CHAPTER VI

A SURVEY OF FURTHER DEVELOPMENTS

Linear systems consisting of \( \infty^3 \) and \( \infty^4 \) two-dimensional contravariant aggregates in \( S_3 \) have also been investigated by means of the theory developed in Chapters I, II and III. A brief survey of these investigations follows.

\[ \text{§ 1. The systems of aggregates } ]3|2|_0 \cap S_3. \]

The series \( ]3|2|_0 \cap S_3 \) contains \( \infty^4 \) simply degenerated aggregates \( ]2| \). Then the series \( ]2|3|_0 \cap S_3 \) resting on the former series contains \( \infty^4 \) simply degenerated \( ]3| \). The vertices of these stars \( ]3| \) form the manifold \( ]2|3|_0 \cap S_3 \) which is a curve of order 6 and genus 3, \( \mathcal{C}^1 \).

The vertices of the degenerated \( ]2| \) of \( ]3|2|_0 \cap S_3 \) are trisecants of this curve and generate a surface of order 8 containing \( \mathcal{C}^1 \) counted three times.

The series \( ]2|3|_0 \cap S_3 \) possesses \( \infty^3 \) subseries \( ]1|3|_0 \cap S_3 \). A series \( ]1|3|_0 \cap S_3 \) rests on a manifold \( ]3|1|_0 \cap S_3 \) which gives a tetrahedral complex of lines. Such a complex contains as singularities four sheaves of lines and four planes of lines, the carriers of which are respectively the vertices and sides of a tetrahedron, inscribed in \( \mathcal{C}^1 \). The surface of trisecants of \( \mathcal{C}^1 \) turns out to be the intersection of \( \infty^3 \) tetrahedral complexes of lines. The planes of the \( \infty^4 \) simply degenerated \( ]2| \) of \( ]2|3|_0 \cap S_3 \) envelop a surface of class 8. The planes containing two trisecants are the double planes of this prime-surface. Every point of \( \mathcal{C}^1 \) belongs to 18 such planes. The above-mentioned prime-surface has a pencil of class 6 in common with every \( ]3| \) of \( ]2|3|_0 \cap S_3 \). It turns out that this pencil is projectively equivalent to the curve \( ]2|3|_0 \cap S_3 = \mathcal{C}^1 \). The birational relation between the manifolds \( ]2|3|_0 \cap S_3 \) and \( ]3|3|_0 \cap S_3 \) represents \( \mathcal{C}^1 \) on a plane curve \( \mathcal{C}^1 \). We obtain a clear insight into this relation if we consider it as a part of the representation of the Bordiga surface \( ]2|3|_0 \cap S_3 = \mathbb{F}_2 \) by a plane.
Intersections of $F_3^2$ by primes give a set of $\infty^4$ curves $3C_1^4$ in $S_2$ all passing through 10 base points.

Then the remaining intersection of two curves $3C_1^4$ consisting of 6 points represents an intersection of $3C_1^4$ by a plane. The image curve $3C_1^4$ of $3C_1^4$ is intersected in four points by a line; these points are the images of the four vertices of the tetrahedron of singularities belonging to a subseries $[1][3]_0 \cap S_3$ of $[2][3]_0 \cap S_3$.

Moreover, one side of such a tetrahedron also contains a trisecant; thus by associating this trisecant with the fourth vertex of the tetrahedron we get a one-one correspondence between the points of $3C_1^4$ and the trisecants of $3C_1^4$. The birational relation between $[3][2]_0 \cap S_3$ and $[3][2]_0 \cap S_3^*$ affords an image of $S_3$ on $S_3^*$ whereby the points of $3C_1^4$ in $S_3$ are represented on the trisecants of $3C_1^4$ in $S_3^*$ and vice versa. A surface of order $n$ in $S_3$ is represented by a surface of order $3n$ in $S_3^*$.

By means of a representation of $[3][2]_0 \cap S_3$ in the matrix-space $S_{11}$ we see that $3C_1^4$ in $S_3$ is the projection of a curve of order 10 in $S_{11}$.

§ 2. The system of aggregates $[4][2]_0 \cap S_3$.

Resting on the series $[4][2]_0 \cap S_3$ we have a series $[2][4]_0 \cap S_3$. All aggregates of the latter are degenerated and 10 aggregates are two-ply degenerated; the vertices form the manifold $[2][4]_0 \cap S_3$ consisting of 10 points $K_{11}$, $K_{12}$.

Thus $[4][2]_0 \cap S_3$ also contains $\infty^2$ simply degenerated $[2]_0$, the vertices forming a congruence of lines $[d]$ of order 3 and class 0. The 10 points $K_1$ are the vertices of cones of order 3 of the congruence $[d]$. Paired with every two-ply degenerated $[4]$ is a regulus of lines which are the vertices of degenerated $[2]$. Every one of these reguli defines a quadratic surface containing 9 of the 10 points $K_1$.

We can immediately determine the mutual positions of manifolds which are connected with subseries of $[4][2]_0 \cap S_3$. The system $[2][4]_0 \cap S_3$ contains $\infty^2$ subseries $[1][4]_0 \cap S_3$. The manifold $[4]_0 \cap S_3$ connected with a similar subseries consists of the entire set of lines in $S_3$ as well as a pencil of planes of class 3, the planes representing the degenerated $[1]$ of the series $[4][1]_0 \cap S_3$ which rests on $[1][4]_0 \cap S_3$. If we take a fixed $[4]$ of $[2][4]_0 \cap S_3$, then it belongs to $\infty^1$ subseries $[1][4]_0 \cap S_3$. The $\infty^1$ pencils of planes of class 3 connected with these subseries envelop a surface of class 5.

The birational transformation $[3][2]_0 \cap S_3$ induces the congruence $[3][2]_0 \cap S_3$.

At the same time $[2][3]_0 \cap S_3 = [2][3]_0 \cap S_3$.

(1) Chapter XI

The investigation of the contravariant $[n]_0 \cap S_3$ as an example.

The theory rests on a great importance.

§ 2. The system of aggregates $[4][2]_0 \cap S_3$. The manifold $[4]_0 \cap S_3$ connected with a similar subseries consists of the entire set of lines in $S_3$ as well as a pencil of planes of class 3, the planes representing the degenerated $[1]$ of the series $[4][1]_0 \cap S_3$ which rests on $[1][4]_0 \cap S_3$. If we take a fixed $[4]$ of $[2][4]_0 \cap S_3$, then it belongs to $\infty^1$ subseries $[1][4]_0 \cap S_3$. The $\infty^1$ pencils of planes of class 3 connected with these subseries envelop a surface of class 5.
The birational relation between the manifolds $|4|2|0_0 \cap S_3$ and $|3|2|0_0 \cap S_3$ induces a one-one correspondence between the lines of the congruence $|d|$ and the points of the Bordiga surface $F^2_3$ in $S_4$. At the same time there is a one-one correspondence between $|2|3|0_0 \cap S_4 = F^2_3$ and $|4|3|0_0 \cap S_2 = S_2$ already mentioned in the preceding paragraph. Many properties of $F^2_3$ as deduced in Room (1) Chapter XIV, can now be proved by means of the above-mentioned correspondences. Conversely the Bordiga surface gives a clear representation of the congruence $|d|$ in $S_3$.

The investigation of the linear systems of two-dimensional contravariant aggregates will be complete if all systems $|n|2|0_0 \cap S_3$ ($n = 1, \ldots, 10$) have been investigated.

The theory regarding complementary systems of aggregates is of great importance in this investigation. In this case the systems are $|n|2|0_0 \cap S_3$ and $|10-n|2|0_0 \cap S_3$. For instance let us take $n = 8$ as an example. The system $|8|2|0_0 \cap S_3$ contains a set of $\infty^2$ completely degenerated stars $|2|$. This set of planes is dual to the point-manifold $|2|2|0_0 \cap S_3$. The planes envelop a surface of class 3.