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Cohomology and the Resolution of the Nilpotent Variety

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1. Let G be a split reductive linear algebraic group over a field k of characteristic zero. Consider the variety N of the nilpotent elements of the Lie algebra $\mathfrak{g}$ of G . It is a normal variety cf. [14] Theorem 16. It is isomorphic to the variety of the unipotent elements of G , cf. [17]. The theorem of Brieskorn-Steinberg-Tits states that the rational singularities are dense in the singular locus of N , see [1] and [18] (3.10). Here we shall prove that N has only rational singularities, cf. [12] p. 50, i.e. we prove

Theorem A. *There exists a proper birational morphism $\tau: Y \rightarrow N$ such that Y is smooth over k , that $\tau_*(\mathcal{O}_Y) = \mathcal{O}_N$ and $R^p\tau_*(\mathcal{O}_Y) = 0$ for $p \geq 1$.*

This theorem admits a generalization which will be stated and proved in Section 5. In the complex analytic situation the same assertions follow by Theorem 5 of [3] exp. II. For rational singularities in that case see [2]. In [11] we investigated the local structure of N for the classical groups.

2. By [17] (2.2) we may assume that G is semi-simple and simply connected. Let G be split with respect to a maximal torus T and a Borel group B .

If E is a finite dimensional vector space over k and $\varrho: B \rightarrow GL(E)$ is a morphism of algebraic groups then $E = (E, \varrho)$ is called a B -module. The contracted product $G \times^B E$ is defined as the quotient of $G \times E$ under the right action of B given by $(g, e)b = (gb, \varrho(b^{-1})e)$ for all $g \in G, e \in E, b \in B$. The morphism $\psi: G \times^B E \rightarrow G/B$ given by $\psi(g, e)B = gB$, is a vector bundle over G/B . The sheaf of sections of ψ is a locally free $\mathcal{O}_{G/B}$ -module. It is denoted by $\mathcal{L}(E)$. See [7] p. 55, 56.

Let $\mathfrak{u}$ be the Lie algebra of the unipotent part U of B . As $\mathfrak{u}$ is a B -module we can define $Y = G \times^B \mathfrak{u}$. The adjoint action $\text{Ad}: G \times \mathfrak{g} \rightarrow \mathfrak{g}$ induces a surjective morphism $\tau: Y \rightarrow N$. Since $\psi: Y \rightarrow G/B$ is a vector bundle, Y is an irreducible smooth variety. The morphism τ is easily identified with the G -equivariant proper morphism τ considered in [17] (the proof of 2.1), which is birational, cf. [17] (2.4). Since N is normal, it follows that $\mathcal{O}_N = \tau_*(\mathcal{O}_Y)$. By [10] (1.4.11) it suffices now to prove that $H^p(Y, \mathcal{O}_Y) = 0$ for all $p \geq 1$. This is a special case of the corollary in Section 5.

Remark. The action of B on u is not completely reducible, so we cannot apply the theorem in [13].

3. Let $X(T)$ be the character group of T . Let R be the root system of G with respect to T and let W be the Weyl group. Conform [5] and [6] a root α is called positive if its eigenspace $\mathfrak{g}_\alpha$ is not contained in u . The set of positive roots is denoted by R_+ . For each root α we have an associated co-root α^* in the dual $\mathbb{Z}$ -module $X(T)^*$. A weight $\chi \in X(T)$ is called regular (resp. dominant) if we have $\langle \alpha^*, \chi \rangle \neq 0$ (resp. $\langle \alpha^*, \chi \rangle \geq 0$) for all $\alpha \in R_+$. The set of dominant weights is denoted by $X(T)_+$. If V is a subset of R we write $|V| = \sum_{\alpha \in V} \alpha$. The number of elements of V is denoted by $\#(V)$. The weight ϱ is defined by $2\varrho = |R_+|$. It is well known that $\varrho + X(T)_+$ is the set of regular dominant weights. The length of $w \in W$ is denoted by $n(w)$, cf. [6].

If E is a B -module we write $H^p(E) = H^p(G/B, \mathcal{L}(E))$. As $\mathcal{L}$ is an exact functor, the functors H^p form an exact delta-functor from the category of B -modules to the category of G -modules. For $\chi \in X(T)$ let $E(\chi)$ be the one-dimensional B -module corresponding to the induced morphism $B \rightarrow T \rightarrow Gl(1)$. We shall use the following version of Bott's theorem, cf. [6].

If $H^p(E(\chi)) \neq 0$, then $\chi + \varrho$ is regular and $p = n(w)$, where w is the unique element of W such that $w(\chi + \varrho)$ is dominant (and regular).

4. Definition. If $\mu \in X(T)$, let $p(\mu)$ be the maximal value of $n(w) - \#(V)$, where $w \in W$ and V is a subset of R_+ such that $w(\mu + \varrho - |V|)$ is dominant and regular.

Lemma. Let $\mu \in X(T)$.

(a) $p(\mu)$ is the maximal value of $\#(V \cap -wR_+) - \#(V \cap wR_+)$, where $w \in W$ and V is a subset of R_+ such that $w(\mu) - |V|$ is dominant.

(b) We have $0 \leq p(\mu) \leq \#(R_+)$.

(c) If $\mu = 0$ then $p(\mu) = 0$.

Proof. Consider subsets P of R satisfying $P \cap -P = \emptyset$ and $P \cup -P = R$. The relations $V = R_+ \cap P$, $P = V \cup -(R_+ \setminus V)$ define a one-to-one correspondence between these subsets of R and the subsets V of R_+ . The natural action of W on the collection of the subsets P induces an action of W on the power set of R_+ , which is given by

$$w * V = R_+ \cap (wV \cup -w(R_+ \setminus V)).$$

If V corresponds to P then $\varrho - |V| = -\frac{1}{2}|P|$. This implies

$$w(\varrho - |V|) = \varrho - |w * V|.$$

It follows that $w(\mu + \varrho - |V|)$ is dominant and regular if and only if $w(\mu) - |w * V|$ is dominant. As $n(w) = \#(R_+ \cap -wR_+)$ we have

$$n(w) - \#(V) = \#((w * V) \cap -wR_+) - \#((w * V) \cap wR_+).$$

Now (a) follows immediately. (b) is a consequence of (a). If V is a non-empty subset of R_+ then $-|V|$ is not dominant. So (c) follows from (a). Compare [4] and [15] (2.13).

Remark. It seems that $\mu \in \varrho + X(T)_+$ implies $p(\mu) = 0$. If R is a root system of type A_l and α is the highest root, then we have $\alpha \in X(T)_+$ and $p(\alpha) \geq l - 4$.

5. The notations are as before. G is semi-simple and simply connected. Let $\mathfrak{p}$ be a linear subspace of $\mathfrak{g}$ which is B -invariant and contains u . Consider $Z = G \times^B \mathfrak{p}$ and the canonical morphism $\psi: Z \rightarrow G/B$.

Theorem B. Let $\mu \in X(T)$ and $p > p(\mu)$. Then $H^p(Z, \psi^* \mathcal{L}(E(\mu))) = 0$.

As $\mathcal{O}_Z = \psi^*(\mathcal{O}_{G/B}) = \psi^* \mathcal{L}(E(0))$ we obtain by 4. Lemma (c) immediately the following

Corollary. $H^p(Z, \mathcal{O}_Z) = 0$ for all $p \geq 1$.

Proof of Theorem B. If E is a B -module, let $S(E^*) = \sum_q S_q(E^*)$ be the graded symmetrical algebra on the dual E^* of E . It may be considered as the ring of polynomial functions on E . The summands $S_q(E^*)$ are B -modules. Using [9] (9.4) one verifies that $\psi_* (\mathcal{O}_Z) = S((\mathcal{L}(\mathfrak{p}))^*) = \sum_q \mathcal{L}(S_q(\mathfrak{p}^*))$ and hence

$$\psi_* \psi^* \mathcal{L}(E(\mu)) = \psi_* (\mathcal{O}_Z) \otimes_{G/B} \mathcal{L}(E(\mu)) = \sum_q \mathcal{L}(S_q(\mathfrak{p}^*) \otimes_k E(\mu)).$$

Now it follows from [10] (1.3.3) and [8] Chapter II (3.10) that

$$H^p(Z, \psi^* \mathcal{L}(E(\mu))) = \sum_q H^p(S_q(\mathfrak{p}^*) \otimes_k E(\mu)).$$

Consider the $S(\mathfrak{g}^*)$ -module $M = S(\mathfrak{g}^*) \otimes_k E(\mu)$. Let J be the kernel of the canonical surjection $\mathfrak{g}^* \rightarrow \mathfrak{p}^*$. As x be a basis of J . As x is an M -regular sequence we have the long exact sequence

$$\dots K_2 \rightarrow K_1 \rightarrow K_0 \rightarrow S(\mathfrak{p}^*) \otimes_k E(\mu) \rightarrow 0$$

where $K_* = K(x; M)$ is the exterior complex, cf. [16] IV A2. Intrinsically the complex K_* may be defined by

$$K_n = M \otimes_k \Lambda^n J, \quad d_n: K_{n+1} \rightarrow K_n,$$

$$d_n(m \otimes (a_0 \wedge \dots \wedge a_n)) = \sum_{i=0}^n (-1)^i a_i m \otimes (a_0 \wedge \dots \wedge \hat{a}_i \wedge \dots \wedge a_n).$$

It is canonically graded by $K_* = \sum_q K^q$ where

$$K_n^q = S_{q-n}(\mathfrak{g}^*) \otimes_k E(\mu) \otimes \Lambda^n J.$$

Thus we have the long exact sequences of B -modules

$$\dots \rightarrow K_2^q \rightarrow K_1^q \rightarrow K_0^q \rightarrow S_q(\mathfrak{p}^*) \otimes_k E(\mu) \rightarrow 0.$$

As H^* is an exact delta-functor it suffices now to prove that $H^{p+n}(K_n^q) = 0$ for $q, n \geq 0$.

As $S_{q-n}(\mathfrak{g}^*)$ is a G -module we have the following cartesian square.

$$\begin{array}{ccc} G \times^B S_{q-n}(\mathfrak{g}^*) & \longrightarrow & S_{q-n}(\mathfrak{g}^*) \\ \downarrow & & \downarrow \\ G/B & \longrightarrow & G/G = pt. \end{array}$$

Considering $S_{q-n}(\mathfrak{g}^*)$ as a locally free sheaf on G/G we have therefore $\mathcal{L}(S_{q-n}(\mathfrak{g}^*)) = f^*(S_{q-n}(\mathfrak{g}^*))$ where $f: G/B \rightarrow G/G$ is the canonical morphism. By [10] (0_{III} 12.2.3) this implies

$$H^{p+n}(K_n^q) = S_{q-n}(\mathfrak{g}^*) \otimes_k H^{p+n}(E(\mu) \otimes \Lambda^n J).$$

So it suffices to prove that $H^{p+n}(E(\mu) \otimes \Lambda^n J) = 0$ for $n \geq 0$.

The B -module $E(\mu) \otimes \Lambda^n J$ has a filtration of B -modules F_i , $0 \leq i \leq r$, such that $F_0 = 0$ and $F_i/F_{i-1} \cong E(\chi_i)$ for some enumeration (χ_i) of the weights of $F_r = E(\mu) \otimes \Lambda^n J$. By convention the weights of u are the negative roots. Now dual modules have opposite weights and $\mathfrak{p}$ contains u , so all positive roots are weights of $\mathfrak{p}^*$. Therefore the non-zero weights of J are negative roots with multiplicity one. Thus for every i there is a subset V_i of R_+ such that $\chi_i = \mu - |V_i|$ and $\#(V_i) \leq n$. If $w \in W$ is such that $w(\chi_i + \varrho)$ is dominant and regular, then we have

$$n(w) \leq p(\mu) + \#(V_i) < p + n.$$

By Bott's theorem as quoted in Section 3 this implies $H^{p+n}(E(\chi_i)) = 0$ for all i . It follows that $H^{p+n}(E(\mu) \otimes \Lambda^n J) = 0$.

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