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Approximate Solutions and Cost Error Bounds for Quantity Flexibility Replenishment

Abstract

Under an environment with quantity flexibility replenishment, we explore the approximate solutions to determine ordering quantities when the demand is correlated with dynamic forecast updating. Moreover, we show the optimality of the myopic policy. We further develop the corresponding cost error bounds to evaluate the performance of the approximate solutions. Extensive numerical experiments demonstrate that our approximate solution can achieve a lower cost performance compared with other bounds.

Keywords: Approximation; Bounds; Quantity Flexibility; Stochastic Dynamic Programming.

1. Introduction

Under the increasing pressure of cost reduction in global sourcing, the procurement process plays a significant role in cost-cutting opportunities. To achieve a cost-efficient procurement process, quantity flexibility (QF) replenishment has been proposed and widely used in various industries, such as Toyota Motor Corporation (Lovejoy, 1999), IBM (Connors et al., 1995), Hewlett Packard, and Compaq (Faust, 1996). However, the complexity of QF replenishment hinders its implementation in business practice. Managers acknowledge that it needs more sophisticated tools for making effective and efficient replenishment strategies.

To address this issue, QF replenishment has been extensively studied in the literature. For instance, Tsay (1999) shows that a QF contract can achieve channel coordination in a decentralized supply chain consisting of a supplier and a retailer. Bassok and Anupindi (2008) propose two efficient heuristics to compute ordering quantities under a QF replenishment. Tang and Tomlin (2008) argue that a QF contract is an effective and efficient tool to mitigate

supply chain risks. Lian and Deshmukh (2009) focus on the development of the structure of the optimal replenishment strategy under a QF replenishment. Lian et al. (2010) investigate the performance of QF replenishment under a rolling-horizon environment. Interested readers can refer to the review by Tsay et al. (1998).

However, one of the critical assumptions made in the above literature is that the demand in a replenishment period during the horizon is independent of the demand in successive periods. In practice, the two are more likely correlated to each other. As a result, the current QF literature may not fit the real-life business situation. It is well-known that when demand is correlated in a dynamic inventory model, it is extremely difficult to obtain the optimal ordering quantity both analytically and numerically. Hence, a common method is to develop upper and lower bounds for the optimal solution and evaluate the performance of approximate solutions.

Lovejoy (1990) is the first attempt in the dynamic inventory literature to establish cost error bounds on suboptimal policies when demand distribution is updated by a Bayesian model and a Markov-modulated stochastic process. In a finite horizon, Graves (1999) studies a single-item inventory model by assuming that the demand process is an integrated moving average process and applies an adaptive base-stock policy. Moreover, he characterizes the safety stock requirement and extends the results to a multi-stage case. Toktay and Wein (2001) focus on one type of forecast-corrected inventory policy in a single-server make-to-stock system facing a stationary demand process and rolling-horizon forecast updates. They heuristically combine results from random walks and heavy-traffic approximations of queues to obtain the close-form expression for the (forecast-corrected) base-stock level that minimizes the expected steady-state inventory holding and backorder costs.

Under a context of fluctuating demand, researchers have tried to develop different forecasting mechanisms in order to improve the accuracy of demand forecasting. Among them, the Martingale Model of Forecast Evolution (MMFE) is quite general, flexible and easy to implement. The MMFE was developed independently by Graves et al. (1986) and Heath and Jackson (1994) and can represent nonstationary and time-correlated demand. It can also accommodate commonly used time series models, such as the Autoregressive Moving Average

(ARMA) model. Several authors have adopted the MMFE to study production-inventory problems. For the classic inventory model, Özer and Wei (2004) consider a periodic-review finite and infinite horizon production system with advance demand information. They derive the structure of optimal replenishment policies and numerically investigate the impacts of capacity, fixed costs, and advance demand information on the optimal decision. Iida and Zipkin (2006) study both an additive MMFE model and a multiplicative MMFE model. They show that a forecast-dependent base-stock policy is optimal and develop the upper bound and the lower bound for the optimal base-stock level. They further develop a piecewise-linear approximation of the cost function and a simulation-based expectation evaluation technique to solve the problem and establish the condition under which the myopic policy is optimal. Lu et al. (2006) use a sample-path approach to compute the approximate solution for the base-stock level and the related cost error bounds. By using numerical experiments, the authors demonstrate that their bounds result in a good performance in terms of the expected total cost. Iida and Zipkin (2010) investigate the impact of sharing demand forecasts on the competition and cooperation strategies under a two-stage supply chain. They find that only when the incentives of supply chain participants are aligned is it beneficial to share the information of demand forecasts. Wang et al. (2012) study a multi-period newsvendor problem with a dynamic update of demand forecast according to the MMFE. By considering the trade-off between improving demand forecast and increasing ordering cost, they show that the optimal ordering policy is a state-dependent base-stock policy. Recently, Bicer and Hagspiel (2016) consider a single-period QF model with the MMFE to analyze the impact of lead-time reduction on the value of quantity flexibility for the retailer. Further, they use the “contracts as reference points” theory to examine the effect of supply chain disintermediation problems on the retailer’s profit.

To the best of our knowledge, although some research (e.g., Bicer and Hagspiel, 2016) considers QF replenishment with the correlated demand, the development of the approximated solution and the performance evaluation of the approximation under QF replenishment have not been studied yet. To address these issues, we consider a buyer’s QF replenishment strategy under nonstationary stochastic demands and differential stock acquisition

costs. We formulate a stochastic dynamic programming model with the correlated demand. Under the QF replenishment, the buyer is allowed to adjust the previous order commitment and place an advance order. We prove that the optimal policy for order adjustment is a threshold type and the advance order follows a base-stock policy. Additionally, we obtain the condition under which a myopic policy is optimal and obtain the optimal solutions for the myopic policy. To promote the application, we also develop upper and lower bounds for the thresholds and the base-stock level. Finally, we develop cost error bounds and carry out numerical experiments for evaluating the cost performance between a general heuristic and the optimal policy. The contributions of our paper are twofold. First, we develop the lower and upper bounds for computing the thresholds and the base-stock level under QF replenishment. Second, we develop cost error bounds of quantity flexibility replenishment and perform extensive numerical experiments to show that our bounds are very cost-efficient.

The rest of the paper is organized as follows. Problem description and the properties of the model are presented in Section 2. Section 3 discusses the approximation of optimal replenishment strategies while Section 4 focuses on the development of cost error bounds. Numerical study is presented in Section 5 and conclusions are given in Section 6. All the proofs are presented in the appendix.

2. Problem Description and Optimal Replenishment Policy

First, we assume that N is the planning horizon of a manufacturer's master production schedule. A buyer will determine order quantities based on a quantity flexibility contract that is negotiated with the manufacturer. Due to the production lead time of the manufacturer, there exists a certain length of the frozen horizon where the order quantity cannot be modified. Without loss of generality, we assume that the length of the frozen period is 1 period. Denote $Q_{n,m}$ as the order quantity placed in period n for period m , where $n \leq m$. Under the QF contract, the buyer can adjust the previous order commitment and place an advanced order. With the latest inventory status and the updated demand forecast in period n , the buyer can change the previous order for period $n+1$ (i.e., $Q_{n-1,n+1}$) into $Q_{n,n+1}$ and place a new order for period $n+2$, i.e., $Q_{n,n+2}$. Note that the buyer cannot modify $Q_{n-1,n}$

due to the frozen horizon. We define ω as the unit purchasing cost for the additional order quantity $(Q_{n,n+1} - Q_{n-1,n+1})^+$; η as the unit cancellation rebate for the canceled quantity $(Q_{n-1,n+1} - Q_{n,n+1})^+$; c as the unit purchasing cost for the advanced order $Q_{n,n+2}$; h as the unit holding cost, and p as the unit backorder penalty cost. To avoid a trivial solution, we assume that $p > \omega > c > \eta$.

The sequence of events at the beginning of period n is as follows. At the beginning of period n , the buyer first updates demand forecasts for periods $n, \dots, N$. Then, based on the available inventory I_n , the buyer will modify order quantity for period $n+1$ and place a new order for period $n+2$. After the buyer receives order $Q_{n-1,n}$, the actual demand is realized. Finally, the demand is fulfilled by the on-hand stock. The excess stock will be carried to the next period and the shortage will be backordered.

We describe the correlated demand process as follows. The demand forecast is updated every period and can be characterized by a sequence of random variables: $D_{n,n}, D_{n,n+1}, \dots, D_{n,N}$. Here, the first subscript represents the period in which the demand forecast is newly updated, and the second subscript represents the period for which the forecasted demand is made, i.e., $D_{n,n+i}$ is the demand forecast updated in period n for period $n+i$, $i \geq 0$. Because we assume that the demand forecasts are updated after the demand information of the current period is realized, $D_{n,n}$ represents the actual demand of period n . For example, for a planning horizon with 4 periods, in period 1, we have made a forecast for period 3, i.e., $D_{1,3}$. When we know the actual demand in period 2, we update the demand forecast for period 3, i.e., $D_{2,3}$. $D_{2,3}$ represents the new demand forecast for period 3 after the actual demand in period 2 becomes known. We do not make any other specific assumption on the demand forecasts. In other words, the demand forecasts can have different general distributions and can be correlated. Let us define Ω_n as the demand forecast vector for period n , which represents both the length of the information horizon and the forecast mechanism.

The decision problem can be stated as follows. For period n , the buyer determines the order quantities $Q_{n,n+1}$ and $Q_{n,n+2}$, using the updated forecast information at the beginning of period n , so that the expected total cost from period n to the end of the planning horizon is minimized. Before we give the mathematical formulation, the following notations are

introduced. Let I_n be the inventory level in period n and ζ_n be the inventory position before ordering in stage n . Denote S_n as the inventory available for periods n and $n + 1$ and Z_n as the inventory available for periods n , $n + 1$, and $n + 2$ after ordering in period n as our decision variables, where $S_n = I_n + Q_{n,n+1}$ and $Z_n = S_n + Q_{n,n+2}$. Let $V_n(I_n, \zeta_n, \Omega_n)$ be the minimum expected total cost from period n to N given I_n , ζ_n and Ω_n . Let $H(x)$ be the expected inventory carrying cost for a period with an ending inventory level x , i.e., $H(x) = hE(x)^+ + pE(x)^-$. We now formulate the decision problem as a stochastic dynamic programming (SDP) formulation. For $1 \leq n \leq N - 2$, the optimality equation is given by

$$V_n(I_n, \zeta_n, \Omega_n) = \min_{S_n \geq I_n, Z_n \geq S_n} \{A_n(I_n, \zeta_n, \Omega_n, S_n, Z_n)\}, \quad (1)$$

where

$$A_n(I_n, \zeta_n, S_n, Z_n, \Omega_n) = C_n(S_n, Z_n, \Omega_n) + EV_{n+1}(I_{n+1}, \zeta_{n+1}, \Omega_{n+1}), \quad (2)$$

and

$$C_n(S_n, Z_n, \Omega_n) = \xi(S_n - \zeta_n) + c(Z_n - S_n) + H(S_n - D_{n,n} - D_{n,n+1}), \quad (3)$$

where $\xi(x) = \omega x^+ - \eta(-x)^+$.

Equation (2) consists of two terms. The first term $C_n(S_n, Z_n, \Omega_n)$ represents the costs in the current period (period n) while the second term $EV_{n+1}(I_{n+1}, \zeta_{n+1}, \Omega_{n+1})$ represents the cost-to-go function that indicates the impact of the current decision on the costs in the remaining periods. Equation (3) represents the costs in period n , including the cost of adjusting the order quantity, the cost of the advanced order, and the expected inventory carrying cost.

For $n = N - 1$, Z_N vanishes due to the one-period frozen horizon. Then, we have

$$C_{N-1}(S_{N-1}, \Omega_{N-1}) = \xi(S_{N-1} - \zeta_{N-1}) + H(I_{N-1} - D_{N-1,N-1}) + H(I_N - D_{N,N}).$$

$C_{N-1}(S_{N-1}, \Omega_{N-1})$ represents the terminal condition of the SDP formulation.

By induction, we can show that $A_n(I_n, \zeta_n, S_n, Z_n, \Omega_n)$ is jointly convex in (S_n, Z_n) . Then, we can first optimize Z_n for any given S_n and then optimize S_n . Thus, we can rewrite A_n as

$$A_n(I_n, \zeta_n, S_n, Z_n, \Omega_n) = C_n(S_n, Z_n, \Omega_n) + \bar{A}_n(S_n, \Omega_n), \quad (4)$$

where $\bar{A}_n(S_n, \Omega_n) = \min_{Z_n \geq S_n} \{cZ_n + EV_{n+1}(I_{n+1}, \zeta_{n+1}, \Omega_{n+1})\}$.

Now, we show the properties of the optimality equation given by (1) that are important for the development of approximate solutions and cost error bounds.

The following proposition shows the structure of the optimal replenishment policy.

Proposition 2.1. *For a given period n , the optimal decisions are given by*

$$S_n^* = \begin{cases} L_n(\Omega_n), & \text{if } \zeta_n < L_n(\Omega_n), \\ \zeta_n, & \text{if } L_n(\Omega_n) \leq \zeta_n \leq U_n(\Omega_n), \\ U_n(\Omega_n), & \text{if } \zeta_n > U_n(\Omega_n), I_n \leq U_n(\Omega_n), \\ I_n, & \text{if } \zeta_n > U_n(\Omega_n), I_n > U_n(\Omega_n), \end{cases} \quad (5)$$

where $L_n(\Omega_n) = \arg \min_{S_n} \{(\omega - c)S_n + H(S_n - D_{n,n} - D_{n,n+1}) + \bar{A}_n(S_n, \Omega_n)\}$ and $U_n(\Omega_n) = \arg \min_{S_n} \{(\eta - c)S_n + H(S_n - D_{n,n} - D_{n,n+1}) + \bar{A}_n(S_n, \Omega_n)\}$

And,

$$Z_n^* = \max\{S_n^*, \bar{Z}_n(\Omega_{n+1})\}, \quad (6)$$

where $\bar{Z}_n(\Omega_{n+1}) = \arg \min_{Z_n \geq S_n^*} \{c(Z_n - S_n) + EV_{n+1}(I_{n+1}, \zeta_{n+1}, \Omega_{n+1})\}$.

Equation (5) represents a dual-threshold policy for the updated order quantity for period $n + 1$, where the lower threshold L_n and the upper threshold U_n depend on the demand information (Ω_n) and $L_n(\Omega_n) \leq U_n(\Omega_n)$. By following this dual-threshold policy, the buyer should order more to raise the inventory position up to $L_n(\Omega_n)$ since there is insufficient amount of stock available for the next period when $\zeta_n < L_n(\Omega_n)$. When $L_n(\Omega_n) \leq \zeta_n \leq U_n(\Omega_n)$, the buyer has a sufficient amount of stock available. Then, no change needs to be made for the previous commitment. When $\zeta_n > U_n(\Omega_n)$, there is excess on-hand stock. Thus, the buyer has to cancel part of the previous committed order or even cancel the entire order if $I_n > U_n(\Omega_n)$.

Equation (6) indicates that the optimal policy for the advanced order follows a base-stock policy where the base-stock level is given by $\bar{Z}_n(\Omega_{n+1})$. Thus, in the current period n , the buyer should raise the inventory position of period $n + 2$ at least to such a base-stock level.

3. Approximation of Optimal Replenishment

In this section, we first analyze the myopic policy and give the necessary and sufficient condition under which the myopic policy is optimal. Moreover, we develop both lower and upper bounds for the thresholds and the base-stock level. From now on, for the exposition purpose, we ignore Ω_n for the expressions of L_n, U_n , and $\bar{Z}_n$. Readers should keep in mind that L_n, U_n , and $\bar{Z}_n$ depend on Ω_n .

3.1. Analysis of the First-Order Conditions and the Myopic Policy

According to the first-order conditions of the optimality equation in period n , we have

$$\begin{aligned} \frac{\partial A_n}{\partial S_n} &= \omega + \frac{\partial G_n}{\partial S_n}, & \text{for } S_n > \zeta_n, \\ \frac{\partial A_n}{\partial S_n} &= \eta + \frac{\partial G_n}{\partial S_n}, & \text{for } S_n < \zeta_n, \\ \frac{\partial G_n}{\partial S_n} &= -c + H'(S_n - D_{n,n} - D_{n+1,n+1}) \\ &\quad + E \frac{\partial V_{n+1}(S_n - D_{n,n}, Z_n - D_{n,n}, \Omega_{n+1})}{\partial S_n}, \\ \frac{\partial A_n}{\partial Z_n} &= \frac{\partial G_n}{\partial Z_n} = c + E \frac{\partial V_{n+1}(S_n - D_{n,n}, Z_n - D_{n,n}, \Omega_{n+1})}{\partial Z_n}, \end{aligned}$$

where we can switch the order of the expectation and the differentiation by the dominated convergence theorem.

From the above first-order conditions, we observe that since $I_{n+1} = S_n - D_{n,n}$ and $\zeta_{n+1} = Z_n - D_{n,n}$, the current decisions S_n and Z_n may affect the future decisions, which causes the difficulty to obtain the optimal solution for the current period.

By (5), we find that only when $I_{n+1} > U_{n+1}$ does S_n play a role in EV_{n+1} . For other cases, S_n^* is only determined by the cost in the current period. Therefore, the myopic policy is optimal for $I_{n+1} \leq U_{n+1}$. Similarly, when $L_{n+1} < \zeta_{n+1} < U_{n+1}$, Z_n affects the future cost. As a result, we require $\zeta_{n+1} \leq L_{n+1}$ or $I_{n+1} < U_{n+1} < \zeta_{n+1}$ to guarantee the optimality of the myopic policy.

Under the myopic policy, we can correspondingly obtain $U_n^{(m)}$, $L_n^{(m)}$ and $\bar{Z}_n^{(m)}$, where

$$U_n^{(m)} = \arg \min_{S_n} \{(\eta - c)S_n + H(S_n - D_{n,n} - D_{n+1,n+1})\}, \quad (7)$$

$$L_n^{(m)} = \arg \min_{S_n} \{(\omega - c)S_n + H(S_n - D_{n,n} - D_{n+1,n+1})\}, \quad (8)$$

$$\bar{Z}_n^{(m)} = \arg \min_{Z_n} \{cZ_n + E(\xi(S_{n+1}^{(m)} - Z_n - D_{n,n}))\}. \quad (9)$$

Equations (7), (8), and (9) indicate that when the myopic thresholds and the base-stock level at period n are reached, S_n^* and Z_n^* have no effect on future optimal decisions. The following proposition gives the condition under which the myopic policy is optimal.

Proposition 3.1. *The myopic policy is optimal if and only if either (10) or (11) is satisfied for $n = 1, 2, \dots, N - 1$.*

$$P(S_n^{(m)} - U_{n+1}^{(m)} \leq D_{n,n} \leq Z_n^{(m)} - U_{n+1}^{(m)}) = 1, \quad (10)$$

$$P(Z_n^{(m)} - L_{n+1}^{(m)} \leq D_{n,n}) = 1, \quad (11)$$

where $S_n^{(m)}$ can be either $U_n^{(m)}$ or $L_n^{(m)}$ and $Z_n^{(m)} = \max\{S_n^{(m)}, \bar{Z}_n^{(m)}\}$.

After examining the first-order conditions, we identify that both $\partial V_{n+1}/\partial S_n$ and $\partial V_{n+1}/\partial Z_n$ depend on the entire forecast evolution. It is very difficult to obtain them either analytically or numerically. Hence, we try to develop bounds for the optimal policy. To achieve this goal, we have to develop bounds for these two partial derivatives. Because the myopic policy approximates them by 0, it provides upper bounds for the optimal thresholds and the base-stock level. However, the bounds are rather loose. Thus, we intend to develop tighter bounds. Before we do so, let us further analyze the property of the first-order conditions. Define

$$Y_{n+i} = \{S_n - D[n, n+i] > U_{n+i}\}, \quad (12)$$

$$\tilde{Y}_{n+i} = \{L_{n+i} < Z_n - D[n, n+i] < U_{n+i}\}, \quad (13)$$

where $D[n, n+i] = \sum_{k=0}^{i-1} D_{n,n+k}$.

Let $I(Y)$ be the indicator function of Y and $I(Y^c) = 1 - I(Y)$. Note that the decisions S_n and Z_n at period n affects period $n+k$ and beyond only if Y_{n+i} happens for all $i = 1, \dots, k$.

Proposition 3.2. *For any period n ,*

$$\frac{\partial A_n}{\partial S_n} = C'_n(S_n) + \sum_{i=1}^{N-n+2} E[I(Y_{n+1} \cdots Y_{n+i}) C'_{n+i}(S_n - D[n, n+i])], \quad (14)$$

$$\frac{\partial A_n}{\partial Z_n} = C'_n(Z_n) + \sum_{i=1}^{N-n+2} E[I(\tilde{Y}_{n+1} \cdots \tilde{Y}_{n+i}) C'_{n+i}(Z_n - D[n, n+i])]. \quad (15)$$

Proposition 3.2 explicitly characterizes the impacts of S_n and Z_n on the future cost in a sample-path way. We will develop bounds based on equations (14) and (15) in the next section.

3.2. Bounds of the Optimal Policy

First, we will consider the upper bounds and the lower bounds for L_n and U_n . For any period n , we denote $L_n^{(u)}(U_n^{(u)})$ and $L_n^{(\ell)}(U_n^{(\ell)})$ as upper and lower bounds for $L_n(U_n)$, respectively. Recall that $L_n^{(m)}(U_n^{(m)})$ is the myopic solution for period n .

Define

$$\begin{aligned}
Y_{n+i}^{(u)} &= \{S_n - D[n, n+i] \geq U_{n+i}^{(u)}\}, \\
Y_{n+i}^{(\ell)} &= \{S_n - D[n, n+i] \geq U_{n+i}^{(\ell)}\}, \\
Y_{n+i}^{(m)} &= \{S_n - D[n, n+i] \geq U_{n+i}^{(m)}\}, \\
\alpha_n^{(\ell)}(S_n, \Omega_n) &= \sum_{i=1}^{N-n+2} E[I(Y_{n+1}^{(u)} \cdots Y_{n+i}^{(u)}) C'_{n+i}(S_n - D[n, n+i])], \\
\alpha_n^{(u)}(S_n, \Omega_n) &= E[I(Y_{n+1}^{(u)}) C'_{n+1}(S_n - D[n, n+1])] \\
&\quad + \sum_{i=2}^{N-n+2} E[I(Y_{n+1}^{(\ell)} \cdots Y_{n+i-1}^{(\ell)} Y_{n+i}^{(m)}) C'_{n+i}(S_n - D[n, n+i])], \\
L_n^{(\ell)} &= \text{the zero point of } \omega - c + H'(S_n - D_{n,n} - D_{n+1,n+1}) \\
&\quad + \alpha_n^{(u)}(S_n, \Omega_n), \\
L_n^{(u)} &= \text{the zero point of } \omega - c + H'(S_n - D_{n,n} - D_{n+1,n+1}) \\
&\quad + \alpha_n^{(\ell)}(S_n, \Omega_n), \\
U_n^{(\ell)} &= \text{the zero point of } \eta - c + H'(S_n - D_{n,n} - D_{n+1,n+1}) \\
&\quad + \alpha_n^{(u)}(S_n, \Omega_n), \\
U_n^{(u)} &= \text{the zero point of } \eta - c + H'(S_n - D_{n,n} - D_{n+1,n+1}) \\
&\quad + \alpha_n^{(\ell)}(S_n, \Omega_n).
\end{aligned}$$

Clearly, we have $L_n^{(u)} \leq L_n^{(m)}$, $L_n^{(\ell)} \leq L_n^{(m)}$, $U_n^{(u)} \leq U_n^{(m)}$, and $U_n^{(\ell)} \leq U_n^{(m)}$. Since the myopic policy has already provided upper bounds, we only consider the upper bounds that are no larger than the values given by the myopic policy.

Theorem 1. *For any given n , $n = 1, \dots, N$, we have $L_n^{(\ell)} \leq L_n \leq L_n^{(u)}$ and $U_n^{(\ell)} \leq U_n \leq U_n^{(u)}$.*

From Theorem 1, the only difference between these bounds and the myopic policy is the adjustments that determine the magnitude of the safety stock. The adjustments are $\alpha_n^{(u)}$ and $\alpha_n^{(\ell)}$. The myopic policy is a special case in which we set $\alpha_n^{(\ell)} = 0$. Moreover, we can show that both $\alpha_n^{(u)}(S_n, \Omega_n)$ and $\alpha_n^{(\ell)}(S_n, \Omega_n)$ are increasing in S_n . Thus, there exist unique solutions for $L_n^{(\ell)}$, $L_n^{(u)}$, $U_n^{(\ell)}$ and $U_n^{(u)}$. Since $L_n \leq U_n$, if $L_n^{(u)} > U_n^{(\ell)}$, we set both the upper bound of L_n and the lower bound of U_n to be equal to $\max\{L_n^{(u)}, U_n^{(\ell)}\}$.

To develop the upper and lower bounds for the base-stock level, we need to redefine the follows sets as

$$\begin{aligned}\tilde{Y}_{n+i}^{(u)} &= \{L_{n+i}^{(u)} < S_n - D[n, n+i] < U_{n+i}^{(\ell)}\}, \\ \tilde{Y}_{n+i}^{(\ell)} &= \{L_{n+i}^{(\ell)} < S_n - D[n, n+i] < U_{n+i}^{(u)}\}, \\ \tilde{Y}_{n+i}^{(m)} &= \{L_{n+i}^{(m)} < S_n - D[n, n+i] < U_{n+i}^{(\ell)}\}.\end{aligned}$$

Then, we can show that $\bar{Z}_n^{(\ell)} \leq \bar{Z}_n \leq \bar{Z}_n^{(u)}$ for $i = 1, \dots, N$, where

$$\begin{aligned}\tilde{\alpha}_n^{(u)}(Z_n, \Omega_n) &= \sum_{i=1}^{N-n+2} E[I(\tilde{Y}_{n+1}^{(\ell)} \cdots \tilde{Y}_{n+i}^{(\ell)}) C'_{n+i}(Z_n - D[n, n+i])], \\ \tilde{\alpha}_n^{(\ell)}(Z_n, \Omega_n) &= E[I(\tilde{Y}_{n+1}^{(\ell)}) C'_{n+1}(Z_n - D[n, n+1])] \\ &\quad + \sum_{i=2}^{N-n+2} E[I(\tilde{Y}_{n+1}^{(u)} \tilde{Y}_{n+i}^{(m)}) C'_{n+i}(Z_n - D[n, n+i])], \\ \bar{Z}_n^{(\ell)} &= \text{the zero point of } c + \tilde{\alpha}_n^{(u)}(Z_n, \Omega_n), \\ \bar{Z}_n^{(u)} &= \text{the zero point of } c + \tilde{\alpha}_n^{(\ell)}(Z_n, \Omega_n).\end{aligned}$$

4. Cost Error Bounds

In this section, we consider how to evaluate the cost performance of a heuristic policy compared with the the minimum cost given by the optimal policy. Based on the lower and upper bounds derived in Section 3, the heuristic policy is defined as

$$L_n^{(h)} = \beta_1 L_n^{(u)} + (1 - \beta_1) L_n^{(\ell)}, U_n^{(h)} = \beta_2 U_n^{(u)} + (1 - \beta_2) U_n^{(\ell)}, \bar{Z}_n^{(h)} = \beta_3 \bar{Z}_n^{(u)} + (1 - \beta_3) \bar{Z}_n^{(\ell)},$$

where $\beta_i \in [0, 1]$ for $i = 1, 2, 3$.

Let $V_n^{(h)}$ be the total expected cost of any given heuristic policy h starting from period n . The cost error of a heuristic policy relative to the minimum cost is defined by

$$\Delta = \frac{V_1^{(h)}(I_1, \zeta_1, \Omega_1) - V_1(I_1, \zeta_1, \Omega_1)}{V_1(I_1, \zeta_1, \Omega_1)} \times 100\%. \quad (16)$$

Since $V_n(I_n, \zeta_n, \Omega_n)$ is usually computationally impossible to obtain, we intend to obtain an upper bound for Δ . Our approach is to develop an upper bound for $V_n^{(h)}(I_n, \zeta_n, \Omega_n) -$

$V_n(I_n, \zeta_n, \Omega_n)$ and a lower bound for $V_n(I_n, \zeta_n, \Omega_n)$. From now on, we denote $\mu \vee \nu = \max\{\mu, \nu\}$ and $\mu \wedge \nu = \min\{\mu, \nu\}$ for any real numbers μ and ν .

First, the derivation of a lower bound for $V_n(I_n, \zeta_n, \Omega_n)$ is given as follows.

Proposition 4.1. *For $n = 1, \dots, N$, we have*

$$\begin{aligned} V_n(I_n, \zeta_n, \Omega_n) &\geq \min\{C_n(\zeta_n \vee L_n^{(u)}, \bar{Z}_n^{(\ell)}, \Omega_n), C_n(I_n \vee U_n^{(u)}, \bar{Z}_n^{(\ell)}, \Omega_n)\} \\ &\quad + \sum_{i=1}^{N-n+2} EC_{n+i}(U_{n+i}^{(u)}, \bar{Z}_{n+i}^{(\ell)}, \Omega_{n+i}). \end{aligned} \quad (17)$$

Next, to construct the upper bound, we consider two cases: $Z_n^{(h)} \leq Z_n^*$ and $Z_n^{(h)} > Z_n^*$. We present the results for each case in the following lemmas. Based on these results, we develop the upper bound for $V_n^{(h)}(I_n, \zeta_n, \Omega_n) - V_n(I_n, \zeta_n, \Omega_n)$. We denote $Z_n^{(h)} = S_n^{(h)} \vee \bar{Z}_n^{(h)}$, where $S_n^{(h)}$ is given by (5) except that L_n and U_n are replaced by $L_n^{(h)}$ and $U_n^{(h)}$, respectively. Here, we assume that $L_n^{(u)} \leq U_n^{(\ell)}$ without loss of generality. For brevity of notations, we use D_n to represent $D_{n,n}$ in this section.

Lemma 1. *If $Z_n^{(h)} > Z_n^*$, we have*

$$\begin{aligned} V_n^{(h)}(I_n, \zeta_n, \Omega_n) - V_n(I_n, \zeta_n, \Omega_n) &\leq \Gamma_1(n, \Omega_n) + EV_{n+1}^{(h)}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) \\ &\quad - EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}), \end{aligned} \quad (18)$$

where

$$\begin{aligned} \Gamma_1(n, \Omega_n) &= g_1(n, \Omega_n) + \max\{g_2(n, \Omega_{n+1}), B_1(L_n^{(h)})g_2(n, \Omega_{n+1}) + B_1^c(L_n^{(u)})g_3(n, L_n^{(u)}, \Omega_{n+1}), \\ &\quad [B_1^c(U_n^{(h)}) + B_2(U_n^{(h)})]g_2(n, \Omega_{n+1}) + B_1(U_n^{(h)})B_2^c(U_n^{(u)})g_3(n, U_n^{(u)}, \Omega_{n+1}), \\ &\quad [B_1(U_n^{(h)})B_1^c(L_n^{(h)}) + B_2(U_n^{(h)})]g_2(n, \Omega_{n+1}) + B_1^c(L_n^{(u)})g_3(n, L_n^{(u)}, \Omega_{n+1}) \\ &\quad + B_1(U_n^{(h)})B_2^c(U_n^{(u)})g_3(n, U_n^{(u)}, \Omega_{n+1})\}. \end{aligned}$$

Here, $B_1(x) = \Pr(x \leq \zeta_n)$, $B_2(x) = \Pr(x \leq I_n)$, $B_i^c(x) = 1 - B_i(x)$, and

$$g_1(n, \Omega_n) = C_n(S_n^{(h)}, Z_n^{(h)}, \Omega_n) - \min\{C_n(L_n^{(u)}, L_n^{(\ell)} \vee \bar{Z}_n^{(\ell)}, \Omega_n), C_n(U_n^{(u)}, L_n^{(\ell)} \vee \bar{Z}_n^{(\ell)}, \Omega_n)\},$$

$$g_2(n, \Omega_{n+1}) = (S_n^{(h)} - S_n^{(\ell)})\alpha_{n+1}^{(u)}(S_n^{(h)} - D_n, \Omega_{n+1}),$$

$$g_3(n, x, \Omega_{n+1}) = -[x \vee \bar{Z}_n^{(h)} - S_n^{(h)} \vee \bar{Z}_n^{(h)}]\tilde{\alpha}_{n+1}^{(\ell)}(x \vee \bar{Z}_n^{(h)}, \Omega_{n+1}).$$

Lemma 2. If $Z_n^{(h)} \leq Z_n^*$, we have

$$\begin{aligned} V_n^{(h)}(I_n, \zeta_n, \Omega_n) - V_n(I_n, \zeta_n, \Omega_n) &\leq \Gamma_2(n, \Omega_n) + EV_{n+1}^{(h)}(S_n^h - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) \\ &\quad - EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}), \end{aligned} \quad (19)$$

where

$$\begin{aligned} \Gamma_2(n, \Omega_n) &= g_1(n, \Omega_n) + \max\{h_1(n, \Omega_{n+1}), B_1(L_n^{(h)})h_1(n, \Omega_{n+1}) \\ &\quad + B_1^c(L_n^{(h)})h_2(n, \Omega_{n+1}), [B_1^c(U_n^{(h)}) + B_2(U_n^{(h)})]h_1(n, \Omega_{n+1}) \\ &\quad + B_1(U_n^{(\ell)})B_2^c(U_n^{(h)})h_3(n, \Omega_{n+1}), \\ &\quad [B_1^c(L_n^{(h)})B_1(U_n^{(h)}) + B_2(U_n^{(h)})]h_1(n, \Omega_{n+1}) + B_1^c(L_n^{(h)})h_2(n, \Omega_{n+1}) \\ &\quad + B_1(U_n^{(\ell)})B_2^c(U_n^{(h)})h_3(n, \Omega_{n+1})\}, \\ h_1(n, \Omega_{n+1}) &= -(S_n^{(h)} \vee \bar{Z}_n^{(u)} - Z_n^{(h)})\tilde{\alpha}_{n+1}^{(\ell)}(S_n^{(h)} \vee \bar{Z}_n^{(u)}, \Omega_{n+1}), \\ h_2(n, \Omega_{n+1}) &= (\bar{Z}_n^{(u)} - \bar{Z}_n^{(h)})\tilde{\alpha}_{n+1}^{(u)}(L_n^{(h)} \vee \bar{Z}_n^{(u)}, \Omega_{n+1}), \\ h_3(n, \Omega_{n+1}) &= -(U_n^{(u)} \vee \bar{Z}_n^{(u)} - U_n^{(\ell)} \vee \bar{Z}_n^{(h)})\tilde{\alpha}_{n+2}^{(\ell)}(U_n^{(u)} \vee \bar{Z}_n^{(u)}, \Omega_{n+1}). \end{aligned}$$

We summarize the results of the above two lemmas in the following proposition.

Proposition 4.2. For a heuristic policy h and period n , we have

$$\begin{aligned} V_n^{(h)}(I_n, \zeta_n, \Omega_n) - V_n(I_n, \zeta_n, \Omega_n) &\leq \Gamma_1(n, \Omega_n) \vee \Gamma_2(n, \Omega_n) \\ &\quad + E \sum_{i=n+1}^{N+2} \Gamma_1(i, \Omega_i) \vee \Gamma_2(i, \Omega_i). \end{aligned}$$

Based on the cost error bounds derived in Propositions 4.1 and 4.2, we can construct an upper bound of Δ by (16) which can be used as a performance indicator for the bounds of the replenishment policy given in Section 3. In the next section, we perform numerical experiments to compute such an upper bound of Δ .

5. Numerical Studies

Now, we design numerical experiments to demonstrate the results developed in the above sections. We also compare our cost bounds with the bounds developed by Lovejoy (1992)

and Lu et al. (2004). To make a comprehensive study, we perform experiments with respect to two types of forecast updates: additive and multiplicative.

For the additive update, we use the same demand forecast model as Lu et al. (2004), i.e.,

$$D_{n,n} = \mu_n + \rho(D_{n-1,n-1}) + \epsilon_n,$$

where $E[D_{n,n}] = \mu_n$, the coefficient of correlation of demands in two successive periods is ρ , and ϵ_n follows i.i.d. $N(0, \sigma^2)$. At period n , after $D_{n,n}$ is realized, we generate a new forecast for the demand in period $n+1$ as $D_{n,n+1} = \mu_{n+1} + \rho(D_{n,n} - \mu_n)$. In a similar way, we obtain the new forecast for period $n+i$ as $D_{n,n+i} = \mu_{n+i} + \rho^i(D_{n,n} - \mu_n)$ for $0 \leq i \leq N - n + 2$.

For the multiplicative update, we use $D_{n,n+i} = D_{n-1,n+i}\epsilon_{n,n+i}$, where $E(\epsilon_{n,n+i}) = 1$. We assume that forecast updates have multi-dimensional log-normal distributions. In practice, only the updates within in are correlated. Here, we assume that the updates within 4 consecutive periods are correlated, including $\epsilon_{n,n}, \epsilon_{n,n+1}, \epsilon_{n,n+2}, \epsilon_{n,n+3}$. The covariance matrix is given by

$$\begin{bmatrix} 0.04 & & & \\ & 0.08 & & \\ & & 0.02 & \\ & & & 0.05 \end{bmatrix}.$$

The rest of updates is independent. We assume that the corresponding standard deviation is 0.2.

For each type of forecast updates, we further consider four scenarios of the initial demand forecasts: zero trend, positive trend, negative trend, and cycles. For zero trend, $\mu_n = 100$ for all n ; for positive trend, $\mu_n = \mu_{n-1} + 5$, $\mu_1 = 100$; for negative trend, $\mu_n = \mu_{n-1} - 5$, $\mu_1 = 100$; for cycles, we use $100 \sin(\pi n/12)$. In addition, we set the initial demand forecast as $D_{0,i} = \mu_i$ and $\sigma = 0.25\mu_1$.

To investigate the impact of demand correlation on the cost performance, we perform the corresponding sensitivity analysis with respect to ρ that can be treated as an indicator of demand correlation.

Furthermore, because we focus on the role of the demand characteristics, we consider two sets of stationary cost parameters as follows: I. $\omega = 20$, $c = 19.5$, $\eta = 19$, $h = 0.25$ and $p = 16$; II. $\omega = 10$, $c = 9.8$, $\eta = 9.6$, $h = 0.1$ and $p = 12$. In addition, we select $N = 12$.

In our numerical studies, we compare upper bounds on relative cost error $\Delta^{(u)}$ with other bounds. $\Delta^{(u)}$ is the ratio of the upper bound of $V_1^{(h)} - V_1$ given by Proposition 4.2 and the lower bound of V_1 given by Proposition 4.1. For parameters in the heuristic policy, we set $\beta_i = 0.5$. We carry out the numerical experiments by using Matlab version 7.0 and set the maximum error to be 0.0001 for precision. Note that data reports in the following tables are percentages.

From Tables 1, 2 and 3, we have the following observations: (i) The cost error between the myopic policy and the optimal increases in ρ . Moreover, there may exist a large gap even for stationary cost parameters. The results show that the cost error bound given by the myopic policy can be at most 20% higher than $\Delta^{(u)}$. Thus, it is necessary to develop better bounds to reduce cost errors. (ii) The performance of Lovejoy's bound could be much worse. This is because by ordering up to the myopic base-stock level given by Lovejoy's method, the overstocking cost is much less than the total cost for holding those items from the current period till the end of the planning horizon. The phenomenon is also consistent with the observation by Lu et al. (2004). (iii) Our bounds perform as closely as the bounds of Lu et al. (2004) indicated by the data presented in the three tables. In particular, when the trend of the demand is positive and a cycle pattern exists, our bounds are tighter than theirs. This is because the buyer may place a large advanced order more likely under this situation. Moreover, since the supplier offers a price discount for the buyer, the total cost is also reduced. (iv) We find that the cost errors of all the four bounds become smaller when ρ decreases. In summary, the heuristic policy based on the upper and lower bounds of the optimal policy provides a good approximation for the optimal solution in a cost-efficient way.

Table 1: Upper bound on relative cost error for the zero trend

Addi.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.12	0.13	0.07	0.08	0.17	0.13	0.06	0.08
-0.6	0.54	0.44	0.22	0.16	0.7	0.52	0.23	0.15
-0.3	0.92	0.91	0.46	0.46	1.25	1.63	0.53	0.46
0	1.98	3.14	0.87	0.9	2.88	3.6	0.89	0.92
0.3	3.36	5.65	2.01	2.02	4.5	6.38	2.15	2.13
0.6	10.26	15.48	4.29	4.29	13.73	17.38	4.53	4.61
0.9	18.41	24.38	9.01	9.13	31.3	33.65	9.33	9.42

Mult.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.23	0.25	0.16	0.1	0.22	0.21	0.15	0.11
-0.6	0.7	0.89	0.67	0.51	0.74	0.79	0.69	0.55
-0.3	1.35	1.57	0.98	0.87	1.45	1.89	1.01	0.86
0	2.11	3.87	1.47	1.2	3.01	3.99	1.52	1.31
0.3	4.98	6.12	2.98	2.14	4.78	5.41	3.03	2.27
0.6	13.01	17.89	5.29	4.62	14.21	18.04	5.42	4.7
0.9	20.19	28.51	10.01	9.21	33.01	35.33	9.98	9.19

Table 2: Upper bound on relative cost error for the positive trend

Addi.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.5	0.57	0.34	0.31	0.66	0.58	0.37	0.34
-0.6	0.97	1.01	0.51	0.51	1.3	1.09	0.6	0.57
-0.3	2.34	3.09	1.14	1.08	3.01	3.4	1.21	1.13
0	6.78	9	3.07	2.89	9.11	10.09	3.19	3.03
0.3	13.72	18.28	7.91	7.19	18.34	20.41	8.38	7.57
0.6	21.15	30.11	10.35	9.89	28.15	33.61	10.78	10.1
0.9	31.22	49.23	15.4	11.78	40.5	55	15.96	12.32

Mult.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.67	0.77	0.56	0.5	0.69	0.69	0.61	0.51
-0.6	1.09	1.34	0.95	0.81	1.2	1.43	0.89	0.79
-0.3	2.65	3.89	1.99	1.24	3.14	3.68	1.79	1.3
0	7.1	9.21	4.67	4.05	9.12	10.42	4.97	3.98
0.3	14.5	20.01	9.18	7.89	18.56	21.78	8.88	7.77
0.6	23.01	32.11	13.67	11.21	28.98	35.31	12.67	11.23
0.9	35.12	53.02	18.11	13.66	36.33	55.22	17.98	13.73

Table 3: Upper bound on relative cost error for the negative trend

Addi.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.36	0.79	0.27	0.27	0.47	0.9	0.27	0.27
-0.6	0.94	2.1	0.91	0.92	1.27	2.27	0.93	0.94
-0.3	2.75	6.18	1.8	2.03	3.42	6.91	1.94	2.1
0	5.5	10.51	4.99	5	7.34	11.71	5.17	5.16
0.3	10.89	23.83	9.59	9.59	14.48	26.6	9.9	9.9
0.6	22.24	38.41	14.51	15.05	29.68	42.81	15.01	15.03
0.9	35.11	57.34	18.12	18.81	46.8	63.87	18.68	18.8

Mult.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.5	0.81	0.43	0.41	0.51	0.91	0.4	0.4
-0.6	1.23	2.56	1.1	0.99	1.32	2.32	0.99	0.95
-0.3	3.41	7.45	2.87	2.42	3.66	6.98	2.48	2.28
0	6.11	11.98	5.46	5.02	7.98	12.16	5.44	5.11
0.3	11.87	24.53	10.02	9.87	14.99	26.79	10.13	9.91
0.6	24.65	39.12	15.09	14.88	30.01	43.12	15.22	15.1
0.9	36.98	58.99	19.12	18.97	47.11	64.02	19.19	19.05

Table 4: Upper bound on relative cost error for cycles

Addi.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.35	0.99	0.34	0.29	0.34	0.82	0.33	0.29
-0.6	1.27	3.1	1.01	0.89	1.3	2.82	1.21	1.08
-0.3	2.89	6.87	2.31	2.08	2.66	6.67	2.43	2.18
0	7.01	9.56	5.99	4.75	6.89	11.35	6.44	5.78
0.3	12.33	19.79	10.45	9.16	13.54	21.98	12.01	10.89
0.6	23.15	36.23	20.67	15.01	22.99	35.23	19.11	15.92
0.9	33.02	55.44	29.88	20.98	36.1	53.01	22.12	18.69

Mult.	Cost Parameters I				Cost Parameters II			
ρ	Myopic	Lovejoy	Lu	$\Delta^{(u)}$	Myopic	Lovejoy	Lu	$\Delta^{(u)}$
-0.9	0.48	1.21	0.39	0.38	0.47	1.19	0.41	0.4
-0.6	1.48	3.26	1.19	1.03	1.51	3.2	1.41	1.39
-0.3	3.15	7.18	2.91	2.5	2.98	6.67	2.78	2.58
0	8	10.01	6.58	5.32	7.89	12.35	6.98	6.01
0.3	13.89	20.31	12.01	9.79	14.21	21.98	13.98	11.34
0.6	24.56	36.89	20.98	16.13	24.01	37.23	20.33	17.01
0.9	35.01	56.12	31.02	21.65	36.32	56.01	23.45	19.99

6. Conclusion

We study the finite-horizon and periodic-review inventory model under quantity flexibility replenishment with dynamic forecast updates. Because of the computational complexity of the thresholds and the base-stock level with the correlated demand, we first establish the sufficient and necessary condition under which the myopic policy is optimal. In general, we develop upper and lower bounds for the thresholds and the base-stock level. Moreover, the related cost error bounds are derived to estimate the cost performance of heuristic policies. Finally, by numerical experiments, we demonstrate that our bounds result in a good cost performance compared with other literature.

There are a number of directions for further research. First, we may relax the assumption that the demand is independent of the sale price. When the external market is price-sensitive, it is interesting to simultaneously study the buyer's pricing and ordering behaviour under such a quantity flexibility contract. Second, we may consider the impact of supply disruption on the system performance and investigate whether quantity flexibility is valuable to reduce the risk of supply uncertainty.

Appendix

Proof of Proposition 2.1:

First, by the definition of C_{N-1} , we can show C_{N-1} is convex in S_{N-1} . Then by induction, we also show that $n = 1, 2, \dots, N-2$, $A_n(I_n, \zeta_n, \Omega_n, S_n, Z_n)$ is jointly convex in (I_n, ζ_n, S_n, Z_n) and $V_n(I_n, \zeta_n, \Omega_n)$ is jointly convex in (I_n, ζ_n) .

Because of the joint convexity, we can show that S^* is given by (5) and Z_n^* is given by (6) by following the similar arguments discussed by in Porteus (2002, page 115).

Proof of Proposition 3.1: The sufficient condition can be shown easily by induction. To show the necessary condition, we firstly assume that $D_{n,n}$ lie in the interval $[0, S_n^{(m)} - U_{n+1}^{(m)})$ or $(Z_n^{(m)} - U_{n+1}^{(m)}, Z_n^{(m)} - L_{n+1}^{(m)})$, i.e., neither (10) or (11) can be satisfied. Note that $L_{n+1}^{(m)} < U_{n+1}^{(m)}$ and $S_n \leq Z_n$. This implies that $\partial A_n / \partial S_n > 0$. Namely, the myopic policy cannot be optimal, which yields a contradiction.

Proof of Proposition 3.2: By the definition of A_n , we need to analyze the partial derivative for EV_{n+1} , i.e.,

$$\begin{aligned} E \left[\frac{\partial V_{n+1}}{\partial S_n} \right] &= E \left[(I(Y_{n+1}) + I^c(Y_{n+1})) \frac{\partial V_{n+1}}{\partial S_n} \right] \\ &= E \left[I(Y_{n+1}) \frac{\partial V_{n+1}}{\partial S_n} \right] \\ &= E \left[I(Y_{n+1}) C'_{n+1}(S_n - D[n, n+1]) \right] \\ &\quad + E \left[I(Y_{n+1} Y_{n+2}) \frac{\partial V_{n+2}}{\partial S_n} \right]. \end{aligned}$$

Then, we can obtain (14) after expanding $\partial V_{n+2}/\partial S_n$ till $\partial V_{N+2}/\partial S_n$. Similarly, we can also have (15).

Proof of Theorem 1: We intend to show that

$$\alpha_n^{(\ell)}(S_n, \Omega_n) \leq \alpha_n(S_n, \Omega_n) \leq \alpha_n^{(u)}(S_n, \Omega_n). \quad (20)$$

Clearly, for $n = N$, (20) becomes an equality, which indicates $L_n^{(\ell)} = L_n^{(u)} = L_n^{(m)}$. Assume that (20) holds for $n = k+1$, we now show it also holds for $n = k$ by induction. Note that $Y_{n+i}^{(u)} \subseteq Y_{n+i} \subseteq Y_{n+i}^{(\ell)}$ and $Y_{n+i}^{(m)} \subseteq Y_{n+i}$. Therefore, $I(Y_{n+i}^{(u)}) \leq I(Y_{n+i}) \leq I(Y_{n+i}^{(\ell)})$ and $I(Y_{n+i}^{(m)}) \leq I(Y_{n+i})$ for $i \geq 1$.

First, we will show $\alpha_k^{(\ell)}(S_k, \Omega_k) \leq \partial A_k / \partial S_k - C'_k(S_k)$. Since $\partial V_{k+1} / \partial S_k$ is nonnegative, we obtain

$$E[I(Y_{k+1}) \partial V_{k+1} / \partial S_k] \geq E[I(Y_{k+1}^{(u)}) \partial V_{k+1} / \partial S_k].$$

Note that when $Y_{k+1}^{(u)}$ happens, $S_k - D_{k,k}$ is greater than $L_{k+1}^{(u)}$, which implies

$$\begin{aligned} &C'_k(S_k) + E[I(Y_{k+1}^{(u)}) \partial V_{k+1} / \partial S_k] \\ &= C'_k(S_k) + E[I(Y_{k+1}^{(u)}) C'_{k+1}(S_k - D_{k,k})] + E[I(Y_{k+1}^{(u)}) \partial V_{k+2} / \partial S_k] \\ &\geq C'_k(S_k) + E[I(Y_{k+1}^{(u)}) C'_{k+1}(S_k - D[k, k+1])] \\ &\quad + E[I(Y_{k+1}^{(u)} Y_{k+2}^{(u)}) (C'_{k+2}(S_k - D[k, k+2]))] + E[I(Y_{k+1}^{(u)} Y_{k+2}^{(u)}) \partial V_{k+3} / \partial S_k], \end{aligned}$$

where the inequality follows from the nonnegative of $\partial V_{k+2} / \partial S_k$. Because $\partial A_k / \partial S_k = C'_k(S_k) + E[I(Y_{k+1}) \partial V_{k+1} / \partial S_k]$, we obtain $\partial A_k / \partial S_k \geq C'_k(S_k) + \alpha_k^{(u)}(S_k, \Omega_n)$.

Second, we show $\partial A_k / \partial S_k \leq C'_k(S_k) + \alpha_k^{(u)}(S_k, \Omega_n)$. As before, we have

$$\begin{aligned} E[I(Y_{k+1})C'_{k+1}(S_k - D[k, k+1])] &= E[I(Y_{k+1} - I(Y_{k+1}^u))C'_{k+1}(S_k - D[k, k+1])] \\ &\quad + E[I(Y_{k+1}^{(u)})C'_{k+1}(S_k - D[k, k+1])] \\ &\leq E[I(Y_{k+1}^{(u)})C'_{k+1}(S_k - D[k, k+1])]. \end{aligned}$$

The inequality follows by the fact that $C'_{k+1}(S_k - D_k) < 0$ for $L_{k+1} < S_k + Q_{k,k+2} - D_{k,k} < L_{k+1}^{(u)} \leq L_{k+1}^{(m)}$.

Using the same reasoning, we have

$$\begin{aligned} E[I(Y_{k+1}Y_{k+2})C'_{k+2}(S_k - D[k, k+2])] &\leq E[I(Y_{k+1}Y_{k+2}^{(m)})C'_{k+2}(S_k - D[k, k+2])] \\ &\leq E[I(Y_{k+1}^{(\ell)}Y_{k+2}^{(m)})C'_{k+2}(S_k - D[k, k+2])]. \end{aligned}$$

Repeating the same techniques, we can obtain an upper bound for $\partial A / \partial S_k$, i.e.,

$$\partial A / \partial S_k - C'_k(S_k) \leq \alpha_k^{(u)}(S_k, \Omega_k).$$

By following the same idea, we can show that $U_n^{(\ell)} \leq U_n \leq U_n^{(u)}$.

Proof of Theorem 4.1: Based on the structure of the one-period cost function, we have

$$\begin{aligned} V_n(I_n, \zeta_n, \Omega_n) &= C_n(S_n^*, Z_n^*, \Omega_n) + \sum_{i=1}^{N-n+2} EC_{n+i}(S_{n+i}^*, Z_{n+i}^*, \Omega_{n+i}) \\ &\geq C_n(S_n^*, \bar{Z}_n^{(\ell)}, \Omega_n) + \sum_{i=1}^{N-n+2} EC_{n+i}(U_{n+i}^{(u)}, \bar{Z}_{n+i}^{(\ell)}, \Omega_{n+i}) \\ &\geq \min\{C_n(\zeta_n \vee L_n, \bar{Z}_n^{(\ell)}, \Omega_n), C_n(I_n \vee U_n, \bar{Z}_n^{(\ell)}, \Omega_n)\} \\ &\quad + \sum_{i=1}^{N-n+2} EC_{n+i}(U_{n+i}^{(u)}, \bar{Z}_{n+i}^{(\ell)}, \Omega_{n+i}) \\ &\geq \min\{C_n(\zeta_n \vee L_n^{(u)}, \bar{Z}_n^{(\ell)}, \Omega_n), C_n(I_n \vee U_n^{(u)}, \bar{Z}_n^{(\ell)}, \Omega_n)\} \\ &\quad + \sum_{i=1}^{N-n+2} EC_{n+i}(U_{n+i}^{(u)}, \bar{Z}_{n+i}^{(\ell)}, \Omega_{n+i}). \end{aligned}$$

Proof of Lemma 1: Note that

$$\begin{aligned} V_n^{(h)}(I_n, \zeta_n, \Omega_n) &= C_n(S_n^{(h)}, Z_n^{(h)}, \Omega_n) + EV_{n+1}^{(h)}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) \\ V_n(I_n, \zeta_n, \Omega_n) &= C_n(S_n^*, Z_n^*, \Omega_n) + EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}). \end{aligned}$$

For the first items of the above two equations, we have

$$C_n(S_n^{(h)}, Z_n^{(h)}, \Omega_n) - C_n(S_n^*, Z_n^*, \Omega_n) = g_1(n, \Omega_n). \quad (21)$$

For the second items of the above two equations, we have

$$\begin{aligned} & EV_{n+1}^{(h)}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) \\ = & EV_{n+1}^{(h)}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) \\ & + EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}). \end{aligned}$$

To compare the last two terms for the above equation, we have to consider four possible scenarios.

If $L_n^{(h)} \geq L_n$ and $U_n^{(h)} \geq U_n$, we have $S_n^{(h)} \geq S_n^*$. Since $V_n(I_n, \zeta_n, \Omega_n)$ is increasing in I_n and decreasing in ζ_n , we have $V_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) \leq V_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1})$.

$$\begin{aligned} & EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) \\ = & (S_n^{(h)} - S_n^*) E \frac{\partial V_{n+1}(y - D_n, Z_n^{(h)} - D_n, \Omega_{n+1})}{\partial y} \Big|_{S_n^* \leq y \leq S_n^{(h)}} \\ \leq & (S_n^{(h)} - S_n^{(\ell)}) E \frac{\partial V_{n+1}(y - D_n, Z_n^{(h)} - D_n, \Omega_{n+1})}{\partial y} \Big|_{y=S_n^{(h)}} \\ \leq & g_2(n, \Omega_{n+1}), \end{aligned}$$

where the first equality is from the mean-value theorem.

If $L_n^{(h)} < L_n$ and $U_n^{(h)} \geq U_n$, we have

$$\begin{aligned} EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) &= B_1^c(L_n) EV_{n+1}(L_n - D_n, L_n \vee \bar{Z}_n^{(h)} - D_n, \Omega_{n+1}) \\ &\quad + B_1(L_n) EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) \\ EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) &= B_1^c(L_n) EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) \\ &\quad + B_1(L_n) EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}). \end{aligned}$$

Then, we obtain

$$\begin{aligned} & B_1(L_n) [EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1})] \\ \leq & B_1(L_n) [EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^{(h)} - D_n, \Omega_{n+1})] \\ \leq & B_1(L_n) g_2(n, \Omega_{n+1}) \leq B_1(L_n^{(h)}) g_2(n, \Omega_{n+1}), \end{aligned}$$

where the first inequality is true since $EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) \geq EV_{n+1}(S_n^* - D_n, Z_n^{(h)} - D_n, \Omega_{n+1})$.

For the second half, we have

$$\begin{aligned}
& B_1^c(L_n)[V_{n+1}(S_n^{(h)} - D_n, S_n^{(h)} \vee \bar{Z}_n^{(h)} - D_n, \Omega_{n+1}) - V_{n+1}(L_n - D_n, L_n \vee \bar{Z}_n - D_n, \Omega_{n+1})] \\
& \leq B_1^c(L_n)[V_{n+1}(L_n - D_n, S_n^{(h)} \vee \bar{Z}_n^{(h)} - D_n, \Omega_{n+1}) - V_{n+1}(L_n - D_n, L_n \vee \bar{Z}_n^{(h)} - D_n, \Omega_{n+1})] \\
& = B_1^c(L_n)(L_n \vee \bar{Z}_n^{(h)} - S_n^{(h)} \vee \bar{Z}_n^{(h)}) \left[-E \frac{\partial V_{n+1}(L_n - D_n, y - D_n, \Omega_{n+1})}{\partial Z_n} \Big|_{S_n^{(h)} \vee \bar{Z}_n^{(h)} \leq y \leq L_n \vee \bar{Z}_n^{(h)}} \right] \\
& \leq B_1^c(L_n)(L_n^{(u)} \vee \bar{Z}_n^{(h)} - S_n^{(h)} \vee \bar{Z}_n^{(h)}) \left[-E \frac{\partial V_{n+1}(L_n - D_n, y - D_n, \Omega_{n+1})}{\partial Z_n} \Big|_{y=L_n^{(u)} \vee \bar{Z}_n^{(h)}} \right] \\
& \leq B_1^c(L_n^{(u)})g_3(n, L_n^{(u)}, \Omega_{n+1}).
\end{aligned}$$

Combining the above results, for $L_n^{(h)} < L_n$ and $U_n^{(h)} \geq U_n$, we have

$$\begin{aligned}
& EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) \\
& \leq B_1(L_n^{(h)})g_2(n, \Omega_{n+1}) + B_1^c(L_n^{(u)})g_3(n, L_n^{(u)}, \Omega_{n+1}).
\end{aligned}$$

Similarly, for $L_n^{(h)} \geq L_n$ and $U_n^{(h)} < U_n$, we have

$$\begin{aligned}
& EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) \\
& \leq [B_1^c(U_n^{(h)}) + B_2(U_n^{(h)})]g_2(n, \Omega_{n+1}) + B_1(U_n^{(h)})B_2^c(U_n^{(u)})g_3(n, U_n^{(u)}, \Omega_{n+1}).
\end{aligned}$$

Finally, for $L_n^{(h)} < L_n$ and $U_n^{(h)} < U_n$, we have

$$\begin{aligned}
& EV_{n+1}(S_n^{(h)} - D_n, Z_n^{(h)} - D_n, \Omega_{n+1}) - EV_{n+1}(S_n^* - D_n, Z_n^* - D_n, \Omega_{n+1}) \\
& \leq [B_1(U_n^{(h)})B_1^c(L_n^{(h)}) + B_2(U_n^{(h)})]g_2(n, \Omega_{n+1}) + B_1^c(L_n^{(u)})g_3(n, L_n^{(u)}, \Omega_{n+1}) \\
& \quad + B_1(U_n^{(h)})B_2^c(U_n^{(u)})g_3(n, U_n^{(u)}, \Omega_{n+1}).
\end{aligned}$$

Then, for the upper bound, we need to find the largest value from the four cases.

Proof of Lemma 2: By following a similar procedure as that of Lemma 1, we can derive (19).

Proof of Proposition 4.2: By Lemma 1 and 2, the proof is followed by induction.

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