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Modeling and Control of Heat Networks With Storage: The Single-Producer Multiple-Consumer Case

Tjardo Scholten, Claudio De Persis, and Pietro Tesi

Abstract—In heat networks, energy storage in the form of hot water in a tank is a viable approach to balancing supply and demand. In order to store a desired amount of energy, both the volume and temperature of the water in the tank need to converge to desired setpoints. To this end, we provide a provably correct internal model controller that guarantees these tracking goals and is robust against parameter uncertainties. In order to design this controller and analyze the closed-loop system, we derive a nonlinear model from first principles. This model describes the temperature and volume dynamics of a setup consisting of a single producer with a storage tank and multiple consumers. We show that the control goal can be achieved by measuring only the aggregated flow rate of the consumers, the volume of the storage device, and the corresponding temperature.

Index Terms—District heating systems (DHSs), heat exchanger networks (HENs), output regulation, thermal storage.

NOMENCLATURE

V	Volume ^{1, 2} (m ³).
q	Flow rate ¹ (m ³ /s).
W	Work (J).
H	Enthalpy (J).
Φ	Heat exchange (J).
P	Heat exchange rate ¹ (J/s).
E	Total energy ^{2, 3} (J).
$\hat{E}$	Relative energy ² (J).
T	Temperature ^{1, 2} (°C).
T^{ref}	Reference temperature (°C).
$\hat{T}$	Relative temperature ^{1, 2} (°C).

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¹Variables denoted by superscript p refer to producers, while c refers to a consumer. In the case of a superscript c , the subscript i is used to distinguish between consumers.

²The hot and cold parts of the storage tank are denoted by superscripts s_h and s_c , respectively.

³For any control volume, the superscripts *in* and *out* mean the inflow and outflow, respectively.

C_p	Specific heat [$J/(kg \text{ } ^\circ C)$].
ρ	Density (kg/m^3).
m	Mass (kg).
U	Heat transfer coefficient [$W/(m^2 \text{ } ^\circ C)$].
A_h	Area (m^2).

I. INTRODUCTION

ENERGY demand is rising, and a considerable fraction of the energy is consumed in the form of heat. While high-quality resources such as natural gas are used for heating, waste heat is left unutilized in most cases. Since this increase in energy consumption has a negative impact on the environment, there is a need for more energy-efficient systems. One of the proposed solutions is the integration of heat networks into the existent infrastructure. Such heat networks are commonly referred to as district heating systems (DHSs) when found in an urban area or heat exchanger networks (HENs) in industrial environments.

Although DHSs have been around since the early 1900s, recent years have witnessed a renewed interest in heating systems for several reasons. One of these reasons is the recent advances in geothermal energy production methods. Another reason is the change in governmental policies with the intention to reduce carbon dioxide emissions. Since adding a DHS to the existing infrastructure often increases the energy efficiency of the overall system, the total carbon dioxide emissions drop. Two examples of higher energy efficiencies due to a DHS are utilizing waste heat and incorporating environmentally friendly sources such as biomass incinerators. A third example is the integration of combined heat power (CHP) plants into a DHS. Since a CHP can produce both heat and electricity, it has a better energy efficiency than conventional power plants that produce only electricity.

One of the reasons why waste heat is not yet shared between different companies on a large scale is the mismatch in supply and demand. Heat may be produced when it is not needed, and conversely, may not be available when needed. Thus, it is imperative to have an efficient system to store heat and provide it when needed. For this reason, storage elements are included in DHSs. Furthermore, if prices in such a network are time varying, a storage device can be used to store energy until prices rise, thus increasing revenues for all parties.

A. Literature Review

General modeling principles of chemical and thermodynamic systems can be found in [1]. Models for heat exchangers are widely available; see [2], where controllability and observability are also investigated. There is a wide variety of possibilities for thermal storage. A survey of different techniques can be found in [3]. The most common way is to use the water tanks with a fixed volume that can be heated and cooled. In such tanks, there are three layers, one with hot water, one with cold water, and a separation layer called a thermocline, which has a steep thermal gradient. This type of storage is referred to as stratification and is studied in detail in [4] and [5] using a numerical evaluation. An alternative is to have an empty tank that can be filled/drained with hot water, but has the disadvantages of lower efficiency and higher dissipation rates. Recent interest has shifted toward heat storage in phase-changing materials, which have some interesting properties such as low dissipation rates.

In the 1980s, the synthesis and control of DHSs and HENs were mostly focused on the steady-state optimal design of heat exchangers, which resulted in simple engineering techniques such as the pinch method [6]. With the introduction of linear optimization techniques such as model predictive control, the focus later shifted to the design of optimal controllers where optimal steady states were considered. An example of this approach can be found in [7]. The same approach applied to cooling systems can be found in [8] and [9]. Although cooling systems serve a different purpose, the principles and dynamics that describe heating systems also apply. Thus, the control methodologies used in this paper can be easily applied to cooling systems. A linear programming model for optimal resource management of a CHP plant, in combination with a distribution network, is presented in [10]. The modeling and control of a similar system is provided in [11]. In case the demand is not known in advance, prediction methods, as studied in [12] and [13], can be applied. Recently, [14] solved an optimization problem to maximize the heat transfer in HENs with stream splits. Note that most of these methods give optimal values for steady-state behavior, but the dynamics of these systems are often neglected.

In [15] and [16], a pressure regulation problem was solved for nonlinear hydraulic networks; however, the temperature dynamics were neglected. Due to consumers that require fixed temperatures, the stability of the temperatures in DHSs is important. A common way to provide this stability is to control the flow and heat injection by PID controllers [17]. Examples of stability analysis in connection with proportional-integral-differential (PID) control and energy systems are given in [2], where a heat exchanger is considered, and [18], where a linearized model of a thermal network is studied.

B. Main Contribution

In this paper, we propose a provably correct internal model (IM) controller for a DHS with a storage device. To this end, we derive a dynamic model of a DHS with a single producer, a storage device, and multiple consumers. This model is obtained by interconnecting models of components

derived from first principles. The model is nonlinear due to the contribution of flow rates, volumes, and temperatures in the storage tank. Consumers can each extract a desired amount of heat from a heat exchanger and set their own flow rate. To guarantee that the demand of these consumers is met, two controllers are introduced. The first controller is a proportional controller that regulates the flow rate of the producer, depending on the aggregated flow of the consumers. The second controller regulates the heat injection in a heat exchanger located at the producer. The two controllers in closed loop to the nonlinear model are shown to guarantee the desired convergence property of the overall system. The controller is robust to parametric uncertainties possibly present in the model. An additional advantage of the proposed approach is that the control goal is achieved without any measurement of the consumers' demand.

C. Outline

This paper is organized as follows. In Section II, we introduce the framework of interest. This is followed by the problem formulation in Section III and the controller design in Section IV. The main result is stated in Section V followed by a case study with corresponding simulations in Section VI. The conclusions and future work are given in Section VII. A number of proofs, some model derivations, and the controllers that are designed for the case study can be found in the Appendix.

D. Notation

We denote by $\mathbb{R}$ and $\mathbb{R}_{>0}$ the set of real numbers and strictly positive real numbers, respectively. Given a vector $x \in \mathbb{R}^n$, x^T is considered its transpose. If the entries of x are functions of time, then the time derivative of x is denoted by $\dot{x} := (dx/dt)$ unless stated otherwise. For the sake of notational simplicity, for time-dependent variables, we drop the explicit dependence on t as long as no confusion arises. We define the operator $\Lambda(x) := \text{diag}(x_1 \ x_2 \ \dots \ x_n)$ as the diagonal matrix of elements x_i , and $\text{block.diag}(A_1 \ A_2 \ \dots \ A_n)$ as the block-diagonal matrix for which the block-diagonal matrices are A_i for $i = 1, \dots, n$. The identity matrix of dimension n is given by I_n , and an $n \times m$ matrix containing only zeros is denoted by $\mathbf{0}_{n \times m}$. Finally, we denote the vector of all ones by $\mathbf{1}_n := (1 \ \dots \ 1)^T$.

II. SYSTEM MODEL

Motivated by industrial waste-to-energy plants (see [19]) and CHP, we consider a setup with a single producer of heat. Such a producer provides steam that is converted to electricity by a turbine and can extract steam or hot water at several locations. This heated water or steam is fed through a heat exchanger that separates two flows with a heat conductor. The heat exchanged between these two streams is considered the injection of heat to the network. The heat is delivered to n consumers that come in the form of industrial or residential buildings, each having its own demand, which is modeled as a combination of periodic and constant signals. Due to

Fig. 1. Topology of a district heating network.

possible variable heat prices and demand patterns, there might be situations in which, in addition to balancing demand and supply, it is desired to store energy. To this end, we introduce a storage tank containing heated water.

Remark 1: Note that the heat exchangers separate the fluid in the network from both the fluids of the consumers and producer. This separation ensures that no contamination can enter the network and storage device. In certain cases, it can, however, be beneficial to bypass the heat exchangers (and storage) by adding direct connections between a producer and a consumer. For example, when the producers generate steam, such a connection can be used to further heat up the hot water delivered to the consumer.

A. Topology of a District Heating System

The topology of the DHS considered in this paper is shown in Fig. 1. A producer supplies heated water or steam³ to a heat exchanger that is connected to a network. The heat transfer in the heat exchanger is considered to be the power injection P^p of the producer. Water in the network is heated in the heat exchanger and transported to a storage tank that has two layers of water with variable volume. The top layer is fed by the water that comes from the heat exchanger of the producer. Each consumer also has a heat exchanger from which it extracts the heat. This heat exchanger connects the outlet of the hot layer and the inlet of the cold layer of the storage tank. In order to derive a model of this topology, we first introduce each component.

B. Storage Tank

The storage tank that we consider uses a stratification principle where hot water on the top is separated from cold water at the bottom by a thermocline. This device has four valves, two at the top and two at the bottom. These valves

are used as inlets and outlets of the hot and cold parts of the storage device. The temperatures of the top and bottom layers are approximately constant with respect to the height of the tank. On the contrary, the thermocline is a thin layer with a steep temperature gradient [5], [8]. We neglect the thermocline, which allows us to model the storage device as two separate storage tanks that are placed on top of each other.⁴ This is motivated by a low heat exchange rate between the hot and cold layers of the storage tank. In addition, it is possible to add an insulation layer that decreases the heat exchange even further. Such a layer can also prevent mixing during long discharging or charging periods, which causes a more severe heat exchange rate. Furthermore, we assume that each layer is perfectly stirred. We denote the total capacity of the storage tank by $V^{\max}$, and consider the storage device to be always completely filled with water. To this end, we assume the initial condition to satisfy

$$V^{sh}(0) + V^{sc}(0) = V^{\max} \quad (1)$$

where V^{sh} and V^{sc} are the volumes of the hot and cold layers, respectively. Furthermore, the following constraint is fulfilled:⁵

$$V^{sh}(t) + V^{sc}(t) = V^{\max} \quad \text{for all } t > 0. \quad (2)$$

In addition, let

$$\mathcal{V} := [V^{\min}, V^{\max} - V^{\min}] \quad (3)$$

be the set admissible volumes of both the layers in the storage with $0 < 2V^{\min} \leq V^{\max} < \infty$. We furthermore assume both $V^{sh}(0)$ and $V^{sc}(0) \in \mathcal{V}$. For reasons of simplicity, the heat exchanges between both the two layers and the environment are neglected for the time being. We define the temperatures in the hot and cold layers of the storage tank as T^{sh} and T^{sc} , respectively. The temperature in the heat exchanger of the producer is denoted by T^p , while its volume is defined as V^p . The temperature and volume in the heat exchanger of consumer j are given by T_j^c and V_j^c , respectively. These variables are collected in the vectors

$$T^c := (T_1^c \ T_2^c \ \dots \ T_n^c)^T$$

$$V^c := (V_1^c \ V_2^c \ \dots \ V_n^c)^T$$

where n is the number of consumers.

C. Producers and Consumers

The power demand of the consumers is given by

$$P_j^c \in \mathbb{R}_{>0} \quad j = 1, 2, \dots, n.$$

Assuming that the used fluid is incompressible and since there is no storage or loss of mass in a heat exchanger, the inflow rate equals the outflow rate. Hence, it is enough to associate with each heat exchanger a single flow variable. The flow

⁴Under this assumption, a topology with separate hot and cold water storage tanks results in exactly the same model. The results presented in this paper can therefore also be applied to such a network.

⁵Details of how this constraint is fulfilled can be found in the proof of Lemma 1.

³In the case of steam delivery to the heat exchanger, a phase-changer heat exchanger can be used and fits in this approach.

through the heat exchanger of consumer j is denoted by q_j^c . By aggregating these entries, we have

$$q^c := (q_1^c \ q_2^c \ \dots \ q_n^c)^T. \quad (4)$$

Similarly, we define the flow passing through the heat exchanger of the producer by

$$q^p \in \mathbb{R}_{>0}. \quad (5)$$

D. Sensors

We restrict ourselves to a setup that allows only for measurements that are located at the producers and storage devices. This implies that there is no need for communication between the controllers and the consumers. We assume that both the volume and the temperature of the hot layer in the storage tank are measured, and these measurements are communicated to the producer. Furthermore, we assume that the aggregated return flow of the consumers is measured.

E. Model of the District Heating System

The open-loop dynamics are obtained by interconnecting the models of the individual components (heat exchangers and storage) such that the network of Fig. 1 is obtained. The derivations of the model of the individual components as well the model of the overall system can be found in Appendix A. The resulting model is given as

$$\begin{aligned} V^p \dot{T}^p &= (T^{sc} - T^p)q^p + P^p \\ V^{sh} \dot{T}^{sh} &= (T^p - T^{sh})q^p \\ V^{sc} \dot{T}^{sc} &= \sum_{i=1}^n (T_i^c - T^{sc})q_i^c \\ V_j^c \dot{T}_j^c &= (T^{sh} - T_j^c)q_j^c - P_j^c, \quad j = 1, 2, \dots, n \\ \dot{V}^{sh} &= q^p - \sum_{i=1}^n q_i^c \\ \dot{V}^{sc} &= \sum_{i=1}^n q_i^c - q^p. \end{aligned} \quad (6)$$

Having this model allows us to formally formulate the control problem we would like to solve.

III. PROBLEM FORMULATION

Optimization techniques aiming at maximizing profit, e.g., by shifting loads in time, can be commonly found in the power systems literature [20]. Motivated by these techniques that provide optimal storage levels, we define a setpoint tracking problem with the objective to store a desired amount of energy. To this end, we consider a setpoint tracking problem with setpoints for both the temperature and volume of the hot layer in the storage device. The motivation for this approach is that a combination of a temperature and a volume defines the amount of energy that is stored, as shown in Appendix A-B. Hence, the regulation problem is defined as follows.

Problem 1: For system (6), design a controller that, given the unmeasured power demand P_j^c , $j = 1, 2, \dots, n$, regulates

the heat injection P^p and the flow rates q in such a way that any solution of the closed-loop system satisfies

$$\lim_{t \rightarrow \infty} (V^{sh} - V^{sh*}) = 0 \quad (7)$$

$$\lim_{t \rightarrow \infty} (T^{sh} - T^{sh*}) = 0 \quad (8)$$

where T^{sh*} and V^{sh*} are the desired setpoints for the temperature and volume, respectively. $\square$

Since the setpoint V^{sh*} cannot exceed the capacity of the storage tank, we introduce the following standing assumption.

Assumption 1: We assume $V^{sh*} \in \mathcal{V}$ with V^{sh*} being a setpoint and $\mathcal{V}$ is defined as in (3).

A. Model Power Demand

Each consumer j extracts an unknown demand $P_j^c(t)$, which can be regarded as a disturbance. For Problem 1 to be solvable, this disturbance needs to be rejected. In the scenario considered in this paper, the sensors are placed only at the storage device, and therefore $P_j^c(t)$ is not available to the controller. We assume that the disturbance consists of a linear combination of constants and sinusoidal signals with unknown amplitudes and phases. A large class of disturbance signals can be modeled by resorting to a sufficiently large number of frequencies [21]. In order to design a controller, the frequencies of the sinusoidal signals must be known. Such frequencies can be obtained from historical data or heat demand predictions based on weather forecasts. If these historical data are not available, adaptive and robust methods to deal with the case of unknown frequencies are available [21]. The investigation of these methods in the present context, however, goes beyond the scope of this paper.

Before introducing the model representing the demand, we collect the power demand P_j^c of each consumer j and the setpoint T^{sh*} in one vector $d \in \mathbb{R}^{n+1}$ such that

$$d := (-P_1^c \ \dots \ -P_n^c \ T^{sh*})^T.$$

We assume that the demand P_j^c for each consumer j is not measured and also T^{sh*} is not communicated to the controller. Therefore, d is not available to our controller, but we do impose some structure on it. To this end, we assume that each d_i is produced by an exosystem, which is described by

$$\dot{w}_i = S_i w_i \quad d_i = \Gamma_i w_i \quad i = 1, \dots, n+1 \quad (9)$$

where $\Gamma_i \in \mathbb{R}^{1 \times s_i}$, $S_i \in \mathbb{R}^{s_i \times s_i}$, and s_i is the dimension of w_i . Furthermore, we set $S_{n+1} = 0$ and $\Gamma_{n+1} = 1$, and initialize $w_n(0) = T^{sh*}$ such that $d_{n+1} = T^{sh*}$. We define $S := \text{block.diag}(S_1 \ \dots \ S_{n+1})$ and $\Gamma := \text{block.diag}(\Gamma_1 \ \dots \ \Gamma_{n+1})$, and collect

$$w := (w_1^T \ \dots \ w_{n+1}^T)^T$$

so that

$$\dot{w} = Sw \quad (10)$$

with $S \in \mathbb{R}^{s \times s}$ and $s = \sum_{i=1}^{n+1} s_i$. We impose that all the eigenvalues of S have zero real part and multiplicity one in the minimal polynomial. This assumption implies that all the trajectories of (10) are bounded and none of them decay to

zero as $t \rightarrow \infty$. Note that this structure imposed on d allows for constant and sinusoidal signals as well as any linear combination of them. Finally, we assume that matrix S is available for the design of the controller.

Remark 2: Note that taking $S_{n+1} = 0$ implies that T^{sh*} is a constant setpoint. Although it is possible in our approach to allow time-varying trajectories for T^{sh*} , this possibility is not considered due to the lack of practical relevance.

B. Compact Form

We now write system (6) in a more compact form. Before doing this, we need to define the inputs and outputs of the system. Because we can control both the heat injection and the flow rates in the pipes, these are taken as inputs and are given as

$$u = P^p \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} q^p \\ q^c \end{pmatrix}$$

respectively. Note here that the heat injection u is a single input, while the flows in the pipes v are aggregated in a vector. In line with Problem 1, we define the tracking error of the system as

$$e = T^{sh} - T^{sh*}$$

which is available to the controller. Furthermore, we collect temperatures and volumes in the vectors

$$x := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} T^p \\ T^{sh} \\ T^{sc} \\ T^c \end{pmatrix} \quad z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} V^{sh} \\ V^{sc} \end{pmatrix}$$

so that (6) becomes

$$M(z)\dot{x} = A(v)x + B_1u + Pw \quad (11)$$

$$\dot{z} = B_2v \quad (12)$$

$$e = Cx + Qw \quad (13)$$

with the matrices given by

$$M(z) = \text{block.diag}(V^p \quad \Lambda(z) \quad \Lambda(V^c)) \quad (14)$$

$$A(v) = \begin{pmatrix} -v_1 & 0 & v_1 & \mathbf{0}_{1 \times n} \\ v_1 & -v_1 & 0 & \mathbf{0}_{1 \times n} \\ 0 & 0 & -\mathbf{1}_n^T v_2 & v_2^T \\ \mathbf{0}_{n \times 1} & v_2 & \mathbf{0}_{n \times 1} & -\Lambda(v_2) \end{pmatrix} \quad (15)$$

$$B_1 = \begin{pmatrix} 1 \\ \mathbf{0}_{(n+2) \times 1} \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & -\mathbf{1}_n^T \\ -1 & \mathbf{1}_n^T \end{pmatrix} \quad (16)$$

$$P = \begin{pmatrix} \mathbf{0}_{3 \times (s-1)} & \mathbf{0}_{3 \times 1} \\ -\Gamma & \mathbf{0}_{n \times 1} \end{pmatrix} \quad Q = (\mathbf{0}_{1 \times n} \quad 1) \quad (17)$$

$$C = (0 \quad 1 \quad 0 \quad \mathbf{0}_{1 \times n}). \quad (18)$$

Observe now that $A(v)$ is a time-varying matrix due to the dynamics of $v(t)$. Also note that (8) and (7) are equivalent to

$$\lim_{t \rightarrow \infty} e = 0 \quad (19)$$

$$\lim_{t \rightarrow \infty} (z_1 - z_1^*) = 0 \quad (20)$$

where $z_1^* := V^{sh*}$.

IV. CONTROLLER DESIGN

The design of the controller consists of two parts. The first part derives the controller that regulates the flows in the pipes such that (7), i.e., (20), is satisfied. We assume that the consumers set their flow rate to a constant depending on their demand. This flow rate can be set by the network operator, a local controller, or the consumer itself, but is not communicated to the producer or the storage device. However, the aggregated flow that returns to the cold layer in the storage device is measured and available for the controllers. This measurement along with a measurement of the volume in the hot storage layer is communicated to the producer. The second part derives the controller of u , which achieves (8), i.e., (19). We show that the temperature dynamics (11) in closed loop with the flow controller asymptotically converge to a linear time-invariant system with unknown parameters. We design an IM controller for such a linear time-invariant system. Finally, we combine the two controllers and analyze the stability of the closed-loop system.

A. Control of Flow Rates

We now design a control input for the flow rate v such that $\lim_{t \rightarrow \infty} (z_1 - z_1^*) = 0$. The proposed controller for the flow rates is given as

$$v_1 = \mathbf{1}_n^T v_2^* + \alpha(z_1^* - z_1) \quad v_2 = v_2^* \quad (21)$$

where $\alpha \in \mathbb{R}_{>0}$ is the gain of the controller and $v_2^* \in \mathbb{R}_{>0}^n$ is the vector of flow rates at the consumers.

Lemma 1: System (12) with controller (21) has solution $z(t)$ such that

$$z_1(t) = e^{-\alpha t} (z_1(0) - z_1^*) + z_1^* \quad (22)$$

$$z(t) \in \mathcal{V} \times \mathcal{V} \quad (23)$$

for all $t \geq 0$ and constraint (2) is satisfied.

Proof: Evaluating (12) with input (21) provides

$$\dot{z}_1 = \alpha(z_1^* - z_1) \quad (24)$$

the solution of which yields (22). Taken together with Assumption 1, it is easily verified that (23) is satisfied. Observe that by (12), we have

$$\dot{z}_2(t) + \dot{z}_1(t) = 0 \quad (25)$$

and also $z_2(0) + z_1(0) = V^{\max}$, which implies that (2) is satisfied. $\square$

Observe furthermore that controller (21) is completely independent of any temperature dynamics. Using Lemma 1, we conclude that (20) is satisfied.

Remark 3: Note that controller (21) sets $v_2 = v_2^*$, which is a constant flow rate. Since this flow rate is in practice set by a separate controller located at the consumers, it is restrictive to assume this flow rate to be constant. However, if the timesteps are taken small enough, these flow rates can be approximated by a slowly varying signal. The effect of this slow variation is not taken into account in the analysis.

B. Control of Heat Injection

Next, we provide a controller that regulates the heat injection u , such that (8), i.e., (19), is satisfied. To do this, in this section, we restrict ourselves to the case where flows and volumes are constant, i.e., $\dot{z} = 0$. Note that this restriction is used only for the design of the controller and will be relaxed in the analysis of the closed-loop system. In order to satisfy $\dot{z} = 0$, we set $v_1^* = \mathbf{1}_n^T v_2^*$, where we recall that v_2^* are the flow rates that are set by the consumers. Since these flow rates are strictly positive, this also implies that $v_1^* > 0$. We now define

$$\begin{aligned}\bar{A} &= M(z^*)^{-1} A(v^*) \\ \bar{B} &= M(z^*)^{-1} B_1 \\ \bar{P} &= M(z^*)^{-1} P\end{aligned}\quad (26)$$

so that (11)–(13) take the form of the linear time-invariant system

$$\dot{x} = \bar{A}x + \bar{B}u + \bar{P}w \quad e = Cx + Qw. \quad (27)$$

For this system, we consider a controller of the form

$$\dot{\zeta} = F\zeta + Ge \quad u = H\zeta + Ke \quad (28)$$

with $\zeta \in \mathbb{R}^l$. Furthermore, let F , G , H , and K be the matrices such that there exist matrices Π , Σ , and R that solve

$$\begin{aligned}\Sigma S &= F\Sigma \\ R &= H\Sigma\end{aligned}\quad (29)$$

$$\begin{aligned}\Pi S &= \bar{A}\Pi + \bar{B}R + \bar{P} \\ 0 &= C\Pi + Q.\end{aligned}\quad (30)$$

These are the celebrated regulator equations. We now prove that a controller of the form (28) exists and is able to solve the output regulation problem.

Theorem 1: There exist F , G , H , and K such that the solutions of the closed-loop system (27)–(28) with exosystem (10) are bounded and satisfy (19)

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

Proof: The proof can be found in Appendix B-A. $\square$

Remark 4: Knowledge of matrix $\bar{A}$ in (26) requires knowledge of vector v^* related to the steady-state flow rates. In practice, v^* need not be perfectly known. As shown in [21], the controller in (28) turns out to be robust against model parametric uncertainties and, hence, against uncertainty in the estimate of v^* .

Remark 5: Note that $\zeta \in \mathbb{R}^l$ with $l = s + r$ (for details, see [21]) with s the dimension of S and r the relative degree⁶ of system (27). Therefore, we see that the state of the controller linearly scales with s . The intuition behind this is that the controller needs to keep track of all the, possibly different, demand patterns of the consumers.

⁶For large-scale systems, it holds that $s \gg r$.

Fig. 2. Block diagram of the closed-loop system.

V. MAIN RESULT

Having designed u and v in (21) and (28), respectively, we combine the two controllers and show that both volume and temperature converge to the desired setpoints (see Problem 1). To do this, we write system (11)–(13) in closed loop with the two controllers, which leads to a nonlinear system whose block diagram is given in Fig. 2. This is followed by an analysis of the stability and convergence of the entire system.

The first step is to write (11) in closed loop with controller (28) to obtain

$$\begin{aligned}M(z)\dot{x} &= (A(v) + B_1KC)x + B_1H\zeta + (B_1KQ + P)w \\ \dot{\zeta} &= GCx + F\zeta + GQw.\end{aligned}\quad (31)$$

Note that applying controller (21) ensures that $M(z)$ is invertible due to Lemma 1. Therefore, we can define

$$\begin{aligned}A_e &= M(z)^{-1}A(v) - \bar{A} \\ B_e &= M(z)^{-1}B_1 - \bar{B} \\ P_e &= M(z)^{-1}P - \bar{P}\end{aligned}\quad (32)$$

with $\bar{A}$, $\bar{B}$, and $\bar{P}$ being as in (26). Now we perform the coordinate change

$$\zeta := \begin{pmatrix} x - \Pi w \\ \zeta - \Sigma w \end{pmatrix}. \quad (33)$$

Bearing in mind (29) and (30), (33) allows us to write closed-loop system (31) as

$$\dot{\zeta} = (A' + F'(t))\zeta + B'(t)w \quad (34)$$

with

$$A' = \begin{pmatrix} \bar{A} + \bar{B}K\bar{C} & \bar{B}H \\ GC & F \end{pmatrix} \quad (35)$$

$$F'(t) = \begin{pmatrix} A_e(t) + B_e(t)KC & B_e(t)H \\ 0 & 0 \end{pmatrix} \quad (36)$$

$$B'(t) = \begin{pmatrix} P_e(t) + B_e(t)R + A_e(t)\Pi \\ 0 \end{pmatrix}. \quad (37)$$

Note that in contrast to what happens in linear time-invariant systems, (37) only vanishes in case $z = z^*$ and $v = v^*$. Consider now system (34) with (33) and observe that $\zeta = 0$

implies that $x = \Pi w$. Due to (30), we know that $C\Pi + Q = 0$ and therefore also that

$$e = Cx + Qw = 0.$$

Consider any solution to (34) with exosystem (10). Proving that these solutions are bounded and $\lim_{t \rightarrow \infty} \zeta = 0$ implies that (19) is satisfied. Since in Section IV-A, it was already proven that (20) holds, this would also imply that Problem 1 is solved.

Theorem 2: Consider system (34) where w is as in (10). Then, for any initial condition, ζ is bounded and satisfies

$$\lim_{t \rightarrow \infty} \zeta(t) = 0.$$

Proof: The proof can be found in Appendix B-B. $\square$

VI. CASE STUDIES

In order to illustrate the model and the proposed controller, we consider three case studies. In the first case study, we consider three different consumers, each having a constant heat demand. Two time intervals are considered with different setpoints for the volume of the storage, while the temperature in the storage device is kept to a fixed level. The volume setpoints of the hot layer are set such that water is stored in the first time interval, while it is drained during the second time interval. As a result, energy is stored in the first time interval, while it is drained in the second interval. An incentive to do this can be a higher cost for production in the second time intervals. Another reason could be to secure supply in case of unforeseen interruptions of the heat production. The second case study is identical to the first one with the exception that a time-varying demand is considered. Finally, a real demand pattern is considered.

A. Constant Demand

In the first case study, we consider two intervals with piecewise constant demand and different setpoints as motivated in Section III. We use a thermal storage that is designed to operate at 85 °C. The temperature should be kept around this value as much as possible, while the change in volume modifies the stored amount of energy. For this reason, we take the setpoint and initial condition of the temperature in the hot layer of the storage tank equal to $T^{sh*} = T^{sh}(0) = 85$ °C. The density and specific heat are given by $\rho = 975$ kg/m³ and $C_p = 4190$ J/kg °C, respectively. The total volume of the storage tank is $V^{\max} = 1000$ m³, and the volume of the hot layer is initialized at $V^{sh}(0) = 100$ m³. Each heat exchanger has a volume of $V_i^c = V^p = (2/100)$ m³. The temperatures of the heat exchangers are initialized to $T^p(0) = 70$ °C for the producer and $T_i^c(0) = 30$ °C for each consumer i . Controller (28) is given in Appendix C, and the gain α in (21) regulating the flow is set to $\alpha = 0.005$. To avoid nonpositive and too high flow rates, we impose a saturation on the corresponding control input. This saturation is given by $0.05 \text{ m}^3/\text{s} \leq q^p \leq 0.25 \text{ m}^3/\text{s}$. The extension of the present analysis to the case of input saturations is left for further research.

The two time intervals we consider are given by $0 \leq t < 8000$ s and $8000 \leq t < 16000$ s. The volume

Fig. 3. Piecewise constant demand pattern.

Fig. 4. Three consumers with constant demand.

Fig. 5. Magnification of Fig. 4. The storage temperature of the hot layer and corresponding power injection.

Fig. 6. Temperatures of the heat exchangers.

setpoints are $V^{sh*} = 900$ m³ and $V^{sh*} = 100$ m³ for the first and second time intervals, respectively. Since the setpoint for temperature in the hot storage tank is fixed, this implies that the energy is stored in the first time interval, while it is drained in the latter time interval. The demands of the three consumers are given in Fig. 3. The flow rates of the consumers in the first time interval are given by $q_1^c = 0.024$ m³/s, $q_2^c = 0.033$ m³/s, and $q_3^c = 0.041$ m³/s, and in the second time interval, they are set to $q_1^c = 0.049$ m³/s, $q_2^c = 0.057$ m³/s, and $q_3^c = 0.065$ m³/s.

The results are shown in Figs. 4–7; Figs. 6 and 7 display the temperatures in the heat exchangers. Fig. 5 shows the evolution of the temperature of the hot layer, which converges to the

Fig. 7. Magnification of Fig. 6. Temperature of all the heat exchangers at $t = 0$ and during the transition between two time intervals.

Fig. 8. Time-varying demand pattern.

desired $T^{sh*} = 85$ °C, and the corresponding power input of the producer. As a consequence of the initialization and the choice of the setpoints of the volume of the hot layer, we see from Fig. 4 that the storage is filled in the first time interval and drained in the second time interval. The changes in volume are a consequence of a difference in the aggregated flow rate of the consumers and the flow rate of the producer. At the end of both the time intervals, these flow rates are equal to each other causing a constant volume of the hot and cold layers in the storage tank. Furthermore, it can be seen that in both the time intervals, the setpoints of the volume are achieved.

It can be seen that the heat injection is approximately proportional to the flow rate of the producer. This can be understood from the dynamics of the heat exchanger located at the producer as in (6). From this equation, it follows that at steady state (e.g., $\dot{T}^p = 0$), the power injection is proportional to $q^p(T^{Sc} - T^p)$. Since the consumer has both a constant flow rate and a constant heat demand, T^{Sc} also converges to a constant value. Furthermore, since $T^p = 85$ °C, it follows that the heat injection is proportional to the flow rate of the producer. Draining the storage tank results in a lower production despite the increased demand.

In Fig. 6, it can be seen that the temperatures in the heat exchangers stay constant during both the time intervals. However, Fig. 7 shows some transient behavior at $t = 0$ s, and some oscillations at $t = 8000$ s. These oscillations are due to the transient behavior because of new setpoints, flow rates, and heat extraction rates.

B. Time-Varying Demand

The second case study we consider is equivalent to the case study in Section VI-A with the exception that a time-varying demand is considered, as shown in Fig. 8. For this reason, we take all the initializations, setpoints, and parameters identical to the previous case study. However, the controller (28) is different due to the time-varying demand, and is given in Appendix C. In Figs. 9–12, we see the outcome of the simulation. Fig. 10 shows that approximately $T^{sh*} = 85$ °C

Fig. 9. Three consumers with time-varying demand.

Fig. 10. Magnification of Fig. 9. Storage temperature and the power injection from $t = 0$ s and $t = 8000$ s.

Fig. 11. Temperatures of the heat exchangers under time-varying demand.

Fig. 12. Magnification of Fig. 11. Temperatures of the heat exchangers from $t = 0$ s and $t = 8000$ s.

within 10 s despite the time-varying demand. Furthermore, Fig. 9 shows that the volume setpoints are also achieved at the end of each time interval. Since the time-varying demand does not influence the flow rates, we see the same behavior of the volumes as in the case study in Section VI-A. Similar to the previous case study, we see a spike in the heat injection

Fig. 13. Heat demand pattern of a whole day.

at $t = 8000$ s due to new setpoints, flow rates, and extraction rates. In Fig. 11, the temperatures of the heat exchangers of the consumers clearly show the impact of the time-varying demand. Finally, in Fig. 12, we see a similar transient behavior at $t = 0$ s and $t = 8000$ s as in the previous case study due to new setpoints, flow rates, and demand patterns.

Remark 6: Additional simulations have shown that the performance of the controller is not significantly deteriorated in the presence of small-range frequency uncertainties (around 20%) in the consumer demand pattern. This is partly due to the large integral gain in the controller (see Appendix C) and partly due to the slow dynamics of the system. As pointed out in Section III-A, the adaptive and robust methods [21] do exist that can be used to compensate for larger uncertainties.

C. Real Demand

In the final case study, we investigate the performance of the controller in the presence of a real demand pattern. We simulate a complete day with a storage device of $V^{\max} = 2000$ m³ where heat is stored during low demand and drained during high demand. To this end, we let $0 \leq t \leq 86400$ s and use a demand pattern that is provided in [5], to which we applied a discretization with a resolution of 1 h. This demand pattern can be found in Fig. 13. The aggregated flow rate of the consumers is set proportional to the demand at all time.

Again we set $\rho = 975$ kg/m³ and $C_p = 4190$ J/kg °C, and the volumes of the heat exchangers are given by $V^p = V^c = 0.02$ m³. The initialization of the temperature of the heat exchangers is given by $T^p(0) = 50$ °C and $T^c(0) = 50$ °C. The temperature setpoint throughout the day is $T^{Sh*} = 85$ °C, and therefore the initial temperature is also set to $T^{Sh}(0) = 85$ °C. The initial temperature for the cold layer is $T^{Sc}(0) = 30$ °C, and the volume of the hot layer is initialized at $V^{Sh} = 600$ m³. The gain for the controller of the flow rate is set to $a = 0.005$. Again we assume a nonnegative flow rate and set a maximum flow rate to avoid flow rates that are too large, which is given by 0.05 m³/s $\leq q^p \leq 0.5$ m³/s. The controller (28) that is used remains identical throughout the whole simulations and given in Appendix C.

The setpoints for the volume of the hot layer are given in Table I. In contrast to the previous case studies, we chose these setpoints such that most time intervals are too short to

TABLE I
VOLUME SETPOINTS WITH THE TIME IN HOURS

Time	1	2	3	4	5	6	7
Setpoint	7.25	7.5	7.75	8.5	10	15.5	30
8	9	10	11	12	13	14	15
40	43.5	36.5	30	35	34	33.5	27.5
17	18	19	20	21	22	23	24
30	34.5	32	31	30.75	25.5	15	6.5

Fig. 16. Comparison of the power injection between two setups, one with and one without storage.

is set proportional to this demand, the flow rate has the same behavior. This implies that the controller receives a step function as input at the beginning of each hour. As a consequence, we observe in Fig. 15 some small spikes of the temperature in the heat exchanger located at the producer, which is the transient response to these step inputs.

VII. CONCLUSION

We presented a model of a DHS with a storage device, single producer, and multiple consumers. The proposed controller is able to regulate the energy level in the storage device to a desired setpoint in spite of unmeasured, possibly time-varying, heat demand.

We plan to investigate the possibility of extending our findings to a topology with multiple storage tanks. In that case, each storage tank is connected to multiple consumers. In addition, we would like to investigate the case where one has different setpoints for both volume and temperature in each of those storage tanks. There is a recent interest from the industry into temperature cascading, in which the leftover heat coming from a consumer is reused by other consumers. This motivates us to interconnect multiple network clusters with each other, where each cluster corresponds to a network as in this paper. In this situation, the output of the consumers in a cluster is connected to another cluster where it is used as an input. Finally, motivated by practical reasons, it is of interest to study the controller under the presence of heat dissipation and time delays.

APPENDIX A MODEL COMPONENTS

In this section, we derive the adopted models for the heat exchanger and storage device to make this paper as self-contained as possible. Furthermore, we derive a relation between temperature, volume, and energy in order to motivate Problem 1. In order to provide this relation, we first introduce the notion of enthalpy.

A. Enthalpy

Consider a control volume of an open system where the change in energy ΔE is given by

$$\Delta E = E^{\text{in}} - E^{\text{out}} + W + \Phi \quad (38)$$

with E^{in} and E^{out} the total inflow and outflow of energy by mass streams. The total work supplied by the surroundings is given by W , while Φ is the heat exchanged with the surroundings. The total amount of energy in a control volume can be expressed in terms of *enthalpy*, which is defined as

$$H = E + pV \quad (39)$$

where E is the energy, p is the pressure, and V is the volume. Since we deal with the temperatures between 0 °C and 100 °C at atmospheric pressure, water exists only in liquid form and no phase changes or chemical reactions occur. Under the additional assumption of a constant heat capacity c_p , enthalpy is only a function of the temperature (see [1]). This implies that

$$H - H|_{T=T^{\text{ref}}} = mc_p(T - T^{\text{ref}}) \quad (40)$$

where m is the mass of the control volume, T is its temperature, T^{ref} is an arbitrary fixed reference temperature, and $H|_{T=T^{\text{ref}}}$ is the enthalpy associated with this reference temperature. The mass satisfies $m = \rho V$, with ρ being the density.

B. Relative Energy

In this section, we provide a relation between a volume, temperature, and stored energy. This relation provides the main motivation for the formulation of Problem 1. In order to provide such a relation, we consider a volume of heated water. Since such a body of water can be used only for heating if its temperature is lower than the outside temperature, we compare the heat of the water to the outside temperature. To this end, we introduce a relative temperature as

$$\hat{T} = T - T^{\text{ref}} \quad (41)$$

where T^{ref} is a constant reference temperature, which can be taken equal to the average outside temperature. In addition, we define a relative energy as

$$\hat{E} := E - E|_{T=T^{\text{ref}}} \quad (42)$$

where $E|_{T=T^{\text{ref}}}$ is the energy of a control volume V^{sh} at temperature T^{ref} . Due to definition (42), together with (39) and (40), we can write

$$\begin{aligned} \hat{E} &= H - pV - (H - pV)|_{T=T^{\text{ref}}} \\ &= mc_p(T - T^{\text{ref}}) + H|_{T=T^{\text{ref}}} \\ &\quad - pV - (H|_{T=T^{\text{ref}}} - pV) \\ &= c_p \rho V \hat{T}. \end{aligned} \quad (43)$$

As a consequence, $\hat{E}$ is expressed as a product of a volume and a relative temperature. Using the identity

$$\hat{T}^{sh} - \hat{T}^{sh*} = T^{sh} - T^{sh*} \quad (44)$$

and bearing in mind relation (43), we observe that solving Problem 1 guarantees

$$\lim_{t \rightarrow \infty} |\hat{E}^{sh} - \hat{E}^{sh*}| = 0 \quad (45)$$

where $\hat{E}^{sh*} := c_p \rho V^{sh*} \hat{T}^{sh*}$. For this reason, we are able to split an energy setpoint tracking problem into a volume and

Fig. 17. Model of a heat exchanger adopted from [2] with m cells with power transfer in cell k given by $P_{[k]}$. Since the inflow equals the outflow, the flow rates q_e and q are defined only once. T_e^{in} and T^{in} are the inflow, and T_e and T are the outflow temperatures.

temperature setpoint tracking problem. In practice, a fixed $\hat{T}^{sh*}$ can be chosen such that a minimal operating temperature will be guaranteed. By varying the volume V^{sh*} , different energy levels can be achieved.

C. Temperature Dynamics

Since we assume perfect mixing within the control volume, the inside temperature T and outflow temperature are equal. The inflow temperature on the other hand is denoted by T^{in} . After straightforward but lengthy derivation (see [1]), the temperature dynamics of this control volume are obtained by (38) and (40) and read as

$$V\dot{T} = q^{\text{in}}(T^{\text{in}} - T) + P \quad (46)$$

where q^{in} is the inflow rate and $P := (1/\rho c_p)\dot{\Phi}$ is the heat injection or extraction rate. Note that $\dot{\Phi}$ is considered to be a rate variable instead of the time derivative. Now (46) can be used to model a heat exchanger and a storage device.

D. Heat Exchanger Model

Heat exchangers are used for energy exchange between at least two fluid phase streams (gas or liquid), a hot and a cold stream. Since heat exchangers are placed between every consumer and the network as well as between the producer and the network, the injection and extraction of power goes only via a heat exchanger. In Fig. 17, the model for a heat exchanger is presented, which consists of m cells connected in series. Each cell has two balance volumes separated by a heat conducting element. We restrict ourselves to a model where $m = 1$, and assume that both the volumes in the cell are perfectly stirred. To simplify the model, we assume the following quantities to be constant in each cell: volume, mass, specific heat c_p , density ρ , heat transfer coefficient U , and contact area of the heat conducting element A_h . Using (46), the dynamics for the temperatures of the two volumes are then given by

$$\dot{T}_e = \frac{q_e}{V_e^h}(T_e^{\text{in}} - T_e) - \frac{1}{V_e^h}P \quad (47)$$

$$\dot{T} = \frac{q}{V^h}(T^{\text{in}} - T) + \frac{1}{V^h}P \quad (48)$$

where $V_e^h > 0$ and $V^h > 0$ are the volumes in the upper and lower cells, respectively. The heat transfer rate P can be

explicitly made by

$$P = \frac{UA_h}{c_p\rho}(T_e - T). \quad (49)$$

At the producer side, we make the assumption that, instead of adjusting temperature T_e^{in} and flow rate q_e , we are able to control the heat injection P directly. This results in a heat exchanger being modeled solely by (48).

Remark 7: From (49), we observe that the extraction rate linearly depends on $T_e - T$. Since the consumer can control only q_e and not T_e^{in} , we see from (47) that the heat exchange becomes $P = 0$ when $T_e = T = T_e^{\text{in}} = T^{\text{in}}$. Since this is independent of q_e , this situation should be avoided by providing sufficient heated water T^{in} compared with T_e^{in} , but by also setting the flow rate q sufficiently high. A similar remark is made in [2], where it is stated that a heat exchanger is controllable with the exception of the singular point $T_e = T_e^{\text{in}}$, $T = T^{\text{in}}$.

E. Storage Model

To model the heat storage tank, we again refer to (46) with $P = 0$ due to the lack of heat generation or absorption (loss). This implies

$$V^{Sh}\dot{T}^{Sh} = q^{\text{in}}(T^{\text{in}} - T^{Sh}) \quad (50)$$

where V^{Sh} and T^{Sh} are the storage volume and temperature of the hot layer. The volume dynamics of the hot layer are

$$\dot{V}^{Sh} = q^{\text{in}} - q^{\text{out}} \quad (51)$$

where q^{in} is the inflow and q^{out} is the outflow. The dynamics for T^{Sc} and V^{Sc} are obtained by replacing S_c with S_h in (50) and (51), respectively.

F. Model Derivation

When an inlet of a component is fed by multiple pipes each having different flow rates and temperatures, the resulting temperature and flow rate are obtained by mass and energy conservation laws. These are given by

$$q^{\text{in}} = \sum_{i=1}^n q_i^{\text{in}} \quad (52)$$

$$q^{\text{in}}T^{\text{in}} = \sum_{i=1}^n q_i^{\text{in}}T_i^{\text{in}} \quad (53)$$

where q_i^{in} is the flow rate and T_i^{in} is the temperature of the fluid passing through pipe i , while n is the total number of pipes. By combining (48)–(53) and considering the topology of Fig. 1, we obtain model (6).

APPENDIX B PROOFS

A. Proof of Theorem 1

For system (27), it is well known (see [21]) that a controller of the form (28) solves the robust output regulation problem if there exist Π , Σ , and R such that (29) and (30) are satisfied. Reference [21, Lemma 1.4.2] states that the regulator equations have a solution for all $\bar{A}$, $\bar{B}$, $\bar{P}$, C , and Q if

and only if

$$\det \begin{pmatrix} \bar{A} - \lambda I & \bar{B} \\ C & 0 \end{pmatrix} \neq 0 \quad (54)$$

for all λ , which are eigenvalues of S . The matrix on the left-hand side of (54) is given by

$$\begin{pmatrix} -v_1 - \lambda & 0 & v_1 & \mathbf{0}_{1 \times n} & 1 \\ v_1 & -v_1 - \lambda & 0 & \mathbf{0}_{1 \times n} & 0 \\ 0 & 0 & -\mathbf{1}_n^T v_2 - \lambda & v_2^T & 0 \\ \mathbf{0}_{n \times 1} & v_2 & \mathbf{0}_{n \times 1} & -\Lambda(v_2) - \lambda I_n & \mathbf{0}_{n \times 1} \\ 0 & 1 & 0 & \mathbf{0}_{1 \times n} & 0 \end{pmatrix}. \quad (55)$$

Due to the sparsity of this matrix, we observe that the last row and column have only one nonzero entry. Therefore, condition (54) holds, if and only if

$$\det \begin{pmatrix} v_1 & 0 & \mathbf{0}_{1 \times n} \\ 0 & -\mathbf{1}_n^T v_2 - \lambda & v_2^T \\ \mathbf{0}_{n \times 1} & \mathbf{0}_{n \times 1} & -\Lambda(v_2) - \lambda I_n \end{pmatrix} \neq 0. \quad (56)$$

By assumption, we have that all the eigenvalues of S have zero real part, which implies that $\lambda = bi$, where $b \in \mathbb{R}$. Also by the upper triangular form of (56), we clearly see that this is equivalent to

$$v_1 (\mathbf{1}_n^T v_2 + bi) \prod_{j=1}^n ((v_2)_j + bi) \neq 0 \quad (57)$$

where $(v_2)_j$ is the j th element of vector v_2 . Due to $v_1 > 0$ and $(v_2)_j > 0$, it follows that (57) clearly holds, which implies there exist Π , Σ , and R such that (29) and (30) are satisfied, and therefore proves Theorem 1.

B. Proof of Theorem 2

In order to prove Theorem 2, we consider volumes and flow rates that are not equal to their reference values. We regard the discrepancy between the actual quantities and the reference values as a disturbance in the dynamics. We next investigate the stability properties of these dynamics. First, we prove exponential stability of the origin of the system without the disturbance, and then prove that convergence to the origin is preserved for the perturbed system. We now introduce some helpful lemmas that substantiate the proof.

Lemma 2: Suppose $\dot{x}(t) = A(t)x(t)$ with $x(t_0) = x_0$ is uniformly exponentially stable.⁷ Also suppose there exists a finite constant β such that for all τ , we have that

$$\int_{\tau}^{\infty} \|F(\sigma)\| d\sigma \leq \beta \quad (58)$$

holds for all $\tau > t_0$. This implies that

$$\dot{z}(t) = [A(t) + F(t)]z(t) \quad (59)$$

is also uniformly exponentially stable.

Proof: See [22, Th. 8.5]. $\square$

⁷A linear time-varying system is uniformly exponentially stable if there exist finite positive constants γ and λ such that its solution satisfies

$$\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)} \|x_0\| \quad t \geq t_0 \quad (60)$$

for any initial condition x_0 and any $t_0 > 0$.

Lemma 3: Consider system (12) with v_1 as in (21), then there exist $\beta_1, \beta_2, \beta_3 \in \mathbb{R}_{>0}$ such that

$$\int_{\tau}^{\infty} \left| \frac{1}{z_1(\sigma)} - \frac{1}{z_1^*} \right| d\sigma \leq \beta_1 \quad (61)$$

$$\int_{\tau}^{\infty} \left| \frac{1}{z_1(\sigma)} (v_1(\sigma) - v_1^*) \right| d\sigma \leq \beta_2 \quad (62)$$

and

$$\int_{\tau}^{\infty} |z_1(\sigma) - z_1^*| d\sigma \leq \beta_3. \quad (63)$$

Proof: Observe that (63) is equivalent to finding β_3 such that

$$|(z_1(0) - z_1^*)| \int_{\tau}^{\infty} e^{-a\sigma} d\sigma \leq \beta_3. \quad (64)$$

Since

$$|(z_1(0) - z_1^*)| \leq V^{\max} - 2V^{\min} \quad (65)$$

β_3 can be taken as

$$\beta_3 = \frac{V^{\max} - 2V^{\min}}{a}. \quad (66)$$

Consider now (61) for which it holds that

$$\int_{\tau}^{\infty} \left| \frac{1}{z_1(\sigma)} - \frac{1}{z_1^*} \right| d\sigma \leq \int_0^{\infty} \left| \frac{z_1^* - z_1(\sigma)}{z_1^* z_1(\sigma)} \right| d\sigma. \quad (67)$$

By Lemma 1, we know that we can bound

$$\frac{1}{z_1^* z_1(\sigma)} \leq \frac{1}{(V^{\min})^2} \quad \text{for all } \sigma \geq 0 \quad (68)$$

which, in light of (22), (65), and (67), gives

$$\int_{\tau}^{\infty} \left| \frac{1}{z_1(\sigma)} - \frac{1}{z_1^*} \right| d\sigma \leq \frac{1}{(V^{\min})^2} \int_0^{\infty} e^{-a\sigma} |z_1^* - z_1(0)| d\sigma.$$

Therefore, taking

$$\beta_1 = \frac{\eta}{a} \quad (69)$$

with $\eta = (V^{\max} - 2V^{\min}/(V^{\min})^2)$ implies that (61) is satisfied. Now finally consider (62) and observe that by definition (21), we have that

$$\frac{1}{\alpha z_1^*} \int_{\tau}^{\infty} \left| \frac{1}{z_1(\sigma)} (v_1(\sigma) - v_1^*) \right| d\sigma = \int_{\tau}^{\infty} \left| \frac{1}{z_1(\sigma)} - \frac{1}{z_1^*} \right| d\sigma.$$

For this reason along with (69), taking $\beta_2 := \eta/(\alpha^2 V^{\min})$ implies that (61) is satisfied. This concludes the proof. $\square$

Lemma 4: The origin of

$$\dot{z} = (A' + F'(t))z \quad (70)$$

is uniformly exponentially stable with A' and F' given in (35) and (36).

Proof: Let $A(t)$ and $F(t)$ from Lemma 2 be equal to A' and $F'(t)$ as given in (35) and (36). Therefore, it is trivially uniformly exponentially stable¹⁰. Clearly, A' is a Hurwitz matrix due to the designed IM controller. By defining

$$G = \begin{pmatrix} KC & H \\ 0 & 0 \end{pmatrix}$$

and using $\|AB\| \leq \|A\|\|B\|$, condition (58) is satisfied if

$$\begin{aligned} & \int_{\tau}^{\infty} \|F'(\sigma)\| d\sigma \\ & \leq \int_{\tau}^{\infty} \|A_e(\sigma)\| d\sigma + \|G\| \int_{\tau}^{\infty} \|B_e(\sigma)\| d\sigma \leq \beta. \end{aligned} \quad (71)$$

Since $v_2 = v_2^*$, we can write $A(v) = \Lambda(v^e)A(v^*)$, where

$$v^e = \begin{pmatrix} \frac{v_1}{v_1^*} & \frac{v_1}{v_1^*} & 1 & \mathbf{1}_n^T \end{pmatrix}^T \quad (72)$$

which implies

$$\|A_e\| \leq \|M(z)^{-1}\Lambda(v^e) - M(z^*)^{-1}\| \|A(v^*)\| \quad (73)$$

and similarly

$$\|B_e\| \leq \|M(z)^{-1} - M(z^*)^{-1}\| \|B\|. \quad (74)$$

Since all time-varying entries on the right-hand sides of both (73) and (74) enter in diagonal form, (71) is satisfied if there exist $\beta_{1,i}$ and $\beta_{2,i}$ such that

$$\begin{aligned} & \int_{\tau}^{\infty} \left| \frac{v_i^e(\sigma)}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}} \right| d\sigma \leq \beta_{1,i} \\ & \int_{\tau}^{\infty} \left| \frac{1}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}} \right| d\sigma \leq \beta_{2,i} \end{aligned} \quad (75)$$

is satisfied for all i , where $M(z(\sigma))_{ii}$ is the i th diagonal component of $M(z(\sigma))$. Note that

$$\frac{v_i^e(\sigma)}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}} = \begin{cases} \frac{1}{v_1^* V^P} (v_1 - v_1^*), & \text{for } i = 1 \\ \frac{1}{v_1^*} \left(\frac{v_1}{z_1} - \frac{v_1^*}{z_1^*} \right), & \text{for } i = 2 \\ \frac{1}{z_2} - \frac{1}{z_2^*}, & \text{for } i = 3 \\ 0, & \text{for } i \geq 4 \end{cases} \quad (76)$$

$$\frac{1}{M(z(\sigma))_{ii}} - \frac{1}{M(z^*)_{ii}} = \begin{cases} 0, & \text{for } i = 1 \\ \frac{1}{z_1} - \frac{1}{z_1^*}, & \text{for } i = 2 \\ \frac{1}{z_2} - \frac{1}{z_2^*}, & \text{for } i = 3 \\ 0, & \text{for } i \geq 4. \end{cases} \quad (77)$$

Recalling that $v_1^* = \mathbf{1}_n^T v_2^*$, by the definition of (21), we know

$$\frac{1}{v_1^* V^P} (v_1 - v_1^*) = \frac{\alpha}{v_1^* V^P} (z_1^* - z_1) \quad (78)$$

$$\frac{1}{v_1^*} \left(\frac{v_1}{z_1} - \frac{v_1^*}{z_1^*} \right) = \frac{\alpha}{\mathbf{1}_n^T v_2^*} \frac{z_1^* - z_1}{z_1} + \frac{1}{z_1} - \frac{1}{z_1^*}. \quad (79)$$

Evaluating (75) in light of (76)–(79), we see it is sufficient to prove there exist $\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3$, and $\tilde{\beta}_4$ such that

$$\int_{\tau}^{\infty} \left| \frac{z_1^* - z_1}{z_1} \right| d\sigma \leq \tilde{\beta}_1 \quad (80)$$

$$\int_{\tau}^{\infty} |z_1 - z_1^*| d\sigma \leq \tilde{\beta}_2 \quad (81)$$

$$\int_{\tau}^{\infty} \left| \frac{1}{z_1} - \frac{1}{z_1^*} \right| d\sigma \leq \tilde{\beta}_3 \quad (82)$$

$$\int_{\tau}^{\infty} \left| \frac{1}{z_2} - \frac{1}{z_2^*} \right| d\sigma \leq \tilde{\beta}_4. \quad (83)$$

By Lemma 3, we have that (80)–(82) are satisfied. Due to the similar dynamics of z_1 and z_2 given by (12), it is easily checked that (83) is also satisfied. Therefore, we can conclude that there exists β that satisfies condition (58), which concludes the proof. $\square$

Now we are ready to prove Theorem 1. To this end, we define

$$u' := B'w \quad (84)$$

such that system (34) becomes

$$\dot{\zeta} = (A' + F'(t))\zeta + u'. \quad (85)$$

We know by Lemma 4 that the origin of (85) is uniformly exponentially stable when $u' = 0$. For this reason (see [22]), there exists a state transition matrix $\Phi(t, s)$ and parameters $\mu \in \mathbb{R}_{>0}$ and $\lambda \in \mathbb{R}_{>0}$ such that

$$\|\Phi(t, t_0)\| \leq \mu e^{-\lambda(t-t_0)}. \quad (86)$$

We also know that the solution $\zeta(t)$ is given by

$$\zeta(t) = \Phi(t, t_0)\zeta(t_0) + \int_{t_0}^t \Phi(t, \tau)u'(\tau)d\tau.$$

Since we can bound $\|B'(t)w(t)\| < \gamma < \infty$ for any $t > 0$, we can also bound $\|\zeta\|$ by (see [23])

$$\begin{aligned} \|\zeta\| & \leq \|\Phi(t, t_0)\zeta(t_0) + \int_{t_0}^t \Phi(t, \tau)u'(\tau)d\tau\| \\ & \leq \mu e^{-\lambda(t-t_0)}\|\zeta(t_0)\| + \int_{t_0}^t \mu e^{-\lambda(t-\tau)}\|u'(\tau)\|d\tau \\ & \leq \mu e^{-\lambda(t-t_0)}\|\zeta(t_0)\| + \frac{\mu\gamma}{\lambda} \sup_{\tau \in [t_0, t]} \|u'(\tau)\| \end{aligned}$$

implying that $\|\zeta\|$ is bounded. Now consider the defined input (84). Since $\lim_{t \rightarrow \infty} z = z^*$ and $\lim_{t \rightarrow \infty} v = v^*$, we know that $\lim_{t \rightarrow \infty} A_e(t) = 0$, $\lim_{t \rightarrow \infty} B_e(t) = 0$, and $\lim_{t \rightarrow \infty} P_e(t) = 0$. This implies $\lim_{t \rightarrow \infty} B'(t) = 0$, from which we conclude

$$\lim_{t \rightarrow \infty} \|u'\| = \lim_{t \rightarrow \infty} \|B'w\| = 0. \quad (87)$$

By the boundedness of $\|\zeta\|$ and (87), we conclude that $\lim_{t \rightarrow \infty} \|\zeta\| = 0$, which proves Theorem 2.

APPENDIX C CONTROLLER DESIGN

In this section, the numerical values of controller (28) are given for the case studies presented in Section VI. These numerical values have been obtained by following the design procedure detailed in [21, Sec. 1.5]. This controller is given by

$$F = \begin{pmatrix} 2 & -3 & 1 & -40 & -400 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -200 & 1 \\ 0 & 0 & 0 & -10000 & 0 \end{pmatrix}$$

$$G = (0 \ 0 \ 0 \ 600 \ 90000)^T$$

$$H = (3 \ -3 \ 1 \ -30 \ -300) \quad (88)$$

and $K = 0$. The second case study with time-varying demand is presented in Section VI-B. In contrast to (88), which is designed for $S = 0$, we have that $S \neq 0$. As a consequence, a different controller is obtained, and is given by

$$F = \begin{pmatrix} 4 & -10 & 10 & -5 & 1 & -90 & -900 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -600 & 1 \\ 0 & 0 & 0 & 0 & 0 & -90000 & 0 \end{pmatrix}$$

$$G = (0 \ 0 \ 0 \ 0 \ 0 \ 600 \ 90000)^T$$

$$H = (5 \ -10 \ 10 \ -5 \ 1 \ -90 \ -900)$$

with again $K = 0$. Finally, in Section VI-C, the last case study with a real demand pattern is presented. The controller that is used is given by

$$F = \begin{pmatrix} 2 & -3 & 1 & -80 & -800 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1800 & 1 \\ 0 & 0 & 0 & -810000 & 0 \end{pmatrix}$$

$$G = (0 \ 0 \ 0 \ 1800 \ 810000)^T$$

$$H = (3 \ -3 \ 1 \ -80 \ -800) \quad (89)$$

with $K = 0$. The controller in (88) has the same structure as the one in (89) since they are both computed relatively to $S = 0$. However, the gain parameters in (89) have been chosen higher as to account for all the possible volume setpoints that are reported in Table I.

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