

$$\begin{aligned}\dot{\xi}_i &= s_i(\xi_i) - R_i^\top \rho_i, & \dot{\kappa}_g &= \delta_g \varrho_g^\top \varrho_g, & \delta_g &> 0, \\ u_i &= -R_i \xi_i - \sum_{g=1}^E b_{ig} (v_{1g} + v_{2g}), & v_{2g} &= \kappa_g \varrho_g\end{aligned}$$

synchronize asymptotically whenever the solution is bounded.

Proof

Let us consider the function $W(x, \eta, \tilde{\xi}, \tilde{\kappa}) = W_1(x) + \sum_{g=1}^E \Psi_g(\eta_g) + \sum_{i=1}^N \tilde{\xi}_i^\top \tilde{\xi}_i / (2aN) + \sum_{g=1}^E \tilde{\kappa}_g^2 / (2\delta_g)$. Then, noting $u = -R\xi - (B \otimes I_q)v$, $\rho = -aN(\Pi \otimes I_q)y$, and $v_2 = (KB^\top \otimes I_q)y$, we have

$$\begin{aligned}\dot{W}(x, \eta, \tilde{\kappa}) &\leq \sigma y^\top (\Pi \otimes I_q)y + y^\top (\Pi \otimes I_q)(u + d) + v_1^\top \varrho - \tilde{d}^\top \rho / (aN) + \varrho^\top (\tilde{K} \otimes I_q)\varrho \\ &= \sigma y^\top (\Pi \otimes I_q)y - y^\top (B \otimes I_q)v_2 + y^\top (B\tilde{K}B^\top \otimes I_q)y \leq -(\kappa_* \lambda_2 - \sigma) \tilde{y}^\top \tilde{y}.\end{aligned}$$

Thus, the result again follows from similar arguments as in the previous sections. $\square$

The extension of these results to the case of dynamics at the edges satisfying different and more general dissipation inequalities from those assumed previously and to the case of graphs, which are not complete, as well as the relaxation of the strong coupling condition by assigning adaptive laws to the nodes, is left for future investigation.

4. SOLUTION BOUNDEDNESS

In the previous section, the synchronization is guaranteed, provided that the solution of the closed-loop system is bounded. We now discuss on what conditions of the open-loop dynamics (1) such boundedness is ensured under the control (7) or (12). To do this, we further assume that the output map $h(\cdot)$ is globally Lipschitz and there is a compact set $\mathcal{X} \subset \mathbb{R}^n$, invariant for (1) when $u_i \equiv 0$ and $d_i \equiv 0$, such that the system (1) is ISS with respect to $\hat{u}_i := [u_i; d_i]$ relative to $\mathcal{X}$. In other words, there are two functions $\beta(\cdot, \cdot)$ and $\mu(\cdot)$ of class $\mathcal{KL}$ and $\mathcal{K}$, respectively, such that the solution of (1) satisfies

$$\|x_i(t)\|_{\mathcal{X}} \leq \max \left\{ \beta(\|x_i(0)\|_{\mathcal{X}}, t), \mu \left(\sup_{0 \leq \tau \leq t} \|\hat{u}_i(\tau)\| \right) \right\}. \quad (15)$$

We refer the reader to [23, section III] for additional details on the condition (15). A class of systems ensuring both conditions (2) and (15) will be discussed after presenting the following result.

Proposition 3

Let the ISS property (15) be satisfied, and suppose one of the proposed controllers is applied to the system (1). Then, the solutions of the closed-loop system are bounded and satisfy $\lim_{t \rightarrow \infty} \|y_i(t) - \bar{y}(t)\| = 0$, $i = 1, \dots, N$.

Proof

We prove the case for controllers placed at the nodes (7). Other cases in Sections 3.2 and 3.3 can be proven similarly.

Let the variables x and ξ be the ones in the proof of Theorem 1, and define $k := [k_1; \dots; k_N]$. Let $[0, T_u)$, $T_u < +\infty$ be the maximal time interval where the unique solution of the closed-loop system starting at $(x(0), \xi(0), k(0))$ exists. Then (10) holds on this interval. As a result, $\xi(t)$ and $k(t)$ are bounded for all $t \in [0, T_u)$ because $w(t)$ is bounded for $t \geq 0$ from (4). Note that $x_i(t) - x_j(t)$, $i, j \in \mathcal{N}$ are also bounded on the interval because $V_1(x)$ in (9) is positive definite with respect to $\tilde{x}$. Together with the globally Lipschitz output map $h(x_i)$, this implies the boundedness of the control inputs $u_i(t)$, $i \in \mathcal{N}$ on $[0, T_u)$. Therefore, the solution $x_i(t)$ is bounded from the ISS property

Figure 2. A Lur'e-type nonlinear system consisting of a passive linear system Σ and a static nonlinearity $\phi(\cdot)$ in the feedback path.

(15), leading to the existence of a positive constant M such that $\|x(t); \xi(t); k(t)\| \leq M$ for all $t \in [0, T_u)$. This means that the solution of the closed-loop system can be extended to $[0, T]$ for some $T > T_u$, which contradicts the assumption that $[0, T_u)$ is the maximal interval of existence of the solution. Thus, $T_u = +\infty$, and the result follows. $\square$

The results in Section 3 combined with Proposition 3 require the open-loop system (1) to satisfy (2) and (15), simultaneously. We note that such systems indeed exist. To see this, let us consider the Lur'e-type nonlinear systems (Figure 2 or [17, section 7]) of the form

$$\begin{aligned}\dot{x}_i &= Ax_i + B(-\phi(y_i) + u_i + d_i), \\ y_i &= Cx_i,\end{aligned}\tag{16}$$

where (C, A) is detectable and the linear part is passive, that is, there is a matrix $P = P^\top > 0$ such that $A^\top P + PA \leq 0$ and $PB = C^\top$ hold. The nonlinearity $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz and satisfies $\lim_{\tau \rightarrow +\infty} \phi(\tau) = +\infty$ and $\lim_{\tau \rightarrow -\infty} \phi(\tau) = -\infty$. It is also assumed that there is $\tau_\star > 0$ (no matter how large it is) such that $\phi(\tau)$ is monotonically increasing on both intervals $(-\infty, -\tau_\star]$ and $[\tau_\star, +\infty)$. Then, we have the following.

Proposition 4

The Lur'e feedback system (16) satisfies both the iOFP condition (2) and the ISS property (15).

Proof

The satisfaction of the ISS property (15) follows from [31, theorem 2]. To show the iOFP of (16), let us consider a storage function $\Phi(x_i, x'_i) = \frac{1}{2}(x_i - x'_i)^\top P(x_i - x'_i)$ and its time derivative along the solutions x_i and x'_i of (16) as

$$\begin{aligned}\dot{\Phi}(x_i, x'_i) &= (x_i - x'_i)^\top P \{Ax_i + B(-\phi(y_i) + u_i + d_i) - Ax'_i - B(-\phi(y'_i) + u'_i + d'_i)\} \\ &\leq (y_i - y'_i)(-\phi(y_i) + \phi(y'_i) + u_i - u'_i + d_i - d'_i).\end{aligned}$$

This implies that (16) is iOFP if there is $\sigma \in \mathbb{R}$ satisfying $\pi(y_i, y'_i) := -(y_i - y'_i)(\phi(y_i) - \phi(y'_i)) \leq \sigma(y_i - y'_i)^2$. The existence of such σ can be shown as follows.

- (i) $y_i, y'_i \in [-\tau_\star, \tau_\star]$: Because ϕ is locally Lipschitz, $\pi(y_i, y'_i) \leq \varpi(y_i - y'_i)^2$ holds, where $\varpi > 0$ is a Lipschitz constant of ϕ on $[-\tau_\star, \tau_\star]$.
- (ii) $y_i, y'_i \in [\tau_\star, +\infty)$: By the monotonicity of ϕ on $[\tau_\star, +\infty)$, it holds that $\pi(y_i, y'_i) \leq 0 \leq \varpi(y_i - y'_i)^2$.
- (iii) $y_i \in [-\tau_\star, \tau_\star]$ and $y'_i \in [\tau_\star, +\infty)$: From the Lipschitz property and monotonicity of ϕ , it follows that

$$\begin{aligned}\pi(y_i, y'_i) &= -(y_i - y'_i)(\phi(y_i) - \phi(\tau_\star)) - (y_i - y'_i)(\phi(\tau_\star) - \phi(y'_i)) \\ &\leq \varpi|y_i - y'_i||y_i - \tau_\star| \leq \varpi(y_i - y'_i)^2.\end{aligned}$$

[¶]An example of such nonlinearities is the function $\phi(\cdot)$ illustrated in Figure 2. We note that, in contrast to [22, 30], the nonlinearity ϕ considered here is neither incrementally passive (due to possible negative slope at the origin) nor incrementally sector bounded (implying that the slope of the nonlinearity needs not to be uniformly bounded).

Figure 3. The characteristic of the nonlinearity $\phi(\cdot)$ in Chua's circuit.

- (iv) $y_i \in (-\infty, -\tau_*)$ and $y'_i \in [\tau_*, +\infty)$: Again, by the Lipschitzness and monotonicity of ϕ , we have

$$\begin{aligned} \pi(y_i, y'_i) &= -(y_i - y'_i) \{(\phi(y_i) - \phi(-\tau_*)) + (\phi(-\tau_*) - \phi(\tau_*)) + (\phi(\tau_*) - \phi(y'_i))\} \\ &\leq \varpi |y_i - y'_i| |(-\tau_*) - \tau_*| \leq \varpi (y_i - y'_i)^2. \end{aligned}$$

The remaining cases can be proven similarly. Therefore, $\pi(y_i, y'_i) \leq \sigma (y_i - y'_i)^2$ holds for all $y_i, y'_i \in \mathbb{R}$ with $\sigma = \varpi > 0$. $\square$

Examples of such Lur'e-type nonlinear systems include Van der Pol oscillators (see Section 5 for the details) and the Liénard-type dead-zone oscillators (refer to [21]). Another example is the Chua's circuit [32, 33] whose dimensionless form is given by

$$\begin{aligned} \dot{x}_{i,1} &= c_1(x_{i,2} - x_{i,1} - \phi(x_{i,1}) + u_i + d_i), \\ \dot{x}_{i,2} &= x_{i,1} - x_{i,2} + x_{i,3}, \\ \dot{x}_{i,3} &= -c_2 x_{i,2}, \quad y_i = x_{i,1}, \end{aligned} \quad (17)$$

where $x_i := [x_{i,1}; x_{i,2}; x_{i,3}] \in \mathbb{R}^3$, $c_1, c_2 > 0$ and the nonlinearity $\phi(\cdot)$ is shown in Figure 3. It is noted that the unforced system (i.e., (17) with $u_i \equiv 0$ and $d_i \equiv 0$) with particular selection of parameters c_j , τ_b , τ_* , and z_j has the double-scroll (chaotic) attractor (see, for instance, [32, figure 6] and [33, figure 23]). The Chua's circuit (17) is of the form (16), and its linear part satisfies the passivity requirements $A^\top P + PA \leq 0$ and $PB = C^\top$ with $P = \text{diag}(1/c_1, 1, 1/c_2)$. Moreover, the piecewise linear function $\phi(\cdot)$ in Figure 3 guarantees all the required properties of local Lipschitzness, monotonicity, and limiting behaviors at the infinity. Therefore, by Proposition 4, the Chua's circuit (17) ensures both the iOFP and the ISS conditions.

5. COMPUTER SIMULATION: VAN DER POL OSCILLATORS

Let us consider Van der Pol oscillators [17, example 2.9] given by

$$\begin{aligned} \dot{x}_{i,1} &= x_{i,2}, \\ \dot{x}_{i,2} &= -x_{i,1} + \nu(x_{i,2} - x_{i,2}^3/3) + u_i + d_i, \quad i = 1, \dots, 4, \\ y_i &= x_{i,2}, \end{aligned} \quad (18)$$

where $x_i := [x_{i,1}; x_{i,2}] \in \mathbb{R}^2$ and $\nu \in \mathbb{R}$ is a positive constant. If we define $A := [0 \ 1; -1 \ 0]$, $B := [0; 1]$, $C := [0 \ 1]$, and $\phi(y_i) := \nu(x_{i,2}^3/3 - x_{i,2})$, then (18) is of the form (16) and satisfies all the required properties with $P = I_2$. Thus, the Van der Pol oscillator (18) is iOFP and satisfies the ISS property given in (15). Moreover, one can verify that $\sigma \geq \nu$, that is, σ cannot be less than ν . We set $\nu = 1$. Meanwhile, the disturbances d_i , $i = 1, \dots, 4$ in (18) are assumed to be generated by the exosystems (satisfying the requirement (4))

$$\begin{aligned} \dot{w}_i &= 0, \quad d_i = w_i, \quad w_i \in \mathbb{R}, \quad i = 1, 2, \\ \dot{w}_i &= \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{s_i(w_i)} w_i, \quad d_i = \underbrace{[1 \ 0]}_{R_i w_i} w_i, \quad w_i \in \mathbb{R}^2, \quad i = 3, 4. \end{aligned} \quad (19)$$

Figure 4. Simulation results with controllers placed at the nodes (left) and at the edges (right). [Colour figure can be viewed at wileyonlinelibrary.com]

In this setting, we first perform a set of simulations to demonstrate the results in Sections 3.1 and 3.2 with the interconnection structure $\mathcal{G}$ characterized by its Laplacian $L = 0.09\bar{L}$ and incidence matrix $B = 0.3\bar{B}$, where

$$\bar{L} = \begin{bmatrix} 5 & -1 & 0 & -4 \\ -1 & 14 & -9 & -4 \\ 0 & -9 & 10 & -1 \\ -4 & -4 & -1 & 9 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -1 & 3 & 2 & 0 & 0 \\ 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -2 \end{bmatrix}.$$

Note that the strong coupling condition is not satisfied in this case because $\lambda_2 = 0.4$ and $\sigma \geq 1$.

Figure 4(a) and (c) shows a simulation result with the controllers placed at the nodes (7), while Figure 4(b) and (d) corresponds to a simulation result with the controllers at the edges (12). The update gains γ_i and δ_g are selected as $\gamma_i = \delta_g = 1$, $i = 1, \dots, 4$, $g = 1, \dots, 5$. All the elements of the initial conditions of the Van der Pol oscillators and exosystems are randomly chosen within the interval $[-3, 3]$, while those of the controllers are set all to zeros. From Figure 4(c) and (d), it is seen that the outputs of the Van der Pol oscillators (18) synchronize asymptotically as expected. Note however that the remaining states $x_{i,1}(t)$, $i = 1, \dots, 4$ do not synchronize because the proposed controllers guarantee the output synchronization only.

To assess the performance of the adaptive versus the nonadaptive controllers, we consider the same setup as before, except that now the controllers at the nodes (7) are replaced by the controllers (7a) and (7c). The gains k_i are all fixed equal to $k_\star$ that satisfies $k_\star > \frac{\sigma}{\lambda_2} \geq \frac{10}{4}\nu = \frac{5}{2}$. Similarly, we replace the previous controllers at the edges with the controllers (12a) and (12c) where all the gains κ_g are fixed equal to $\kappa_\star$ satisfying $\kappa_\star > \frac{\sigma}{\lambda_2} \geq \frac{10}{4}\nu = \frac{5}{2}$. Figure 5 illustrates the state and output responses of the systems under the effect of the nonadaptive node and edge controllers for values of $k_\star$ and $\kappa_\star$ both fixed equal to 3. The output responses of the system under the adaptive and the nonadaptive controllers have similar behaviors. We note however that λ_2 and σ are generally unknown or hard to compute, so the nonadaptive controllers may not be feasible in practice.

Figure 5. Simulation results with nonadaptive controllers placed at the nodes (left) and at the edges (right). [Colour figure can be viewed at wileyonlinelibrary.com]

For the case of edges with dynamic systems attached as in Section 3.3, let us consider the systems (18) connected over the complete graph with $a_{ij} = 1$ for $i, j = 1, \dots, 4$, that is,

$$L = 4\Pi, \quad B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & -1 \end{bmatrix}.$$

In this case, the strong coupling condition is guaranteed because $\lambda_2 = 4$ and the shortage of passivity, σ , can be taken as $\sigma = \nu = 1$. The edges connected to the first Van der Pol oscillator are modeled by

$$\dot{\eta}_g = \varrho_g, \quad v_g = \eta_g + \varrho_g, \quad g = 1, 2, 3,$$

whereas the rest is given by the dynamical systems

$$\dot{\eta}_g = -\eta_g + \varrho_g, \quad v_g = \eta_g + \varrho_g \quad g = 4, 5, 6.$$

Note that, with the storage function $\Psi(\eta_g) = \frac{1}{2}\eta_g^2$, both dynamics satisfy the dissipation inequality for the input strict passivity given in Section 3.3. The left column of Figure 6 shows a simulation result with the controller (11) in the absence of disturbances. The right column of Figure 6 shows a simulation result with the controller (14) in the presence of the external disturbances generated by the systems (19). Each element of the initial conditions of the Van der Pol oscillators, exosystems, and the dynamics at the edges is randomly chosen within the interval $[-3, 3]$. The initial conditions of controllers are set to all zeros. As expected from Propositions 1 and 2, the outputs of the Van der Pol oscillators (18) synchronize asymptotically. See the middle row of Figure 6, that is, Figure 6(c) and (d).

Figure 6. Simulation results in the presence of dynamical edges without disturbances (left) and with disturbances (right). [Colour figure can be viewed at wileyonlinelibrary.com]

6. CONCLUSION

In this paper, we have considered the asymptotic output synchronization problem of iOFP nonlinear systems (defined over a connected undirected graph) in the presence of external disturbances. Two different structures of distributed controllers are considered: one placed at the nodes and the other placed at the edges. In both cases, the proposed controllers are shown to solve the problem if the solution of the corresponding closed-loop system is bounded. A pair of results has been also dis-

cussed to deal with the case in which the dynamics at the edges are given. A class of iOFP systems that guarantee the boundedness of the closed-loop solutions is then provided, by using the notion of ISS relative to a set. The controllers are synthesized on the basis of the adaptive control technique and the internal model principle. As a consequence, the results of the paper do not require the so-called strong coupling condition.

APPENDIX

Lemma 1

Suppose that $\mathcal{G}$ is connected. Then, there are two class $\mathcal{K}_\infty$ functions $\underline{\eta}$ and $\bar{\eta}$ that satisfy $\underline{\eta}(\|\tilde{x}\|) \leq V_1(x) \leq \bar{\eta}(\|\tilde{x}\|)$, where $V_1(x)$ is in (9) and $\tilde{x} = (\Pi \otimes I_n)x$.

Proof

(Existence of $\bar{\eta}$): Define $\tilde{x}_i := x_i - \bar{x}$. Because

$$\alpha(\tau_1 + \cdots + \tau_N) \leq \alpha(N\tau_1) + \cdots + \alpha(N\tau_N) \quad (20)$$

holds[‡] for any class $\mathcal{K}_\infty$ function α , one has

$$\begin{aligned} V_1(x) &\leq \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} a_{ij} \bar{\alpha}(\|\tilde{x}_i\| + \|\tilde{x}_j\|) \leq \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} a_{ij} (\bar{\alpha}(2\|\tilde{x}_i\|) + \bar{\alpha}(2\|\tilde{x}_j\|)) \\ &= \sum_{i \in \mathcal{N}} \Delta_i \bar{\alpha}(2\|\tilde{x}_i\|) \leq \Delta_{\max} \sum_{i \in \mathcal{N}} \bar{\alpha}(2\|\tilde{x}_i\|), \end{aligned}$$

where $\Delta_{\max} := \max_i \Delta_i > 0$. Defining $\tilde{X} := [\|\tilde{x}_1\|; \dots; \|\tilde{x}_N\|] \in \mathbb{R}^N$, the inequality further becomes

$$V_1(x) \leq \Delta_{\max} \sum_{i \in \mathcal{N}} \bar{\alpha}(2\|\tilde{x}_1\| + \cdots + 2\|\tilde{x}_N\|) = N\Delta_{\max} \bar{\alpha}(2\|\tilde{X}\|_1) \leq N\Delta_{\max} \bar{\alpha}(2\sqrt{N}\|\tilde{X}\|).$$

Observing $\|\tilde{X}\| = \|\tilde{x}\|$, set $\bar{\eta}(\tau) := N\Delta_{\max} \bar{\alpha}(2\sqrt{N}\tau)$.

(Existence of $\underline{\eta}$): For $1 \leq i \leq N-1$ and $j > i$, let $\{p_{ij,1}, p_{ij,2}, \dots, p_{ij,d_{ij}}\}$ be one of the shortest paths connecting the nodes i and j . Its length is $d_{ij} - 1$. The total length of those $N(N-1)/2$ paths satisfies

$$\sum_{i=1}^{N-1} \sum_{j>i} (d_{ij} - 1) \leq \sum_{i=1}^{N-1} \sum_{j>i} (N-1) = \frac{N(N-1)^2}{2}.$$

Define $a_{\min} := \min_{a_{ij} \neq 0} a_{ij} > 0$ and let $\mathcal{N}_i$ be the set of neighbors of node i . Then, using (20) and the triangular inequality, and noting that $p_{ij,1} = i$ and $p_{ij,d_{ij}} = j$, we have

$$\begin{aligned} V_1(x) &\geq \frac{a_{\min}}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} \underline{\alpha}(\|x_i - x_j\|) \geq \frac{a_{\min}}{N(N-1)^2} \sum_{i=1}^{N-1} \sum_{j>i} \sum_{g=1}^{d_{ij}-1} \underline{\alpha}(\|x_{p_{ij,g}} - x_{p_{ij,g+1}}\|) \\ &\geq \frac{a_{\min}}{N(N-1)^2} \sum_{i=1}^{N-1} \sum_{j>i} \underline{\alpha} \left(\frac{\sum_{g=1}^{d_{ij}-1} \|x_{p_{ij,g}} - x_{p_{ij,g+1}}\|}{d_{ij} - 1} \right) \\ &\geq \frac{a_{\min}}{N(N-1)^2} \sum_{i=1}^{N-1} \sum_{j>i} \underline{\alpha} \left(\frac{\|x_i - x_j\|}{N-1} \right) = \frac{a_{\min}}{2N(N-1)^2} \sum_{i=1}^N \sum_{j=1}^N \underline{\alpha} \left(\frac{\|x_i - x_j\|}{N-1} \right). \end{aligned}$$

[‡]Indeed, letting τ_i be one of the largest elements among $\tau_1, \dots, \tau_N$, we have that $\alpha(\tau_1 + \cdots + \tau_N) \leq \alpha(N\tau_i) \leq \alpha(N\tau_1) + \cdots + \alpha(N\tau_N)$.

By using (20) and the triangular inequality again, one finally has

$$\begin{aligned} V_1(x) &\geq \frac{a_{\min}}{2N(N-1)^2} \alpha \left(\frac{1}{N(N-1)^2} \sum_{i=1}^N \sum_{j=1}^N \|x_i - x_j\| \right) \\ &\geq \frac{a_{\min}}{2N(N-1)^2} \alpha \left(\frac{1}{(N-1)^2} \sum_{i=1}^N \|\tilde{x}_i\| \right) \geq \frac{a_{\min}}{2N(N-1)^2} \alpha \left(\frac{\|\tilde{X}\|}{(N-1)^2} \right). \end{aligned}$$

Set $\underline{\eta}(\tau) := a_{\min} \alpha(\tau/(N-1)^2)/\{2N(N-1)^2\}$. $\square$

Lemma 2

Let $\theta_i, \vartheta_i \in \mathbb{R}^q$ for $i = 1, \dots, N$. Then, the following holds.

$$\frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} a_{ij} (\theta_i - \theta_j)^\top (\vartheta_i - \vartheta_j) = \theta^\top (L \otimes I_q) \vartheta,$$

where $\theta := [\theta_1; \dots; \theta_N]$ and $\vartheta := [\vartheta_1; \dots; \vartheta_N]$.

Proof

The following computation proves the lemma.

$$\begin{aligned} \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} a_{ij} (\theta_i - \theta_j)^\top (\vartheta_i - \vartheta_j) &= \sum_{i \in \mathcal{N}} \Delta_i \theta_i^\top \vartheta_i - \frac{1}{2} \sum_{i \in \mathcal{N}} \left(\theta_i^\top (A_i \otimes I_q) \vartheta + \theta^\top (A_i^\top \otimes I_q) \vartheta_i \right) \\ &= \theta^\top (\Delta \otimes I_q) \vartheta - \theta^\top (A \otimes I_q) \vartheta = \theta^\top (L \otimes I_q) \vartheta, \end{aligned}$$

where A_i is the i -th row of the symmetric adjacency matrix A . $\square$

Lemma 3

Let $\mathcal{G}$ be a connected undirected graph. Then, $BB^+ = LL^+ = \Pi$ holds.

Proof

Because $B^+ = B^\top (BB^\top)^+$ by [25, p. 49] and $L = BB^\top$, we have $BB^+ = LL^+$. On the other hand, by [25, p. 60, corollary 7], LL^+ is the unique (orthogonal) projector on $\text{im}(L)$ along $\ker(L^+) = \ker(L^\top) = \ker(L)$, where $\text{im}(L)$ and $\ker(L)$ denote the image and kernel of L , respectively. Therefore, $LL^+ = \Pi$ because Π is also the projector on $1_N^\perp = \text{im}(L)$ along $\text{span}(1_N) = \ker(L^+)$, where $1_N^\perp$ is the orthogonal complement of 1_N . $\square$

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