Balanced Truncation Approach to Linear Network System Model Order Reduction
Cheng, Xiaodong; Scherpen, Jacquelien M.A.

Published in:
IFAC-PapersOnLine

DOI:
10.1016/j.ifacol.2017.08.408

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2017

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment.

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Download date: 23-10-2023
Balanced Truncation Approach to Linear Network System Model Order Reduction

Xiaodong Cheng * Jacquelien M.A. Scherpen *
* Jan C. Willems Center for Systems and Control, Faculty of Mathematics and Natural Sciences, University of Groningen, Nijenborgh 4, 9747 AG Groningen, the Netherlands, (e-mail: {x.cheng,j.m.a.scherpen}@rug.nl).

Abstract:
In this paper, we propose a model reduction method for semistable Laplacian dynamics, which describe the behaviors of network systems. In the method, the original semistable system is split into an average system and asymptotically stable part. We only implement the balanced truncation to reduce the dimension of the stable part and obtain the reduced-order model preserving the semistability. Then, a specific coordinate transform enables to convert the resulting reduced-order model to the lower-dimensional network system that represents a simplified complete network with less vertices. The reduction procedure allows for the a priori computation of a bound on the approximation error between the original and reduced Laplacian dynamics. Finally, the proposed method is illustrated by an example.

© 2017, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.

Keywords: Model reduction, Network systems, Laplacian Dynamics, Balanced truncation, Coordinate transformations, Error estimation

1. INTRODUCTION

In the real world, the behavior of a large number of systems is captured by a series of interacting subsystems, which are interconnected with each other according to some specific topologies. These systems are usually referred to network systems or multi-agent systems that often are described by Laplacian dynamics, see Mirzaev and Gunawardena (2013). A typical example is the power grid of a certain region, which consists of generators and loads interconnected by transmission lines. However, Laplacian dynamics with complex spatial structures are often modeled by large-scale differential equations. For many purposes, including controller design, system simulation and validation, it is crucial to have methodologies to derive lower-order approximations of the original high-dimensional systems. Meanwhile, it is also desirable for the obtained reduced-order model to inherit a network interpretation. More precisely, the simplified model preserves a network structure by defining a reduced graph with fewer vertices, which is useful for further analysis, such as synchronization and distributed controller design.

Conventional reduction techniques, including balanced truncation, Krylov subspace methods and moment matching, have been intensively investigated, see e.g. Moore (1981); Antoulas (2005); Astolfi (2010) and references therein. However, as claimed by Monshizadeh et al. (2014); Besselink et al. (2016); Ishizaki et al. (2014), a direct application of these methods to network systems potentially destroys the network interpretation in the reduced-order model. This is because the obtained projection matrices have no specific structures to preserve the algebraic structure of Laplacian matrices. Towards the network structure preservation, most of the researches focus on the clustering-based approaches (see van der Schaft (2014); Monshizadeh et al. (2014); Ishizaki et al. (2014, 2015); Mlinarić et al. (2015); Cheng et al. (2016c,b,a); Cheng and Scherpen (2016); Cheng et al. (2017)). These approaches naturally preserve the spatial structure of networks and show an insightful physical interpretation for the reduction process. However, the approximation error highly relies on cluster selections, and there are no a priori error bounds given in the above references. Besselink et al. (2014, 2016) considers an edge system and introduces the generalized edge controllability and observability Gramians to identify the importance of edges. Then, vertices linked by the edges of less importance are merged to reduce the dimension of a graph. This approach provides an a priori bound of the approximation error. However, the reduction process and error computation are heavily reliant on the tree topology of the system.

In the current paper, we investigate the reduction problem of network systems with a different way of thinking. In the first step, we derive a lower-dimensional model by the adoption of a standard reduction method, e.g. balanced truncation or Krylov subspace approaches, which only intends to preserve the semistability in the reduced-order model. Then, the second step aims to find a specific coordinate transformation, which converts the resulting reduced-order model to its equivalent network state space realization, which is in the form of Laplacian dynamics again. Thus, the simplified network system can be interpreted as dynamics on undirected completed graph. Naturally, the approximation error from this approach is exactly determined by the first step. Consequently, an a priori error bound is guaranteed if we apply balanced truncation in the first step.
Compared to the existing results, the advantages of the proposed method are stated as follows. First, we consider a general graph which can have various topologies. Second, there are no restrictions on the input and output distributions. Specifically, unlike Ishizaki et al. (2014, 2015); Cheng et al. (2016c,b), we also consider the effort of states’ observability, and we do not require our output matrix to possess special structures as in Monshizadeh et al. (2014); Mlinarić et al. (2015). More importantly, the reduced-order Laplacian dynamics obtained by our method will generally have a smaller approximation error than those found by clustering-based approaches. Instead of using the characteristic matrix of a graph clustering to project the original network system to a reduced space, our method do not restrict the structure of the projection matrix in the first step. Therefore, in principle, we can implement some $H_2$-optimal model order reduction techniques to obtain the finest reduced-order model, and then transform the resulting system to its network realization with the same input-output behavior.

This paper is organized as follows. The model reduction problem of Laplacian dynamics is formulated in Section 2. The main results are presented in Section 3, including the balanced truncation of the stable part of the system and the network realization of the reduced-order model. Finally, the proposed method is compared with two existing methods by a simulation example in Section 4, and conclusions are summarized in Section 5.

In addition, we use the following notation throughout this paper. The set of real numbers is denoted by $\mathbb{R}$. $I_n$ and $1_n$ represent the identity matrix of size $n$ and all-ones vector of $n$ entries, respectively. $|S|$ means the cardinality of set $S$. Moreover, the $H_2$-norm and $H_{\infty}$-norm of a transfer function $T(s)$ is denoted by $\|T(s)\|_{H_2}$ and $\|T(s)\|_{H_\infty}$, respectively.

2. PRELIMINARY AND PROBLEM FORMULATION

A weighted graph is defined by a triplet $G=(V, E, W)$, where $V$ and $E \subseteq V \times V$ are the sets of vertices and edges, respectively. If a graph $G$ consists of $n$ vertices and $m$ edges, then $|V| = n$ and $|E| = m$. Besides, $W \in \mathbb{R}^{n \times n}$ is called weighted adjacency matrix. The $(i, j)$ entry of $W$, denoted by $w_{i,j}$, is positive if edge $(i, j) \in E$, and $w_{i,j} = 0$ otherwise.

Furthermore, the Laplacian matrix of graph $G$ is denoted by $L \in \mathbb{R}^{n \times n}$ whose $(i, j)$ entry satisfies

$$L_{ij} = \begin{cases} \sum_{j=1, j \neq i}^{n} w_{i,j}, & \text{if } i = j, \\ -w_{i,j}, & \text{otherwise}. \end{cases}$$

In case of undirected graphs, we have $w_{i,j} = w_{j,i}$. Besides, let $w_i$ be the the weight associated to the edge $i$, then

$$L = EWE^T, \quad (2)$$

where $W := \text{diag}(w_1, w_2, \ldots, w_m) \in \mathbb{R}^{n \times m}$, and $E$ is called the incidence matrix of $G$, see Mesbahi and Egerstedt (2010) for the above definitions.

Remark 1. For a connected undirected graph, the Laplacian matrix $L$ fulfills the following structural conditions:

- $1^TL = 0$, and $L1 = 0$;
- $L_{ij} \leq 0$ if $i \neq j$, and $L_{ij} > 0$ otherwise;
- $L$ is positive semi-definite with one zero eigenvalue.

Conversely, if a real matrix satisfies the above conditions, it can be interpreted as a Laplacian matrix which uniquely represents a connected undirected graph. ■

A network system is composed of multiple interacting sub-systems which are interconnected according to a certain graph topology. In this paper, we consider the dynamics of each vertex to be an integrator as follows.

$$\dot{x}_i = -\sum_{j=1, j \neq i}^{n} w_{i,j} (x_i - x_j) + \sum_{k=1}^{p} f_{ik} u_k, \quad (3)$$

where $x_i$ and $u_k$ are the state of the $i$-th vertex $(i \in V)$ and the $k$-th external input signal, respectively. $w_{i,j}$ presents the coupling strength between vertices $i$ and $j$. If we consider external outputs $y \in \mathbb{R}^q$, the whole network system is then given by a compact form

$$\dot{\Sigma} : \begin{cases} \dot{x} = -Lx + Fu, \\ y = Hx, \end{cases} \quad (4)$$

where $x \in \mathbb{R}^n$, $F \in \mathbb{R}^{n \times p}$, and $H \in \mathbb{R}^{q \times n}$. $L \in \mathbb{R}^{n \times n}$ is a Laplacian matrix associated with a weighted connected undirected graph $G$. Some typical examples of the network system in (4) can be found in van der Schaft (2014) and the references therein, including interconnected mass-damper systems and single-species reaction networks.

Compared to the systems investigated in Ishizaki et al. (2014); Monshizadeh et al. (2014); Mlinarić et al. (2015); Cheng et al. (2016c), the network model in (4) is more general, since $\Sigma$ in (4) has multiple inputs and outputs, and there are no specific constraints for $F$ and $H$. Due to the singularity of $L$, $\Sigma$ is not asymptotically stable. Instead, it is called semistable (Bernstein and Bhat, 1995), because $\Sigma$ has one eigenvalue at the origin. It means that the model reduction techniques developed for asymptotically stable systems are not applicable in this case.

The purpose of this paper is to present a model reduction approach for the network system $\Sigma$ such that the resulting reduced-order model still inherits a network structure, i.e., it still can be interpreted as network system evolving over a weighted connected undirected graph with less number of vertices. Specifically, this model reduction problem is formulated as follows.

Problem. For an $n$-dimensional network system $\Sigma$ as in (4), find a lower-dimensional model

$$\dot{\tilde{\Sigma}} : \begin{cases} \dot{x} = -\tilde{L}\tilde{x} + \tilde{F}u, \\ \tilde{y} = \tilde{H}\tilde{x}, \end{cases} \quad (5)$$

such that the following requirements are satisfied:

- $\dot{\tilde{x}} \in \mathbb{R}^r$, $\tilde{L} \in \mathbb{R}^{r \times r}$, $\tilde{F} \in \mathbb{R}^{r \times p}$, $\tilde{H} \in \mathbb{R}^{q \times r}$ with $r \ll n$;
- The input-output behavior of $\tilde{\Sigma}$ is similar to that of $\Sigma$, i.e., $\|\Sigma - \tilde{\Sigma}\|_{H_{\infty}}$ is small enough;
- $\tilde{L}$ is a Laplacian matrix representing a connected undirected reduced graph, i.e., $\tilde{L}$ fulfills all the structural conditions in Remark 1;

The network structure is maintained in the sense of the preservation of Laplacian matrix. If $\tilde{L}$ inherits the algebraic structure in Remark 1, then we are able to construct a connected undirected graph with less vertices, and the reduced-order model evolves on this reduced graph.
In this section, we present a novel scheme of network approximation. In the first step, we split the semistable system $\Sigma$ into two subsystems: an average system $\Sigma_a$ and a stable system $\Sigma_s$. Then, we apply the balanced truncation method to approximate $\Sigma_a$. In the second step, we find a coordinate transformation to convert the reduced model to its network realization. We will discuss these two aspects in the following subsections, respectively.

\subsection*{3.1 Balanced Truncation of Network Systems}

Observe that the network system $\Sigma$ is semistable, which means that a direct application of balanced truncation method to the overall system may be not feasible since semistability is not guaranteed to be preserved. Thus, in this section, we first use a coordinate transformation to split $\Sigma$ into two subsystems.

Notice that $L$ is symmetric matrix with a simple zero eigenvalue. We consider the following spectral decomposition

$$L = T\Lambda L^T = [T_1 \ T_2] \begin{bmatrix} \bar{\Lambda}_L & 0 \\ 0 & T_2^T \end{bmatrix}.$$  \hfill (6)

Here, $\bar{\Lambda}_L$ is diagonal and positive definite. $T_2 \in \mathbb{R}^n$ is equal to $1/\sqrt{n}$, which is obtained from Remark 1.

Then, we consider a coordinate transformation $x = Tz$, which yields an equivalent representation of $\Sigma$:

$$\begin{align*}
\dot{z} &= -T^T LTz + T^T Fu, \\
y &= HTz,
\end{align*}$$

which contains two subsystems: an average system

$$\Sigma_a : \begin{align*}
\dot{z}_a &= \frac{1}{\sqrt{n}} T^T Fu, \\
y_a &= \frac{1}{\sqrt{n}} H1 z_a,
\end{align*}$$

and a stable system

$$\Sigma_s : \begin{align*}
\dot{z}_s &= A z_s + Bu, \\
y_s &= C z_s.
\end{align*}$$

where

$$A = A^T = -\bar{\Lambda}_L, \quad B = T_1^T F, \quad \text{and} \quad C = HT_1.$$  

Hereafter, we propose a balanced truncation method, which reduces the dimension of $A$ from $(n-1) \times (n-1)$ to $(r-1) \times (r-1)$ and meanwhile preserves the symmetry of $A$ in the reduced-order model. To this end, we consider the balanced truncation approach using the generalized Gramians (see Dullerud and Paganini (2013) for the definition).

In this paper, the generalized controllability and observability Gramians of $\Sigma_a$, denoted by $\mathcal{X} \in \mathbb{R}^{(n-1) \times (n-1)}$ and $\mathcal{Y} \in \mathbb{R}^{(n-1) \times (n-1)}$, follow the structures

$$\mathcal{X} = \mathcal{X}^T \geq 0, \quad \text{and} \quad \mathcal{Y} \geq 0 \quad \text{is diagonal.} \hfill (10)$$

Furthermore, $\mathcal{X}$ and $\mathcal{Y}$ satisfy

$$\begin{align*}
\mathcal{X} + \mathcal{X} A + B B^T C &\leq 0, \\
\mathcal{Y} + \mathcal{Y} A + C^T C &\leq 0.
\end{align*}$$

The above inequalities can be easily solved: $\mathcal{X}$ is a solution of the corresponding Lyapunov equation of (11), while $\mathcal{Y}$ is computed by LMI with objective function min $\text{trace}(\mathcal{Y})$.

Then, analogous to conventional Lyapunov balancing, $\Sigma_a$ can also be balanced based on the pair $(\mathcal{X}, \mathcal{Y})$. The system $\Sigma_a$ is balanced, if

$$\mathcal{X} = \mathcal{Y} = \Sigma = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}, \hfill (13)$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n-1} \geq 0$ is the so-called generalized Hankel singular values (GHSVs).

The observation of the differences of the vertex states is sometimes of interest. The disagreement among the vertices are embedded in the incidence matrix $E$, therefore, in e.g. Monshizadeh et al. (2014), the output $y = W^{1/2} E^T x$ is considered. The following proposition then discusses the generalized observability Gramian in this case.

\textbf{Proposition 2.} Consider the system $\Sigma$ in (4). If $H = W^{1/2} E^T$, the generalized observability Gramian defined in (12) is given by $\mathcal{Y} = \frac{1}{2} I$.

\textbf{Proof.} In this special case, $C^T C = T_1^T EW E^T T_1 = T_1^T LT_1$. Then, (12) becomes

$$T_1^T LT_1 \mathcal{Y} + \mathcal{Y} T_1^T LT_1 + T_1^T LT_1 \leq 0.$$  \hfill (14)

Clearly, $\mathcal{Y} = \frac{1}{2} I$ is a solution of the above inequality. \hfill \blacksquare
Now, we adopt Petrov-Galerkin projection to project the 
\( n - 1 \) dimension model \( \Sigma_s \) to its \( r - 1 \) reduced space, while 
this projection is obtained by balanced truncation method, and we refer to e.g. Moore (1981); Laub et al. (1987); 
Antoulas (2005) for the details. Consider the singular value 
decompositions (SVDs):
\[
\mathcal{X} = U_s \Sigma_s V_c, \quad \mathcal{Y} = U_o \Sigma_o V_o,
\]
and denote
\[
P := (U_s \Sigma_s^{1/2})^T, \quad Q := (U_o \Sigma_o^{1/2})^T.
\]
Then applying SVD to matrix \( PQ^T \) leads to
\[
PQ^T = U \Sigma V^T = [U_1 \ U_2] \begin{bmatrix} \Sigma_1 & V_1^T \\ \Sigma_2 & V_2^T \end{bmatrix},
\]
where the diagonal entries of \( \Sigma \) are the generalized Hankel singular values, and \( \Sigma_1 = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{r-1}\} \in \mathbb{R}^{(r-1) \times (r-1)} \) is positive definite and \( \Sigma_2 \) captures all the zero singular values. Finally, the projection matrices are 
given as follows.
\[
\psi^T = \Sigma_1^{-1/2} V_1^T Q \quad \text{and} \quad \nu^T = P^T U_1 \Sigma_1^{-1/2},
\]
A reduced-order model is obtained by truncating based on the GHSVs of \( \Sigma_s \):
\[
\tilde{\Sigma}_s : \begin{cases}
\hat{z}_s = \tilde{A}\hat{z}_s + \tilde{B}u, \\
y_s = \tilde{C}\hat{z}_s,
\end{cases}
\]
where \( \hat{z}_s \in \mathbb{R}^{r-1} \), \( \tilde{A} = \psi^T A \psi \), \( \tilde{B} = \psi^T B \), and \( \tilde{C} = C \nu \).

The following lemma implies that there is a coordination transformation such that \( \tilde{A} \) becomes symmetric.

**Lemma 3.** All the eigenvalues of \( \hat{A} \) in (19) are real negative, i.e., \( \hat{A} \) is similar to a symmetric matrix.

**Proof.** Observe that \( \nu = Q^T \psi \). Then, we have
\[
\nu^T \nu = \Sigma_1^{-1/2} U_1^T PQ^T \psi^T U_1 \Sigma_1^{-1/2} = \Sigma_1^{-1/2} U_1^T U \Sigma V^T V \Sigma U_1 \Sigma_1^{-1/2} = \Sigma_1^{-1/2} [I \ 0] \Sigma^2 [I \ 0] \Sigma_1^{-1/2} = \Sigma_1.
\]
It follows that
\[
(\nu^T \nu)^{-1} \nu^T \nu = \Sigma_1^{1/2} \Sigma_1^{-1/2} U_1^T PQ^T \psi^T U_1 \Sigma_1^{-1/2} \nu^T \nu = \psi^T \nu.
\]
Notice that \( \nu A \) is diagonal. Therefore, the reduced system matrix \( \hat{A} = \psi^T A \psi = (\nu^T \nu)^{-1} \nu^T \nu A \nu \) is similar to 
\((\nu^T \nu)^{-1/2} \nu^T (\nu A \nu) \nu^T (\nu^T \nu)^{-1/2} \), which is symmetric negative definite. \( \blacksquare \)

**Remark 4.** The corresponding model reduction error bound is twice the sum of the neglected GHSVs, i.e.,
\[
\|\Sigma_s - \tilde{\Sigma}_s\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r}^{n-1} \sigma_i.
\]
For details see Dullerud and Paganini (2013). \( \blacksquare \)

Finally, combining the systems \( \tilde{\Sigma}_s \) and \( \Sigma_s \) yields a \( r \)-dimensional simplified model of \( \Sigma \):
\[
\tilde{\Sigma} : \begin{cases}
\dot{\hat{z}} = \tilde{A}\hat{z} + \frac{1}{\sqrt{n}}1^T F u, \\
\dot{y} = \tilde{C} \frac{1}{\sqrt{n}} H 1 \hat{z},
\end{cases}
\]
where \( \hat{z} = [\hat{z}_s \ z_o]^T \). Lemma 3 indicates that the reduced-order model \( \tilde{\Sigma} \) has only one pole at the origin, and all the other poles are real and strictly negative. Hence the semistability is preserved in \( \tilde{\Sigma} \).

### 3.2 Network Realization of Reduced-Order Model

Although the reduced-order model \( \tilde{\Sigma} \) is semistable, it is not in the form of (4) and cannot be interpreted as a network system. Therefore, the most crucial problem is whether there always exists a coordinate transform such that \( \tilde{\Sigma} \) can be transformed to the reduced Laplacian dynamics. More precisely, if we denote
\[
\mathcal{A} = \begin{bmatrix} \hat{A} & 0 \end{bmatrix},
\]
we can find a nonsingular matrix \( \mathcal{T} \) such that
\[
\hat{L} = -T^{-1} \mathcal{A} \mathcal{T}
\]
with \( \hat{L} \) a Laplacian matrix representing a reduced connected undirected graph.

**Lemma 3** guarantees that \( -\mathcal{A} \) has exactly one zero eigenvalue while the others are positive. This is an necessary property of a undirected graph Laplacian matrix. Based on this, the solution of the above problem is given by the following theorem.

**Theorem 5.** Consider an \( n \times n \) diagonal matrix
\[
\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)
\]
with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0 \). Then, there always exists an undirected complete graph Laplacian matrix \( \mathcal{L} \) whose eigenvalue decomposition is given by
\[
\mathcal{L} = \mathcal{U} \Lambda \mathcal{U}^T.
\]
The proof will be provided in the full version of this paper.

In fact, Theorem 5 implies that for any real values \( \lambda_1 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0 \), there must be a realization of undirected complete network, i.e., each pair of vertices is connected by an edge with nonzero weight. However, this result does not hold for incomplete graphs.

We explain this by a simple example.

Consider a 3-node incomplete graph. Without loss of generality, we assume vertex 2 is not adjacent to vertex 3, i.e., \( w_{2,3} = 0 \). Then, the Laplacian matrix is given by
\[
\mathcal{L} = \begin{bmatrix} w_{1,2} + w_{1,3} & -w_{1,2} & -w_{1,3} \\ -w_{1,2} & w_{1,2} & 0 \\ -w_{1,3} & 0 & w_{1,3} \end{bmatrix},
\]
and
\[
|\Lambda I_3 - \mathcal{L}| = \lambda [\lambda^2 - 2(w_{1,2} + w_{1,3})\lambda + 3w_{1,2}w_{1,3}],
\]
Suppose the eigenvalues of \( \mathcal{L} \) are assigned to \( \lambda_1 \geq \lambda_2 > \lambda_3 = 0 \). We obtain
\[
\begin{bmatrix} w_{1,2} + w_{1,3} = \frac{1}{2} (\lambda_1 + \lambda_2), \\ w_{1,2}w_{1,3} = \frac{1}{3} \lambda_1 \lambda_2. \end{bmatrix}
\]
Therefore, both of $w_{1,2}$, $w_{1,3}$ satisfy
\[ w^2 - \frac{1}{2}(\lambda_1 + \lambda_2)w + \frac{1}{3} \lambda_1 \lambda_2 = 0 \]
\[ \Leftrightarrow \left[ w - \frac{1}{4}(\lambda_1 + \lambda_2) \right]^2 - \frac{\lambda_1^2 + \lambda_2^2}{16} + \frac{5 \lambda_1 \lambda_2}{24} = 0. \]  
(31)

Equation (31) has a real solution if and only if
\[ \frac{\lambda_1^2 + \lambda_2^2}{2} \leq \frac{5 \lambda_1 \lambda_2}{3}. \]
(32)

This constraint does not hold when $\lambda_1 > 3\lambda_2$, which implies that the eigenvalues of an incomplete graph do not always match a real set $\Omega = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ with $\lambda_1 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$.

Therefore, in the current paper, we find a network realization of the system $\Sigma$ with a complete graph topology. For a given matrix $-A$ in (24) whose eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r-1} > \lambda_r = 0$, Theorem 5 guarantees that there always exists a Laplacian matrix $\hat{L}$ such that $-A$ is similar to $\hat{L}$. Therefore, we apply a coordinate transform $\hat{\xi} = T \hat{x}$ to the system $\hat{\Sigma}$ in (23), which yields a reduced-order network model
\[ \hat{\Sigma} : \begin{cases} \dot{\hat{x}} = -\hat{L}\hat{x} + \hat{F}u, \\ \dot{\hat{y}} = \hat{H}\hat{x}, \end{cases} \]
(35)
with $\hat{F} = T^{-1} \frac{1}{\sqrt{n}} 1^T F$ and $\hat{H} = \left[ \hat{C} \frac{1}{\sqrt{n}} H^T \right] T$.

Then, the following theorem gives the approximation error between the original and reduced-order network systems.

**Theorem 6.** Consider the $n$-dimensional network system $\Sigma$ and its reduced-order network model $\hat{\Sigma}$. The approximation between $\Sigma$ and $\hat{\Sigma}$ is bounded by
\[ \|\Sigma - \hat{\Sigma}\|_{H_\infty} \leq 2 \sum_{i=r}^{n-1} \sigma_i, \]  
(36)
with $\sigma_i$-th GHSV of the system $\Sigma_a$ in (9).

**Proof.** Notice that $\hat{\Sigma}$ is obtained by coordinate transformation from $\Sigma$, we have $\|\Sigma - \hat{\Sigma}\|_{H_\infty} = 0$. Then, it follows from $\Sigma = \Sigma_a + \Sigma_a$, and $\hat{\Sigma} = \hat{\Sigma}_a + \hat{\Sigma}_a$ that
\[ \|\Sigma - \hat{\Sigma}\|_{H_\infty} \leq \|\Sigma_a - \hat{\Sigma}_a\|_{H_\infty} + \|\hat{\Sigma} - \hat{\Sigma}\|_{H_\infty}, \]
(37)
where (22) is used.

4. ILLUSTRATIVE EXAMPLE

For comparison, we consider a network system from Monshizadeh et al. (2014); Mlinarić et al. (2015), where 10 vertices are linearly diffusely coupled according to an weighted undirected connected graph as in Fig. 2.

The original Laplacian and the input and output matrices in (4) are given by
\[ L = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \end{bmatrix}. \]
\[ H = W^{1/2} R^T \]
with
\[ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]
and $W = \text{diag}(5, 3, 2, 1, 2, 3, 5, 2, 6, 7, 6, 7, 1, 1, 1)$.

From Proposition 2, the generalized observability Gramian of its stable system is $\mathcal{Y} = \frac{1}{4} I$. Now, we apply balanced truncation method and Theorem 5 to find a 5-dimensional reduced network model $\hat{\Sigma}$ in (35), whose lower-dimensional Laplacian is computed as
\[ \hat{L} = \begin{bmatrix} 18.4350 & -4.8194 & -1.9862 & -0.2556 & -11.3738 \\ -4.8194 & 11.8806 & -1.9862 & -0.2556 & -4.8194 \\ -1.9862 & -1.9862 & 6.2144 & -0.2556 & -1.9862 \\ -0.2556 & -0.2556 & -0.2556 & 1.0225 & -0.2556 \\ -11.3738 & -4.8194 & -1.9862 & -0.2556 & 18.4350 \end{bmatrix}, \]
This new system can be interpreted as the form of Laplacian dynamics as shown in Fig. 3.

By Theorem 6, we obtain the upper bound of the approximation error: $\|\Sigma - \hat{\Sigma}\|_{H_\infty} \leq 0.0466$, while the real approximation error is given by $\|\Sigma - \hat{\Sigma}\|_{H_\infty} \approx 0.0352$.

Next, we compare the approximation accuracy obtained by our method and the algorithms in Monshizadeh et al. (2014) and Mlinarić et al. (2015) in Table 1. It shows that our method has a much better performance in the sense of relative $H_2$ error.

<table>
<thead>
<tr>
<th>Approaches</th>
<th>Proposed Method</th>
<th>Monshizadeh</th>
<th>Mlinarić</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|\Sigma - \hat{\Sigma}|_{H_2}$</td>
<td>0.0849</td>
<td>0.5270</td>
<td>0.1459</td>
</tr>
<tr>
<td>$|\Sigma - \hat{\Sigma}|<em>{H</em>\infty}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

5. CONCLUSION

We have presented a novel method for model order reduction of Laplacian dynamics. Balanced truncation is applied to derive a lower-order model from the original network system, where the semistability is preserved. Then, a coordinate transformation is found to convert this reduced model to the simpler Laplacian dynamics, i.e., a simpler system defined on a new graph with fewer vertices. Compared to the clustering-based approaches, this method can produce a better-approximated reduced network system,
and allow for an a prior approximation error bound. In the end, the feasibility of this method is illustrated by an example.

Currently, the reduced network system evolves over a complete graph. However, it will be better to find more general graphs (i.e., some incomplete graphs) which may match the desired eigenvalues. Besides, the extensions of this method to second-order networks and directed networks are also under consideration.

REFERENCES


