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Brief paper

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ABSTRACT

Recently, concepts of nonlinear eigenvalues and eigenvectors are introduced. In this paper, we establish connections between the nonlinear eigenvalues and nonlinear accessibility/observability. In particular, we provide a generalization of Popov–Belevitch–Hautus (PBH) controllability/observability test to nonlinear accessibility/observability. Our PBH accessibility/observability test is equivalent to the accessibility/observability rank condition under a stranded assumption.

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1. Introduction

Nonlinear accessibility and observability have been studied with several methods such as the rank conditions (Isidori, 1995; Nijmeijer & van der Schaft, 1990) and energy functions (Kawano & Scherpen, 2017; Scherpen & Gray, 2000). However, Popov–Belevitch–Hautus (PBH) tests have never been generalized to nonlinear systems despite their significance in linear systems and control theory. For instance, PBH tests are useful for studying controllability and observability of closed-loop systems, cascade systems, and n -dimensional systems having specific structures (Franceschelli, Gasparri, Giua, & Seatzu, 2013; Kailath, 1980; O’Sullivan-Greene, Mareels, Kuhlmann, & Burkitt, 2014). Another application is studying so called controllability and observability radius (Eising, 1984). Furthermore, PBH tests are still extended even for linear systems such as multidimensional systems and complementarity systems (Camlibel, 2007; Carmi & Gurfil, 2013; Lomadze, 2013; Shankar, 2014).

Recently, concepts of nonlinear eigenvalues and eigenvectors are introduced (Halás & Moog, 2013; Kawano & Ohtsuka, 2015, 2017). These papers partly extended results on stability analysis, the eigenvalue method of the Riccati equation, and Jordan canonical forms, respectively. According to Halás and Moog (2013) and Kawano and Ohtsuka (2015, 2017), nonlinear eigenvectors are one-dimensional f -invariant distribution and codistribution. Although distribution and codistribution are well-known concepts in nonlinear systems and control theory (Isidori, 1995; Nijmeijer & van der Schaft, 1990), it is not focused on that one-dimensional distribution and codistribution play similar roles to eigenvectors. Thus, by comparing linear eigenvalues, applications and properties of nonlinear eigenvalues are not well studied. At present, there is no universal method of finding nonlinear eigenvalues and eigenvectors because they are solutions to a set of specific nonlinear partial differential equations.

In this paper, we develop a nonlinear accessibility/observability condition in terms of nonlinear eigenvalues and eigenvectors. Under the standard assumption that the accessibility distribution has a constant dimension, our accessibility condition is equivalent to the conventional accessibility rank condition. A similar relationship also holds for observability. Without the standard assumptions, our conditions are necessary conditions at least. It is worth mentioning that computations of nonlinear eigenvalues are not always required when using our conditions as well as the conventional linear PBH tests, which will be demonstrated with an example.

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Our primary purpose in this paper is to establish connections between nonlinear eigenvalues and nonlinear accessibility/observability rather than deriving more practically useful conditions than the rank conditions. For the linear systems, our conditions reduce to conventional linear PBH tests. Moreover, proofs of our conditions are natural extensions of those of the linear PBH tests. That is, we utilize the conventional rank conditions and canonical decompositions. From those facts, our conditions can be viewed as generalizations of the PBH accessibility and observability tests to nonlinear systems. Results in this paper and (Halás & Moog, 2013; Kawano & Ohtsuka, 2015, 2017) indicate nonlinear eigenvalues and eigenvectors play similar roles as eigenvalues and eigenvectors in linear algebra.

Our PBH tests help to develop nonlinear systems and control theory. For instance, in Theorem 3.3(5) of Kawano and Ohtsuka (2017), our PBH tests are essential tools for analysis. The paper (Kawano & Ohtsuka, 2017) extends the eigenvalue method of the Riccati equation to the differential Riccati equation in contraction analysis, and Theorem 3.3(5) gives a characterization of regularity of a solution. Owing to our PBH tests, it is possible to conclude that regularity of a solution to the differential Riccati equation depends on accessibility and observability.

The preliminary results in this paper have been reported in Kawano and Ohtsuka (2013), which gives an observability condition based on left eigenvalues and eigenvectors of pseudo-linear transformation (Lam, Leroy, & Ozturk, 2008; Leroy, 1995). In contrast, the PBH observability test in this paper is described by using right eigenvalues and eigenvectors of a matrix, and thus the derived condition is different from those in the preliminary results.

The remainder of this paper is organized as follows. In Section 2, properties of nonlinear eigenvalues and eigenvectors are discussed. In Section 3, the PBH controllability and observability tests are generalized to nonlinear systems, and duality of our PBH tests is discussed. Concluding remarks are given in Section 4.

2. Nonlinear eigenvalues and eigenvectors

2.1. Definitions

Consider a continuous-time nonlinear system described by

$$\Sigma : \begin{cases} \dot{x}(t) := \frac{dx(t)}{dt} = f(x(t)) + \sum_{i=1}^m g_i(x(t))u_i(t), \\ y(t) = h(x(t)), \end{cases}$$

where $x \in \mathcal{X} \subset \mathbb{C}^n$, $u \in \mathbb{C}^m$, and $y \in \mathbb{C}^p$; $\mathcal{X}$ is an open subset; $f, g_i : \mathcal{X} \rightarrow \mathbb{C}^n$ ($i = 1, \dots, m$), and $h : \mathcal{X} \rightarrow \mathbb{C}^p$ are analytic. In this paper, we use concepts of nonlinear eigenvalues and eigenvectors, which can be complex functions of the state variables even if f, g , and h are real functions. For the sake of simplifying notations, we treat the field of complex numbers and complex vector fields. When we consider real functions f, g , and h , they are supposed to be smooth.

We show the definition of nonlinear eigenvalues and eigenvectors (Halás & Moog, 2013; Kawano & Ohtsuka, 2015, 2017). Let $U \subset \mathcal{X}$ be a sufficiently small domain, i.e. a sufficiently small open and connected subset. A considered domain U varies depending on the context. Let $\mathcal{A}_U$ be the set of complex analytic functions $\varphi : U \rightarrow \mathbb{C}$ with variables $x := [x_1, \dots, x_n]^T$. In addition, let $L_f : \mathcal{A}_U \rightarrow \mathcal{A}_U$ be $L_f a := (\partial a / \partial x)f$, which is the Lie derivative of function a along vector field f . Also, for matrix $A = (a_{ij}) \in \mathcal{A}_U^{n \times n}$, denote $L_f A := (L_f a_{ij})$.

Definition 2.1. A nonzero $v \in \mathcal{A}_U^n$ is a left eigenvector for matrix $A \in \mathcal{A}_U^{n \times n}$ along vector field $f \in \mathcal{A}_U^n$ with a left eigenvalue $\alpha \in \mathcal{A}_U$ if

$$v^T A + (L_f v)^T = \alpha v^T. \quad (1)$$

A nonzero $w \in \mathcal{A}_U^n$ is a right eigenvector for matrix $A \in \mathcal{A}_U^{n \times n}$ along vector field $f \in \mathcal{A}_U^n$ with a right eigenvalue $\beta \in \mathcal{A}_U$ if

$$Aw - L_f w = \beta w. \quad (2)$$

Moreover, if $A = \partial f / \partial x$, then the left (right) eigenvalues and eigenvectors of A are also called nonlinear left (right) eigenvalues and eigenvectors of the nonlinear system Σ . In addition, we simply call them nonlinear left (right) eigenvalues and eigenvectors.

From their definitions, nonlinear eigenvalues and eigenvectors are solutions to nonlinear partial differential equations. At present, there is no universal method to find them. This is an interesting problem to tackle but is beyond the scope of this paper. Our primary purpose is to develop nonlinear systems and control theory in terms of nonlinear eigenvalues as partly done in Halás & Moog (2013) and Kawano and Ohtsuka (2015, 2017). In this paper, we establish connections between nonlinear eigenvalues and nonlinear accessibility/observability.

When $A = \partial f / \partial x$, the left and right eigenvectors correspond to a one-dimensional codistribution and distribution, respectively. By using the Lie derivative of differential one-form, definitions (1) can be rewritten as

$$\begin{aligned} L_f(v^T dx) &= (L_f v)^T dx + v^T L_f dx \\ &= (L_f v)^T dx + v^T \frac{\partial f}{\partial x} dx = \alpha v^T dx, \end{aligned}$$

where $dx := [dx_1 \ \dots \ dx_n]^T$, and dx_i is a differential one-form. Also, by using the Lie bracket of vector fields, (2) can be written as

$$[w, f] = \frac{\partial f}{\partial x} w - \frac{\partial w}{\partial x} f = \beta w.$$

That is, $v^T dx$ and w are the one-dimensional invariant codistribution and distribution, respectively.

Next, we give examples of nonlinear eigenvalues.

Example 2.2. For linear system $\dot{x} = Ax$ with $A \in \mathbb{C}^{n \times n}$, definition (1) holds for a linear eigenvalue $\alpha \in \mathbb{C}$ and left eigenvector $v \in \mathbb{C}^n$. Actually, for constant v , definition (1) reduces to

$$v^T A = \alpha v^T.$$

This is just the definition of eigenvalue and left eigenvector in linear algebra. A similar relationship holds for a right eigenvalue and eigenvector. Thus, the linear eigenvalue and eigenvector of a constant matrix are a nonlinear eigenvalue and eigenvector.

Example 2.3. Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + x_1(t)x_2(t), \\ \dot{x}_2(t) &= x_2(t). \end{aligned}$$

First, we compute some of the left eigenvalues and eigenvectors of the system. That is, we find $\alpha \in \mathcal{A}_U$ and $v := [v_1 \ v_2]^T \in \mathcal{A}_U^2$ for some $U \subset \mathbb{C}^2$ satisfying

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} 1 + x_2 & x_1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} L_f v_1 & L_f v_2 \end{bmatrix} = \alpha \begin{bmatrix} v_1 & v_2 \end{bmatrix}.$$

For example,

$$\begin{aligned} \alpha^1 &= 1 + x_2, & v^1 &= \begin{bmatrix} 1 & -x_1 \end{bmatrix}^T, \\ \alpha^2 &= 1, & v^2 &= \begin{bmatrix} 0 & 1 \end{bmatrix}^T \end{aligned}$$

are left eigenvalues and eigenvectors of the system in $U = \mathbb{C}^2$. Next, we compute some of the right eigenvalues and eigenvectors. By solving the following equality with respect to $\beta \in \mathcal{A}_U$ and $w \in \mathcal{A}_U^2$,

$$\begin{bmatrix} 1+x_2 & x_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} L_f w_1 \\ L_f w_2 \end{bmatrix} = \beta \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

we have, in $U = \mathbb{C}^2$,

$$\beta^1 = 1 + x_2, \quad w^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \quad \beta^2 = 1, \\ w^2 = \begin{bmatrix} x_1 & 1 \end{bmatrix}^T.$$

2.2. Properties in relation to L_f conjugacy

It is well known that linear eigenvalues are invariant under linear coordinate transformations. It is shown in Kawano and Ohtsuka (2015) that nonlinear eigenvalues are invariant under coordinate transformation $z = \varphi(x)$ such that $T := \partial\varphi/\partial x$ is regular in U . In other words, $z = \varphi(x)$ is diffeomorphism in U . We denote the system Σ after the coordinate transformation by $\dot{z} = \bar{f}(z) + \bar{g}(z)u$. By using $f(x)$, the Jacobian matrix of $\bar{f}(z)$ can be represented as

$$\frac{\partial \bar{f}}{\partial z} = \left(T \frac{\partial f}{\partial x} + L_f T \right) T^{-1}. \quad (3)$$

To study the relationship between $\partial \bar{f}/\partial z$ and $\partial f/\partial x$, the concept of L_f -conjugacy (Lam et al., 2008; Leroy, 1995) is given.

Definition 2.4. Two matrices $A, B \in \mathcal{A}_U^{n \times n}$ are L_f -conjugate if there exists a nonsingular matrix $T \in \mathcal{A}_U^{n \times n}$ such that $A = (TB + L_f T)T^{-1}$ holds.

Proposition 2.5 (Lam et al., 2008; Leroy, 1995). For every L_f -conjugate matrix $A, B \in \mathcal{A}_U^{n \times n}$, the set of nonlinear eigenvalues of A and B is identical.

This proposition implies that eigenvalues are invariant under coordinate transformations. Moreover, it can be shown that if $\bar{v} \in \mathcal{A}_U^n$ is a left eigenvector associated with a left eigenvalue $\alpha \in \mathcal{A}_U$ in the z -coordinates, $v = T^T \bar{v}$ is a left eigenvector associated with an eigenvalue $\alpha \in \mathcal{A}_U$ in the x -coordinates. Actually, for left eigenvector $\bar{v} \in \mathcal{A}_U^n$, we have

$$\begin{aligned} v^T \frac{\partial f(x)}{\partial x} + (L_f v)^T &= \bar{v}^T T \frac{\partial f(x)}{\partial x} + (L_f (\bar{v}^T T))^T \\ &= \bar{v}^T T \frac{\partial f(x)}{\partial x} + \bar{v}^T L_f T + (L_f \bar{v})^T T. \end{aligned}$$

From (3),

$$\begin{aligned} \bar{v}^T T \frac{\partial f(x)}{\partial x} + \bar{v}^T L_f T + (L_f \bar{v})^T T &= \bar{v}^T \frac{\partial \bar{f}(z)}{\partial z} T + (L_f \bar{v})^T T \\ &= \alpha \bar{v}^T T = \alpha v^T. \end{aligned} \quad (4)$$

A similar relationship holds for right eigenvectors.

2.3. Simplicity

In linear algebra, a matrix whose eigenvectors span the entire space is called simple. Here, we generalize the concept of simplicity because it will be instrumental in the main theorems.

Definition 2.6. A matrix $A \in \mathcal{A}_U^{n \times n}$ is said to be left simple if there exist n left eigenvectors $v_1, \dots, v_n \in \mathcal{A}_U^n$ such that $\text{span}_{\mathcal{A}_U} \{v_1, \dots, v_n\} = \mathcal{A}_U^n$. A system Σ is called left simple if $\partial f/\partial x$ is left simple.

Remark 2.7. In a similar manner, we can define right simplicity for both a matrix and system. It can be shown that matrix $A \in \mathcal{A}_U^{n \times n}$ is right simple if and only if it is left simple. That is, a system is right simple if and only if it is left simple. Hereafter, a matrix or system that is left or right simple is called simple.

Simplicity assumption holds around a non-equilibrium point of system Σ owing to the straightening theorem (Menini & Tornambè, 2011). Although the following proposition does not hold around an equilibrium point, this fact does not affect accessibility/observability analysis under a standard assumption as will be shown later.

Proposition 2.8 (Kawano & Ohtsuka, 2015). Suppose that $f(x_0) \neq 0$ for $x_0 \in \mathcal{X}$. Then, there exists a domain $U_{x_0} \subset \mathcal{X}$ around x_0 such that system Σ is simple.

For simple and block triangular matrices, we have the following results, which will be used.

Proposition 2.9. Let $A \in \mathcal{A}_U^{n \times n}$ be the following block lower or upper triangular matrix:

$$A := \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad \text{or} \quad A := \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathcal{A}_U^{n_1 \times n_1}$, $A_{12} \in \mathcal{A}_U^{n_1 \times n_2}$, $A_{21} \in \mathcal{A}_U^{n_2 \times n_1}$, $A_{22} \in \mathcal{A}_U^{n_2 \times n_2}$, and $n = n_1 + n_2$. If A is simple, both A_{11} and A_{22} are also simple.

Proof. Here, we prove only for a lower triangular matrix. Since A is simple, there exist $\alpha_i \in \mathcal{A}_U$ and $v_i := [v_{i,1}^T \ v_{i,2}^T]^T \in \mathcal{A}_U^n$ ($i = 1, \dots, n$) such that

$$\begin{aligned} [v_{i,1}^T \ v_{i,2}^T] \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \\ + [(L_f v_{i,1})^T \ (L_f v_{i,2})^T] - \alpha_i [v_{i,1}^T \ v_{i,2}^T] = 0 \end{aligned} \quad (5)$$

and $\text{span}_{\mathcal{A}_U} \{v_1, \dots, v_n\} = \mathcal{A}_U^n$ hold. We show that one of the $v_{i,2}$ can be chosen to be 0. Because $\text{span}_{\mathcal{A}_U} \{v_1, \dots, v_n\} = \mathcal{A}_U^n$, we have $\text{span}_{\mathcal{A}_U} \{v_{1,2}, \dots, v_{n,2}\} = \mathcal{A}_U^{n_2}$. Since the vector space $\mathcal{A}_U^{n_2}$ can always be spanned by n_2 elements in a sufficiently small U , $\{v_{n_1+1,2}, \dots, v_{n,2}\}$ can be chosen as a basis of $\mathcal{A}_U^{n_2}$ without loss of generality. For $v_{n_1+1,2} \in \mathcal{A}_U^{n_2} = \text{span}_{\mathcal{A}_U} \{v_{n_1+1,2}, \dots, v_{n,2}\}$, there exist $a_{n_1+1}, \dots, a_n \in \mathcal{A}_U$ such that

$$v_{n_1+1,2} = a_{n_1+1} v_{n_1+1,2} + \dots + a_n v_{n,2}. \quad (6)$$

From (5), we obtain

$$v_{i,2}^T A_{22} + (L_f v_{i,2})^T - \alpha_i v_{i,2}^T = 0 \quad (i = n_1, \dots, n), \quad (7)$$

which yields

$$\begin{aligned} v_{n_1+1,2}^T A_{22} + (L_f v_{n_1+1,2})^T - \alpha_{n_1+1} v_{n_1+1,2}^T \\ - \sum_{i=n_1+1}^n a_i (v_{i,2}^T A_{22} + (L_f v_{i,2})^T - \alpha_i v_{i,2}^T) \\ = \sum_{i=n_1+1}^n \left((L_f (a_i v_{i,2}))^T - \alpha_{n_1+1} a_i v_{i,2}^T - a_i ((L_f v_{i,2})^T - \alpha_i v_{i,2}^T) \right) \\ = \sum_{i=n_1+1}^n (L_f a_i - \alpha_{n_1+1} a_i + \alpha_i a_i) v_{i,2}^T = 0. \end{aligned}$$

Since $\{v_{2,n_1+1}, \dots, v_{2,n}\}$ is a basis of $\mathcal{A}_U^{n_2}$, we have

$$\alpha_i a_i = \alpha_{n_1+1} a_i - L_f a_i \quad (i = n_1+1, \dots, n). \quad (8)$$

On the other hand, (5) yields

$$v_{i,1}^T A_{11} + v_{i,2}^T A_{21} + (L_f v_{i,1})^T - \alpha_i v_{i,1}^T = 0 \quad (i = n_1, \dots, n).$$

From (6), we have

$$\begin{aligned} & v_{n_1,1}^T A_{11} + v_{n_1,2}^T A_{21} + (L_f v_{n_1,1})^T - \alpha_{n_1} v_{n_1,1}^T \\ & - \sum_{i=n_1+1}^n a_i (v_{i,1}^T A_{11} + v_{i,2}^T A_{21} + (L_f v_{i,1})^T - \alpha_i v_{i,1}^T) \\ & = v_{n_1,1}^T A_{11} + (L_f v_{n_1,1})^T - \alpha_{n_1} v_{n_1,1}^T \\ & - \sum_{i=n_1+1}^n a_i (v_{i,1}^T A_{11} + (L_f v_{i,1})^T - \alpha_i v_{i,1}^T) = 0 \end{aligned}$$

which can be rearranged by using (8) as

$$\begin{aligned} & \hat{v}_{n_1,1}^T A_{11} + (L_f \hat{v}_{n_1,1})^T - \alpha_{n_1} \hat{v}_{n_1,1}^T = 0, \\ & \hat{v}_{n_1,1} := v_{n_1,1} - \sum_{i=n_1+1}^n a_i v_{i,1}. \end{aligned} \quad (9)$$

Note that $\hat{v}_{n_1,1} \neq 0$. Otherwise, from (6), $v_{n_1}, \dots, v_n$ are linearly dependent, which contradicts the fact that $\text{span}_{\mathcal{A}_U}\{v_1, \dots, v_n\} = \mathcal{A}_U^n$. Therefore,

$$\text{span}_{\mathcal{A}_U}\{v_1, \dots, v_{n_1-1}, \hat{v}_{n_1,1}, v_{n_1+1}, \dots, v_n\} = \mathcal{A}_U^n,$$

where $\hat{v}_{n_1,1} = [v_{n_1,1}^T - \sum_{i=n_1+1}^n a_i v_{i,1}^T \ 0^T]^T \in \mathcal{A}_U^n$. Moreover, (9) implies that a_{n_1} and $\hat{v}_{n_1,1}$ are a left eigenvalue and eigenvector of A_{11} , respectively. By repeating the above procedure for v_i ($i = 1, \dots, n_1 - 1$), we have $\hat{v}_i = [\hat{v}_{i,1}^T \ 0^T]^T$ ($i = 1, \dots, n_1 - 1$), where every $\hat{v}_{i,1} \in \mathcal{A}_U^{n_1}$ is a left eigenvector of A_{11} . To prove the simplicity of A_{11} , it remains to show that $\text{span}_{\mathcal{A}_U}\{\hat{v}_{1,1}, \dots, \hat{v}_{n_1,1}\} = \mathcal{A}_U^{n_1}$. Suppose that $\text{span}\{\hat{v}_{1,1}, \dots, \hat{v}_{n_1,1}\} \neq \mathcal{A}_U^{n_1}$, i.e., $\text{span}\{\hat{v}_{1,1}, \dots, \hat{v}_{n_1,1}\} \subsetneq \mathcal{A}_U^{n_1}$. Because $\hat{v}_{i,2} = 0$ ($i = 1, \dots, n_1$), we have $\text{span}_{\mathcal{A}_U}\{\hat{v}_{1,1}, \dots, \hat{v}_{n_1,1}, v_{n_1+1}, \dots, v_{n_1+n_2}\} \subsetneq \mathcal{A}_U^n$ ($n_1 + n_2 = n$), which contradicts $\text{span}_{\mathcal{A}_U}\{v_1, \dots, v_n\} = \mathcal{A}_U^n$.

Finally, (7) implies that $v_{i,2}$ ($i = n_1 + 1, \dots, n$) is a left eigenvector of A_{22} . Also, $\{v_{n_1+1,2}, \dots, v_n\}$ is a basis of $\mathcal{A}_U^{n_2}$, i.e., $\text{span}_{\mathcal{A}_U}\{v_{n_1+1,2}, \dots, v_n\} = \mathcal{A}_U^{n_2}$ holds. Therefore, A_{22} is simple. $\square$

3. PBH tests for nonlinear systems

In this section, we generalize the PBH tests to establish connections between nonlinear eigenvalues and nonlinear accessibility/observability. In particular, we analyze relationship between our PBH tests and the conventional rank conditions (Isidori, 1995; Nijmeijer & van der Schaft, 1990).

3.1. Accessibility and observability rank conditions

Here, we summarize the conventional rank conditions for local strong accessibility and local observability. For definitions of accessibility and observability, see Isidori (1995) and Nijmeijer and van der Schaft (1990).

Denote $ad_f^0 g := g$, $ad_f^1 g = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$, and $ad_f^i g := ad_f(ad_f^{i-1} g)$, $i \geq 2$. The strong accessibility algebra $\mathcal{C}$ is the set of all linear combinations (over $\mathbb{C}$) of $ad_f^i g$ ($i = 1, 2, \dots$). Define the strong accessibility distribution $C(x) := \text{span}\{X(x) : X \in \mathcal{C}\}$.

Proposition 3.1 (Accessibility Rank Condition Isidori, 1995; Nijmeijer & van der Schaft, 1990). Suppose that $C(x)$ has a constant dimension in some U . A system Σ is locally strongly accessible at $x_0 \in U$ if and only if $\dim C(x_0) = n$.

Remark 3.2. If a system Σ is locally strongly accessible at all points in U , we simply say that Σ is locally strongly accessible in U . If $C(x)$ has a constant dimension in U , local strong accessibility at a point $x_0 \in U$ implies that in U . That is, $\dim C(x_0) = n$ is a necessary and sufficient condition for local strong accessibility in U .

Denote $L_f^0 h := h$, $L_f^1 h := \frac{\partial h}{\partial x} f$, and $L_f^i h := \frac{\partial L_f^{i-1} h}{\partial x} f$, $i \geq 2$. The observation space $\mathcal{O}$ is the linear space (over $\mathbb{C}$) of $L_f^i h$ ($i = 1, 2, \dots$). Define the observability codistribution $dO(x) := \text{span}\{dH(x) : H \in \mathcal{O}\}$.

Proposition 3.3 (Observability Rank Condition Isidori, 1995; Nijmeijer & van der Schaft, 1990). Suppose that $dO(x)$ has a constant dimension in some U . A system Σ is locally observable at $x_0 \in U$ if and only if $\dim dO(x_0) = n$.

3.2. PBH accessibility tests

Here, we generalize a PBH controllability test to nonlinear accessibility in terms of nonlinear left eigenvalues and eigenvectors.

Theorem 3.4 (PBH Accessibility Test). The following condition (a) implies condition (b).

- (a) The system Σ satisfies the accessibility rank condition in U .
- (b) There is no left eigenvector $v \in \mathcal{A}_U^n$ of Σ satisfying both of the following (10) and (11).

$$v^T g_i = 0, \quad i = 1, \dots, m, \quad (10)$$

$$v^T \frac{\partial g_i}{\partial z} + (L_{g_i} v)^T = 0, \quad i = 1, \dots, m. \quad (11)$$

The converse is also true if the system is simple, and if accessibility distribution $C(x)$ has a constant dimension in U .

Remark 3.5. If system Σ satisfies condition (b), we say that the system satisfies the PBH accessibility test.

Proof. ((a) $\Rightarrow$ (b)) We prove by contraposition. Due to space limitations, we prove only in the single-input case. We show that (1) and (10) imply

$$v^T adf^j g = 0 \quad (12)$$

for all $j = 0, 1, \dots$ by induction. When $j = 0$, (12) is nothing but (10).

Suppose that (12) holds for $j = k$. By multiplying (1) when $A = \partial f / \partial x$ by $adf^k g$, we have, from (12) for $j = k$,

$$\begin{aligned} & \left(v^T \frac{\partial f}{\partial x} + (L_f v)^T - \alpha v^T \right) adf^k g \\ & = \left(v^T \frac{\partial f}{\partial x} + (L_f v)^T \right) adf^k g = 0. \end{aligned} \quad (13)$$

On the other hand, by premultiplying (12) by L_f for $i = k$, we have

$$(L_f (v^T adf^k g)^T)^T = (L_f v)^T adf^k g + v^T L_f (adf^k g) = 0. \quad (14)$$

By subtracting the left-hand side of (13) from that of (14), we obtain

$$\begin{aligned} & (L_f v)^T adf^k g + v^T L_f (adf^k g) - \left(v^T \frac{\partial f}{\partial x} + (L_f v)^T \right) adf^k g \\ & = v^T L_f (adf^k g) - v^T \frac{\partial f}{\partial x} adf^k g = v^T adf^{k+1} g = 0. \end{aligned}$$

Therefore, (12) holds, which implies that the system does not satisfy the rank condition.

To extend this proof to the multiple-input case, we use (11). In a similar manner with (12), we have $[g_i, adf^k g_j]$ ($i, j = 1, \dots, m$; $k = 1, 2, \dots$) and so on.

((b) $\Rightarrow$ (a)) We prove for a multiple-input system by contraposition. If the system Σ does not satisfy the rank condition then according to (Isidori, 1995; Nijmeijer & van der Schaft, 1990) the

system can be decomposed by a coordinate transformation $z = \varphi(x)$ as

$$\begin{aligned}\dot{\bar{z}}_1 &= \bar{f}_1(\bar{z}_1), \\ \dot{\bar{z}}_2 &= \bar{f}_2(\bar{z}_1, \bar{z}_2) + \sum_{i=1}^m \bar{g}_{i,2}(\bar{z}_1, \bar{z}_2)u_i,\end{aligned}\quad (15)$$

where $\bar{z}_1 := [z_1, \dots, z_r]^T$ and $\bar{z}_2 := [z_{r+1}, \dots, z_n]^T$ are the state variables of non-accessible and accessible subsystems, respectively. Since the system Σ is simple, and since simplicity does not depend on the coordinates, the following Jacobian matrix is simple:

$$\begin{bmatrix} \frac{\partial \bar{f}_1}{\partial \bar{z}_1} & 0 \\ \frac{\partial \bar{f}_2}{\partial \bar{z}_1} & \frac{\partial \bar{f}_2}{\partial \bar{z}_2} \end{bmatrix}. \quad (16)$$

Proposition 2.9 implies that $\partial \bar{f}_1 / \partial \bar{z}_1$ is simple. That is, there exist a left eigenvalue $\alpha \in \mathcal{A}_U$ and eigenvector $\bar{v}_1 \in \mathcal{A}_U^r$ of $\partial \bar{f}_1 / \partial \bar{z}_1$. This α is also a left eigenvalue of matrix (16), where a corresponding left eigenvector is $\bar{v} = [\bar{v}_1^T \ 0]^T \in \mathcal{A}_U^n$. For this $\bar{v}$, we show (10) and (11). First, (10) is clear because $\bar{g}_i := [0 \ \bar{g}_{i,2}^T]^T \in \mathcal{A}_U^n$. Next,

$$\begin{aligned}\bar{v}^T \frac{\partial \bar{g}_i}{\partial z} + (L_{\bar{g}_i} \bar{v})^T \\ = [\bar{v}_1^T \ 0] \begin{bmatrix} 0 & 0 \\ \frac{\partial \bar{g}_{i,2}}{\partial \bar{z}_1} & \frac{\partial \bar{g}_{i,2}}{\partial \bar{z}_2} \end{bmatrix} + \left(\begin{bmatrix} \frac{\partial \bar{v}_1}{\partial \bar{z}_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \bar{g}_{i,2} \end{bmatrix} \right)^T = 0,\end{aligned}$$

i.e., (11) holds in the z -coordinates.

Note that it remains to prove the existence of a left eigenvector $v \in \mathcal{A}_U^n$ satisfying (10) and (11) in the x -coordinate. We can see that, from (4), such a v is $T^T \bar{v}$, where $T := \partial \varphi / \partial x$. Since $g_i = T^{-1} \bar{g}_i$ holds, we obtain $v^T g_i = (\bar{v}^T T)(T^{-1} \bar{g}_i) = \bar{v}^T \bar{g}_i = 0$, i.e., (10) holds in the x -coordinate. Also, (11) follows from (4). $\square$

For the linear system, condition (b) in **Theorem 3.4** reduces to a linear PBH controllability test. Moreover, the proof in **Theorem 3.4** is a natural extension of that for the linear PBH controllability test. Therefore, we call condition (b) a PBH accessibility test.

Like the linear PBH test, our PBH test, i.e., conditions (10) and (11) can readily be checked for obtained nonlinear left eigenvectors although there still remains a challenging problem of computation of nonlinear eigenvalues and eigenvectors. In the linear case, the advantage of the PBH tests in comparison to the rank conditions is that it is possible to study properties of general n -dimensional systems. This fact indicates that our PBH test can be useful for analysis of general properties of nonlinear systems. Actually, **Example 3.9** demonstrates our PBH tests for accessibility analysis of closed-loop systems. Also, by combining our PBH tests and the results in **Kawano and Ohtsuka (2017)**, we can conclude that regularity of solutions to the differential Riccati equation in contraction analysis depends on nonlinear accessibility and observability.

In the first part of the proof in the single-input case, we do not use condition (11). Therefore, we can omit (11) from **Theorem 3.4**. However, in the multiple-input case, (11) is essential as shown in **Example 3.10**.

Corollary 3.6. *If a single-input system Σ satisfies the accessibility rank condition in U , there does not exist a left eigenvector $v \in \mathcal{A}_U^n$ of Σ such that (10) holds. The converse is also true if the system is simple, and if $C(x)$ has a constant dimension in U .*

In the theorem, to show the converse, we assume simplicity and that the strong accessibility distribution $C(x)$ has a constant dimension in U . This assumption for the distribution is standard (**Isidori, 1995; Nijmeijer & van der Schaft, 1990**). Without this standard assumption, even the accessibility rank condition is only a sufficient condition for local strong accessibility (**Isidori, 1995; Nijmeijer & van der Schaft, 1990**), i.e., there is a gap between local strong accessibility and the rank condition. Therefore, it is reasonable to proceed with accessibility and observability analysis under the standard assumptions for the distribution and codistribution. For instance, under these standard assumptions, the paper (**Scherpen & Gray, 2000**) clarified that the existences of kinds of positive definite nonlinear controllability and observability Gramians are sufficient conditions for local strong accessibility and local observability.

According to **Proposition 2.8**, for any non-equilibrium point x_0 , there exists a domain U_{x_0} around x_0 such that system Σ is simple. Therefore, the simplicity assumption is not so restrictive at least when we study nonlinear accessibility/observability under the standard assumption for the distribution/codistribution as will be shown in the next subsection.

In **Theorem 3.4**, simplicity is assumed to guarantee that the non-accessible subsystem has at least an eigenvalue. It is an interesting question whether every nonlinear system has at least one nonlinear eigenvalue. **Proposition 2.8** implies that this is true around a non-equilibrium point. In contrast, this is an open problem around an equilibrium point. On the analogy of linear eigenvalues, it is expected that this is also true. However, this question is not our interest in this paper because our aim is to study relationship between our PBH test and the rank condition.

3.3. Relationship between PBH accessibility test and accessibility rank condition

According to **Theorem 3.4**, nonlinear accessibility is characterized by the left eigenvalues and eigenvectors. Under the standard assumption for the distribution, the PBH accessibility test is equivalent to the accessibility rank condition as in the linear case.

Theorem 3.7. *Consider a domain U such that accessibility distribution $C(x)$ has a constant dimension. Then the following three properties are equivalent.*

- (a) System Σ is locally strongly accessible in U .
- (b) System Σ satisfies the accessibility rank condition in U .
- (c) System Σ satisfies the PBH accessibility test in some domain $V \subset U$.

Proof. From the accessibility rank condition, (a) is equivalent to (b). Here, we prove that (b) holds if and only if (c) holds. First, in the case $f \equiv 0$ in U , the system is simple in U . According to **Theorem 3.4**, these two conditions are equivalent for $V = U$.

Next, we consider the case $f \not\equiv 0$ in U . Then, there exists $x_0 \in U$ such that $f(x_0) \neq 0$. From **Proposition 2.8**, there is a domain $U_{x_0} \subset U$ around x_0 such that the system is simple. From **Remark 3.2**, the accessibility rank condition holds in U if and only if it holds at $x_0 \in U$, or equivalently, if and only if it holds in U_{x_0} . Note that $C(x)$ has a constant dimension in $U_{x_0} \subset U$. Finally, according to **Theorem 3.4**, the accessibility rank condition holds in U_{x_0} if and only if the PBH accessibility test holds in U_{x_0} . $\square$

Remark 3.8. **Theorem 3.7** holds even if U contains an equilibrium point.

Our PBH accessibility test is based on nonlinear eigenvalues and eigenvectors, i.e. solutions to a set of nonlinear partial differential equation. However, computations of them are not always required when we use our PBH test as shown in the following example. This example demonstrates that the PBH test for nonlinear systems is potentially useful for testing accessibility of cascade systems, controlled systems, and n -dimensional systems having specific structures as for the linear PBH controllability test.

Example 3.9. We study the relationship between accessibility of the system Σ and the following closed-loop system:

$$\begin{aligned}\hat{\Sigma} : \dot{x} &= f(x) + g(x)(k(x) + \hat{u}) \\ &= (f(x) + g(x)k(x)) + g(x)\hat{u},\end{aligned}$$

where $\hat{u} \in \mathbb{C}$ and $k \in \mathcal{A}_U$. Suppose that accessibility distribution $C(x)$ has a constant dimension in U . If Σ is not accessible then from Theorem 3.7, there exists a left eigenvector $v \in \mathcal{A}_V^n$ of Σ such that (10) and (11) hold in some $V \subset U$. By multiplying (10) by $\partial k / \partial x$, we have

$$v^T g \frac{\partial k}{\partial x} = 0. \quad (17)$$

Also, by multiplying (11) by k ,

$$v^T \frac{\partial g}{\partial x} k + \left(\frac{\partial v}{\partial x} g \right)^T k = 0. \quad (18)$$

By adding (1) with $A = \partial f / \partial x$, (17), and (18), we obtain

$$v^T \frac{\partial f}{\partial x} + gk + \left(\frac{\partial v}{\partial x} (f + gk) \right)^T = \alpha v^T. \quad (19)$$

Since (10), (11), and (19) hold, Theorem 3.7 implies that $\hat{\Sigma}$ is not accessible for any k . Conversely, it can be shown that Σ is not accessible if $\hat{\Sigma}$ is not accessible. Therefore, Σ is accessible if and only if $\hat{\Sigma}$ is accessible. This example implies that a left eigenvalue whose left eigenvector satisfies (10) and (11) cannot be shifted by the feedback control, which is a generalization of the linear case.

In the above example, we showed that non-accessibility of Σ implies that of $\hat{\Sigma}$, and vice versa. In the linear case, the proof is similar. This proof process follows from the fact that both PBH linear controllability and nonlinear accessibility tests characterize uncontrollable modes rather than controllable modes. Therefore, our PBH accessibility test can be viewed as a non-accessibility condition rather than accessibility.

In the linear case, condition (11) does not appear because it trivially holds. However, for nonlinear multiple-input cases, condition (11) is important.

Example 3.10. We test nonlinear accessibility of the following unicycle (Conte, Moog, & Perdon, 2007)

$$\dot{x} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2.$$

For this system, every nonzero $v \in \mathcal{A}_U^3$ is a nonlinear left eigenvector because the drift term of the unicycle is $f = 0$. We show that a nonzero v satisfying conditions (10) and (11) does not exist. First, the nonlinear eigenvector v satisfying condition (10), i.e.,

$$v^T \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} = 0 \quad \text{and} \quad v^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

is only

$$v = a(x) \begin{bmatrix} -\sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix},$$

where $a(x)$ is a nonzero scalar-valued function. For this v and g_1 , condition (11) does not hold because the left hand side of (11) is

$$\begin{aligned} & \begin{bmatrix} 0 & 0 & a(x) \end{bmatrix} + \left(\frac{\partial a(x)}{\partial x_1} \cos x_3 + \frac{\partial a(x)}{\partial x_2} \sin x_3 \right) \\ & \times \begin{bmatrix} -\sin x_3 & \cos x_3 & 0 \end{bmatrix}. \end{aligned}$$

This cannot be zero for any nonzero $a(x)$. From Theorem 3.7, the unicycle is locally strongly accessible on a domain such that the accessibility distribution has a constant dimension.

3.4. PBH observability test

In a similar manner with accessibility, we obtain a PBH observability test for nonlinear systems.

Theorem 3.11 (PBH Observability Test). Let Σ be an autonomous system. The following condition (a) implies condition (b).

- (a) System Σ satisfies observability rank condition in U .
- (b) There is no right eigenvector $w \in \mathcal{A}_U^n$ of Σ such that

$$\frac{\partial h_i}{\partial x} w = 0, \quad i = 1, \dots, p. \quad (20)$$

The converse is also true if the system is simple, and if observability codistribution $dO(x)$ has a constant dimension in U .

Remark 3.12. If system Σ satisfies condition (b), we say that the system satisfies the PBH observability test.

Proof. ((a) $\Rightarrow$ (b)) We prove by contraposition. We show that (2) and (20) imply

$$\frac{\partial L_f^j h_i}{\partial x} w = 0, \quad i = 1, \dots, p \quad (21)$$

for all $j = 0, 1, \dots$ by induction.

Suppose that (21) holds for $j = k$. By multiplying $\partial L_f^k h_i / \partial x$ by (2) when $A = \partial f / \partial x$, we have

$$\frac{\partial L_f^k h_i}{\partial x} \left(\frac{\partial f}{\partial x} w - L_f w - \beta w \right) = \frac{\partial L_f^k h_i}{\partial x} \left(\frac{\partial f}{\partial x} w - L_f w \right) = 0. \quad (22)$$

On the other hand, by premultiplying L_f by (21) for $j = k$, we have

$$L_f \left(\frac{\partial L_f^k h_i}{\partial x} w \right) = L_f \left(\frac{\partial L_f^k h_i}{\partial x} \right) w + \frac{\partial L_f^k h_i}{\partial x} L_f w = 0. \quad (23)$$

By adding the left-hand side of (22) to the left-hand side of (23), we obtain

$$\begin{aligned} & \frac{\partial L_f^k h_i}{\partial x} \left(\frac{\partial f}{\partial x} w - L_f w \right) + L_f \left(\frac{\partial L_f^k h_i}{\partial x} \right) w + \frac{\partial L_f^k h_i}{\partial x} L_f w \\ & = \frac{\partial L_f^k h_i}{\partial x} \frac{\partial f}{\partial x} w + L_f \left(\frac{\partial L_f^k h_i}{\partial x} \right) w + \frac{\partial L_f^{k+1} h_i}{\partial x} w = 0. \end{aligned}$$

Therefore, (21) holds. That is, the observability rank condition does not hold.

((b) $\Rightarrow$ (a)) We prove by contraposition. If the system Σ is not observable then according to Isidori (1995) and Nijmeijer and van der Schaft (1990) it can be decomposed by a coordinate

transformation $z = \varphi(x)$ as

$$\begin{aligned}\dot{\bar{z}}_1 &= \bar{f}_1(\bar{z}_1), \\ \dot{\bar{z}}_2 &= \bar{f}_2(\bar{z}_1, \bar{z}_2), \\ y &= \bar{h}(\bar{z}_1),\end{aligned}\quad (24)$$

where $\bar{z}_1 := [z_1, \dots, z_r]^T$ and $\bar{z}_2 := [z_{r+1}, \dots, z_n]^T$ are the state variables of observable and unobservable subsystems, respectively. Since the system Σ is simple, and since simplicity does not depend on the coordinates, the Jacobian matrix of (24) is simple. Proposition 2.9 implies that $\partial \bar{f}_2 / \partial \bar{z}_2$ is simple. That is, there exist a right eigenvalue $\beta \in \mathcal{A}_U$ and eigenvector $\bar{w}_2 \in \mathcal{A}_U^{n-r}$ of $\partial \bar{f}_2 / \partial \bar{z}_2$. This β is also a right eigenvalue of matrix (16), where a corresponding right eigenvector is $\bar{w} = [0 \ \bar{w}_2^T]^T \in \mathcal{A}_U^n$. For this $\bar{w}$, we have $(\partial \bar{h}_i / \partial \bar{z}) \bar{w} = 0$ ($i = 1, \dots, p$), where $\partial \bar{h}_i / \partial \bar{z} := [\partial \bar{h}_i / \partial \bar{z}_1 \ 0] \in \mathcal{A}_U^n$, i.e., (20) holds in the z -coordinates. In a similar manner with the proof of Theorem 3.4 for accessibility, it can be shown that (20) also holds in the x -coordinates. $\square$

Theorem 3.13. Consider a domain U such that observability codistribution $dO(x)$ has a constant dimension. Then the following three properties are equivalent for autonomous system Σ .

- (a) System Σ is locally observable in U .
- (b) System Σ satisfies the observability rank condition in U .
- (c) System Σ satisfies the PBH observability test in some domain $V \subset U$.

Example 3.14 (Continuation of Example 2.3). Here, we apply our PBH tests to actuator and sensor location problems of the system in Example 2.3. If we use the accessibility and observability rank conditions, we solve these problems by trial and error. For the actuator location problem, first we choose an arbitrary input vector field g . Then, we verify the rank condition. If the rank condition fails, we choose an arbitrary input vector field g again. We repeat this procedure. Note that it is not easy to get an observation for a new choice of g if the rank condition fails because we compute compositions of Lie brackets, which are partial derivatives of vector fields, when verifying the accessibility rank condition. For the sensor location problem, we compute compositions of Lie derivatives instead. If we use our PBH tests, we can easily find g and h such that a system is locally strongly accessible and locally observable by using nonlinear eigenvectors as demonstrated in this example.

First, we consider finding g such that the system is locally strongly accessible. From the state space equation, it is clear that the system is not locally strongly accessible if $g = [1 \ 0]^T$ because the input does not affect x_2 . In fact, for this g , Theorem 3.4 does not hold, i.e.,

$$(v^2)^T g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

If we choose $g = [x_1 \ 1]^T$ then again the system is not locally strongly accessible because $(v^1)^T g = 0$. Note that it is not obvious from the state space equation that the following system with $g = [x_1 \ 1]^T$ is not locally strongly accessible.

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + x_1(t)x_2(t) + x_1(t)u(t), \\ \dot{x}_2(t) &= x_2(t) + u(t).\end{aligned}$$

However, Theorem 3.4 implies that this system is not locally strongly accessible. On the other hand, for $g = [0 \ 1]^T$, we have $(v^1)^T g \neq 0$ and $(v^2)^T g \neq 0$. Then, the system is locally strongly accessible in a domain such that accessibility distribution $C(x)$ has a constant dimension.

Next, we consider the sensor location problem. The following system does not satisfy condition (20) for w^1 or w^2 .

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + x_1(t)x_2(t), \\ \dot{x}_2(t) &= x_2(t), \\ y(t) &= x_1(t) + x_2(t).\end{aligned}$$

Then, the system is locally observable in a domain such that observability codistribution $dO(x)$ has a constant dimension.

3.5. Duality

Next, we consider duality between our PBH accessibility and observability tests. In similar manners to Theorems 3.4 and 3.11, we have the following.

Theorem 3.15. If system Σ satisfies accessibility rank condition in U , there does not exist a right eigenvector $w \in \mathcal{A}_U^n$ of the matrix $-(\partial f / \partial x)^T \in \mathcal{A}_U^{n \times n}$ such that

$$\begin{aligned}w^T g_i &= 0, \quad i = 1, \dots, m, \\ -\left(\frac{\partial g_i}{\partial x}\right)^T w - L_{g_i} w &= 0, \quad i = 1, \dots, m.\end{aligned}$$

The converse is also true if $-(\partial f / \partial x)^T$ is simple, and if accessibility distribution $C(x)$ has a constant dimension in U .

Theorem 3.16. If the autonomous system Σ satisfies observability rank condition in U , there does not exist a left eigenvector $v \in \mathcal{A}_U^n$ of the matrix $-(\partial f / \partial x)^T \in \mathcal{A}_U^{n \times n}$ such that

$$\frac{\partial h_i}{\partial x} v = 0, \quad i = 1, \dots, p.$$

The converse is also true if $-(\partial f / \partial x)^T$ is simple, and if observability codistribution $dO(x)$ has a constant dimension in U .

Let $\alpha \in \mathcal{A}_U$ and $v \in \mathcal{A}_U^n$ be a left eigenvalue and eigenvector of $-(\partial f / \partial x)^T$, respectively. From Definition 2.1, $-\alpha$ and v are a right eigenvalue and eigenvector of $\partial f / \partial x$, respectively. A similar relationship also holds between a right eigenvalue (and eigenvector) of $-(\partial f / \partial x)^T$ and a left eigenvalue (and eigenvector) of $\partial f / \partial x$. On the basis of these facts, Theorems 3.15 and 3.16 can be proved, and the conditions in Theorems 3.15 and 3.16 can be viewed as dual conditions of nonlinear PBH accessibility and observability tests.

4. Conclusions

In this paper, we generalized the PBH accessibility and observability tests in terms of nonlinear eigenvalues and eigenvectors. Accessibility and observability are characterized in terms of left and right eigenvectors, respectively. These conditions are equivalent to the conventional rank conditions under the standard assumptions. Finally, we also derived accessibility and observability conditions in terms of right and left eigenvectors, respectively, that can be viewed as dual conditions of nonlinear PBH tests.

This paper is the first step of nonlinear systems analysis in terms of nonlinear eigenvalues. Future work includes developing analysis methods. For instance, relating to this paper, extending the rank types of PBH tests is future work. On the other hand, since there are various applications of the linear eigenvalues, if a method for finding nonlinear eigenvalues and eigenvectors is established, we can expect various applications of the nonlinear eigenvalues. In linear algebra, the characteristic polynomial plays a key role in finding eigenvalues, since the eigenvalues are zeros of the characteristic polynomial. In Lam et al. (2008), a non-commutative polynomial corresponding to a companion matrix

has been defined, which may play a similar role to a characteristic polynomial. However, the relationship between this polynomial and nonlinear eigenvalues has not been studied yet. Future work includes studying the relationship.

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