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Hesselink, Wim H.

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Polarizations in the Classical Groups

Wim H. Hesselink

Mathematisch Instituut, Rijksuniversiteit Groningen, Postbus 800,
Groningen, The Netherlands

1. Introduction

1.1. Let G be a connected reductive group over an algebraically closed field k . It has an adjoint action on its Lie algebra $\mathfrak{g}$. Let P be a parabolic subgroup of G . Let $U(P)$ denote its unipotent radical with Lie algebra $\mathfrak{u}(P)$. An element $x \in \mathfrak{u}(P)$ is called a *Richardson element* of $\mathfrak{u}(P)$ if $\dim(Gx) = 2 \dim(G/P)$.

Lemma. Let $x \in \mathfrak{u}(P)$.

(a) $\dim(Gx) \leq 2 \dim(G/P)$.

(b) x is a Richardson element of $\mathfrak{u}(P)$ if and only if Px is dense in $\mathfrak{u}(P)$ and P contains the identity component $Z_G(x)^0$ of the centralizer $Z_G(x)$.

Proof. This follows from the inequality

$$\begin{aligned} \dim(Gx) &= \dim(G) - \dim(Z_G(x)) \leq \dim(G) - \dim(Z_P(x)) = \\ &= \dim(G/P) + \dim(Px) \leq \dim(G/P) + \dim(\mathfrak{u}(P)) = 2 \dim(G/P). \end{aligned}$$

1.2. Assume that $\mathfrak{g}$ has only finitely many orbits of nilpotent elements. It is known that this assumption is fulfilled if G is a classical group or if the characteristic of k is good enough.

Theorem (a) (Richardson [8, 11] (3.9)). Every parabolic group P in G has Richardson elements in $\mathfrak{u}(P)$.

(b) (Borho-Jantzen [3] (5.18)). Let P_1 and P_2 be parabolic groups in G . For $i = 1, 2$ let $x_i \in \mathfrak{u}(P_i)$ be a Richardson element and let L_i be a Levi factor of P_i . If L_1 is conjugate to L_2 in G then x_1 is conjugate to x_2 under G .

1.3. Conversely, let $x \in \mathfrak{g}$ be nilpotent. A parabolic group P in G is called a *polarization* of x if x is a Richardson element of $\mathfrak{u}(P)$. The set of polarizations of x is denoted by $\text{Pol}(x)$.

Remark. So we only consider polarizations of nilpotent elements. A more general definition leads to questions which are usually first reduced to the nilpotent case, cf. the proof of [3] (5.16). The structure of $\text{Pol}(x)$ is interesting for the infinite-dimensional representation theory, cf. [7] and [1] (6.3).

By (1.1) (b) the “disconnected” centralizer $A(x) := Z_G(x)/Z_G(x)^0$ acts on the set $\text{Pol}(x)$. If $P \in \text{Pol}(x)$ its stabilizer $A_P(x)$ in $A(x)$ is equal to $Z_P(x)/Z_G(x)^0$, since parabolic subgroups are self normalizing, cf. [6] (23.1).

Lemma. *Let $\mathcal{P}$ be a conjugacy class of parabolic subgroups of G . If $\mathcal{P} \cap \text{Pol}(x)$ is non-empty then it is one single orbit under the finite group $A(x)$.*

Proof. Let $P, P' \in \mathcal{P} \cap \text{Pol}(x)$, say $P' = \text{int}(g)P$. Both x and $g^{-1}x$ are Richardson elements of $\mathfrak{u}(P)$. Their dense P -orbits in $\mathfrak{u}(P)$ intersect, so there is $p \in P$ with $px = g^{-1}x$. We have $gp \in Z_G(x)$ and $P' = \text{int}(gp)P$.

Corollary. $\text{Pol}(x)$ is a finite set.

Proof. G has only finitely many conjugacy classes of parabolic subgroups.

1.4. We shall describe $\text{Pol}(x)$ for the classical groups: $G = \text{Gl}(n)$, or $\text{char}(k) \neq 2$ and $G = \text{SO}(n)$ or $G = \text{Sp}(n)$. We verify Theorem (1.2) by hand. So it is natural to describe $\text{Pol}(x)$ in the following steps.

Step 1. Describe the conjugacy classes $\mathcal{L}$ of the Levi factors of the elements of $\text{Pol}(x)$.

Step 2. Given class $\mathcal{L}$ of step 1, determine the classes $\mathcal{P}$ of parabolic subgroups with Levi factor in $\mathcal{L}$. The number of classes $\mathcal{P}$ associated to $\mathcal{L}$ is denoted by $N_0 = N_0(\mathcal{L})$.

Step 3. Given a class $\mathcal{P}$ of step 2, describe $\mathcal{P} \cap \text{Pol}(x)$ which set is an orbit under $A(x)$, by a representative $P \in \mathcal{P} \cap \text{Pol}(x)$ together with its stabilizer $A_P(x)$ in $A(x)$. The number of elements of $\mathcal{P} \cap \text{Pol}(x)$ is denoted by $N_1 = N_1(\mathcal{P})$. So N_1 is the index of $A_P(x)$ in $A(x)$, or equivalently of $Z_P(x)$ in $Z_G(x)$.

1.5. For the classical groups the finite group $A(x)$ is always abelian. So the stabilizer $A_P(x)$ is independent of the choice of $P \in \mathcal{P} \cap \text{Pol}(x)$. It turns out that $A_P(x)$ only depends on the class $\mathcal{L}$. More precisely we obtain.

Corollary (7.6). *For $i = 1, 2$ let $P(i) \in \text{Pol}(x)$, say with Levi factor $L(i)$. The following conditions are equivalent:*

- (a) $L(1)$ is conjugate to $L(2)$ in G .
- (b) $A_{P(1)}(x) = A_{P(2)}(x)$ (or equivalently $Z_{P(1)}(x) = Z_{P(2)}(x)$).
- (c) $N_1(P(1)) = N_1(P(2))$.

Note that (b) $\Rightarrow$ (c) is trivial. The implication (a) $\Rightarrow$ (c) holds in general, cf. [4] (7.2). These results lead to the following conjecture for a reductive group G and a nilpotent element $x \in \mathfrak{g}$.

Conjecture. *Let $P(1), P(2) \in \text{Pol}(x)$ have Levi factors $L(1)$ and $L(2)$. Then $L(1)$ is conjugate to $L(2)$ in G if and only if $Z_{P(1)}(x)$ is conjugate to $Z_{P(2)}(x)$ in $Z_G(x)$.*

1.6. The solution of the steps 1 and 2 is due to N. Spaltenstein, cf. [9] Section 5. His methods are sufficient to solve step 3 as well. We give a new proof and more explicit results. Finally we give tables for the groups of type $B_2, B_3, B_4, C_2, C_3, C_4, D_4, D_5, D_6$.

It is a pleasure to express our gratitude to Walter Borho and Hanspeter Kraft for a stimulating correspondence, many discussions and much hospitality.

2. Combinatorial Conventions

$\mathbb{N}$ is the set of natural numbers $1, 2, 3, \dots$. If V is a finite set, its cardinality is denoted by $\#(V)$. If x is a real number $[x]$ denotes the largest integer $\leq x$.

We define a *partition* λ to be a finite subset of $\mathbb{N}^2$ such that if $(p, q) \in \lambda$ and $i \leq p$ and $j \leq q$ then $(i, j) \in \lambda$. So we identify a partition with the corresponding Young diagram. If λ is a partition we define.

$$|\lambda| = \#(\lambda), \quad \lambda_j = \#\{i | (i, j) \in \lambda\}, \quad \lambda^i = \#\{j | (i, j) \in \lambda\}.$$

A partition λ is determined by each of the non-increasing sequences λ_* and λ^* . Clearly $|\lambda| = \sum_j \lambda_j = \sum_i \lambda^i$. One verifies that

$$\sum_j (j-1) \lambda_j = \sum_i \frac{1}{2} \lambda^i (\lambda^i - 1). \quad (2.1)$$

If $(p_1, \dots, p_s)$ is a finite sequence in $\mathbb{N}$ we define the partition $\lambda = \text{ord}(p_1, \dots, p_s)$ by $\lambda_j = \#\{i | p_i \geq j\}$. Then $\lambda_1 = s$ so that $\lambda^s > 0 = \lambda^{s+1}$. There is a permutation σ of $\{1, \dots, s\}$ such that $\lambda^i = p_{\sigma(i)}$ for $1 \leq i \leq s$. In particular $|\lambda| = \sum p_i$.

For a partition λ we define $N(\lambda)$ to be the number of sequences $(p_1, \dots, p_s)$ with $\lambda = \text{ord}(p_1, \dots, p_s)$. Since the above permutation σ is unique up to multiplication by elements of the stabilizer of $(\lambda^1, \dots, \lambda^s)$ we have

$$N(\lambda) = \lambda_1! \prod_{i \geq 1} ((\lambda_i - \lambda_{i+1})!)^{-1}. \quad (2.2)$$

Let G be a reductive group. We define $b(G) = \dim(G/B)$ where B is some (any) Borel group of G . If P is a parabolic subgroup of G with Levi factor L then we have

$$\dim(G/P) = b(G) - b(L). \quad (2.3)$$

For reference below we note the formulas

$$\begin{aligned} b(\text{Gl}(n)) &= \frac{1}{2}n(n-1) \\ b(\text{Sp}(n)) &= \frac{1}{2}(\frac{1}{2}n(n-1) + [\frac{1}{2}n]) \quad (\text{here } n \text{ is even}) \\ b(\text{SO}(n)) &= \frac{1}{2}(\frac{1}{2}n(n-1) - [\frac{1}{2}n]). \end{aligned} \quad (2.4)$$

3. The General Linear Group $\text{Gl}(n)$

3.1. Let V be a vector space over the algebraically closed field k with $\dim(V) = n$. Let G be $\text{Gl}(V)$. Its Lie algebra $\mathfrak{g}$ is identified with $\text{End}(V)$. Let $x \in \mathfrak{g}$ be nilpotent. The endomorphism x has a Jordan normal form. So there is a partition λ with $|\lambda| = n$ and a *Jordan basis* of V consisting of vectors $e(i, j)$, $(i, j) \in \lambda$, such that $x e(i, j) = e(i-1, j)$ if $i > 1$, and $x e(1, j) = 0$. If $m \geq 0$ then $\dim(\text{Im}(x^m)) = \sum_{i > m} \lambda^i$. So the partition λ of x is unique. G acts on $\mathfrak{g}$ by $\text{Ad}(g)x = g x g^{-1}$. It is clear that nilpotent elements x and y of $\mathfrak{g}$ are conjugate under G if

and only if they have the same partition. The centralizer $Z_G(x)$ of an element $x \in \mathfrak{g}$ is open in the linear space $\{y \in \text{End}(V) | yx = xy\}$, and hence connected.

3.2. A (partial) *flag* F in V is a sequence of subspaces $(F_0, \dots, F_s)$ with $F_0 = 0$, $F_s = V$ and $F_{i-1} \subset F_i$, $F_{i-1} \neq F_i$ for $1 \leq i \leq s$. Its *type* is the sequence $(p_1, \dots, p_s)$ given by $p_i = \dim(F_i/F_{i-1})$. Its *stabilizer* P consists of the $g \in G$ with $gF_i = F_i$ for $0 \leq i \leq s$. Now P is a parabolic subgroup of G . Every parabolic subgroup P of G is the stabilizer of a unique flag. The type of its flag is called the *flag type* of P . Parabolic subgroups of G are conjugate if and only if they have the same flag type.

Let P be the stabilizer of the flag $F = (F_0, \dots, F_s)$. Its unipotent radical consists of the elements $g \in P$ with $(g-1)F_i \subset F_{i-1}$ for $1 \leq i \leq s$. So $\mathfrak{u}(P)$ consists of the $x \in \text{End}(V)$ with $x F_i \subset F_{i-1}$ for $1 \leq i \leq s$. If we choose factors L_i with $F_i = F_{i-1} \oplus L_i$ then

$$L = \prod_{i=1}^s \text{Gl}(L_i) \cong \prod_{i=1}^s \text{Gl}(p_i)$$

is a Levi factor of P . So parabolic subgroups P and Q of G with flag types $(p_1, \dots, p_s)$ and $(q_1, \dots, q_t)$ have conjugate Levi factors if and only if $\text{ord}(p_1, \dots, p_s) = \text{ord}(q_1, \dots, q_t)$, cf. Section 2.

3.3. Theorem. Let $x \in \mathfrak{g}$ be nilpotent with partition λ .

(a) The element x has a polarization P with flag type $(p_1, \dots, p_s)$ if and only if $\lambda = \text{ord}(p_1, \dots, p_s)$. In that case P contains $Z_G(x)$ and is unique.

(b) The element x has $N(\lambda)$ polarizations. All its polarizations have conjugate Levi factors, isomorphic to $\prod_{i \geq 1} \text{Gl}(\lambda^i)$.

Proof. Although this is known, cf. [9] (5.3), we indicate a proof as a preparation for the other classical groups.

Assume $\lambda = \text{ord}(p_1, \dots, p_s)$. We construct a flag $F = (F_0, \dots, F_s)$ with flag type $(p_1, \dots, p_s)$ such that $x F_i \subset F_{i-1}$ for $1 \leq i \leq s$. Put $p = p_1$ and $m = \lambda_p - 1$ and $F_1 = \text{Ker}(x) \cap \text{Im}(x^m)$. If $e(i, j)$, $(i, j) \in \lambda$, is a Jordan basis of V with respect to x , cf.

(3.1), then $F_1 = \sum_{j=1}^p ke(1, j)$. It follows that $\dim(F_1) = p_1$. The induced endomorphism x' of V/F_1 has partition μ given by $\mu_j = \lambda_j - 1$ if $j \leq p$, and $\mu_j = \lambda_j$ if $j > p$. One verifies that $\mu = \text{ord}(p'_1, \dots, p'_{s-1})$ where $p'_i = p_{i+1}$. By induction we may assume the existence of a flag $F' = (F'_0, \dots, F'_{s-1})$ in V/F_1 of type $(p'_1, \dots, p'_{s-1})$ such that $x' F'_i \subset F'_{i-1}$ for $1 \leq i \leq s-1$. Writing $F'_i = F_{i+1}/F_1$ we obtain a flag F which satisfies the requirements. Let P be the stabilizer of F . Then we have $x \in \mathfrak{u}(P)$. Using the formulas (2.3), (2.4) and (2.1) we obtain

$$2 \dim(G/P) = n^2 - n - 2 \sum (j-1) \lambda_j.$$

It follows that $2 \dim(G/P) = \dim(Gx)$, as calculated for example in [5] (3.8). So P is a polarization. Since $Z_G(x)$ is connected P contains $Z_G(x)$. Alternatively this may be verified by the induction.

By the conjugacy of the parabolic subgroup of a given flag type, it follows that every parabolic subgroup, say with flag type $(q_1, \dots, q_t)$, is the above constructed polarization of some nilpotent element with partition $\text{ord}(q_1, \dots, q_t)$. By the P -conjugacy of the Richardson elements of $u(P)$ this proves part (a). Part (b) follows immediately. See Section 2 for the definition of $N(\lambda)$.

Remark. In the proof we had to verify (1.2) (a) by hand. The assertion (1.2) (b) follows also.

4. The Symplectic and the (Special) Orthogonal Group

We treat these groups in a unified way, cf. [10] p. 253 and [5] (3.1).

4.1. Conventions. All congruences are modulo 2. We fix a number ε equal to 0 or 1. The field k is algebraically closed and of characteristic $\neq 2$. Let V be a vector space over k with $\dim(V) = n$. Let $\varphi: V \times V \rightarrow k$ be a non-degenerate bilinear form satisfying

$$\varphi(v_1, v_2) = (-1)^\varepsilon \varphi(v_2, v_1) \quad v_1, v_2 \in V.$$

Remark. (a) So $n \equiv 0$ if $\varepsilon = 1$. (b) All forms φ are equivalent.

The group $H = H(V, \varphi)$ consists of the elements $g \in \text{Gl}(V)$ which preserve the form φ . The group $G = G(V, \varphi)$ is the identity component of H . It consists of the elements of H with determinant 1. The common Lie algebra $\mathfrak{g}$ of G and H consists of the elements $x \in \text{End}(V)$ satisfying

$$\varphi(xv_1, v_2) + \varphi(v_1, xv_2) = 0 \quad v_1, v_2 \in V.$$

The groups G and H are reductive of rank $l = [\frac{1}{2}n]$. Their types, Dynkin diagrams and classical names are as follows.

$$\left. \begin{array}{ll} n \not\equiv \varepsilon = 1 & C_l \quad \overset{1}{\circ} - \overset{2}{\circ} - \dots - \overset{l-1}{\circ} \rightleftharpoons \overset{l}{\circ} : \\ n \not\equiv \varepsilon = 0 & B_l \quad \overset{1}{\circ} - \overset{2}{\circ} - \dots - \overset{l-1}{\circ} \rightrightarrows \overset{l}{\circ} \\ n \equiv \varepsilon = 0 & D_l \quad \overset{1}{\circ} - \overset{2}{\circ} - \dots - \overset{l-2}{\circ} \begin{array}{l} \nearrow \overset{l-1}{\circ} \\ \searrow \overset{l}{\circ} \end{array} \end{array} \right\} \begin{array}{l} G = H = \text{Sp}(n) \\ G = \text{SO}(n) \neq H = \text{O}(n). \end{array}$$

4.2. We can choose a basis $e_1, \dots, e_n$ of V such that $\varphi(e_i, e_j) \neq 0$ if and only if $i + j = n + 1$. The torus T in G with weight vectors $e_1, \dots, e_n$ is maximal. Let $\lambda_1, \dots, \lambda_n$ be the corresponding weights. They generate the character group $X(T)$ and are subject to the relations $\lambda_i + \lambda_{n+1-i} = 0$, and $\lambda_{l+1} = 0$ if $n = 2l + 1$.

The root system $\Phi = \Phi(G, T)$ consists of the differences $\lambda_i - \lambda_j$ with $i \neq j$, and $i + j \neq n + 1$ if $\varepsilon = 0$. The simple roots $\alpha_1, \dots, \alpha_l$ are chosen $\alpha_i = \lambda_i - \lambda_{i+1}$ if $1 \leq i < l$, $\alpha_l = \lambda_l - \lambda_{l+1} = (1 + \varepsilon)\lambda_l$ if $n \not\equiv \varepsilon$, and $\alpha_l = \lambda_{l-1} - \lambda_{l+1} = \lambda_{l-1} + \lambda_l$ if $n \equiv \varepsilon = 0$. Their numbers correspond to the numbers in the above Dynkin diagram.

Remark. (a). Assume $n \equiv \varepsilon = 0$. Let $r \in H$ be the reflexion given by $re_i = e_i$ if $i \notin \{l, l+1\}$, and $re_l = e_{l+1}$ and $re_{l+1} = e_l$. This reflexion interchanges the roots α_{l-1}

and α_i . The group H is generated by r and G . (b) If $n \neq \varepsilon = 0$ then H is generated by G and $-\text{id}$, which element usually acts trivial.

4.3. A flag $F = (F_0, \dots, F_s)$ in V is called *isotropic* if $F_i^\perp = F_{s-i}$ for $0 \leq i \leq s$. Let $(p_1, \dots, p_s)$ be its type, cf. (3.2). It satisfies $p_i = p_{s+1-i}$ for $1 \leq i \leq s$. Put $t = [\frac{1}{2}s]$, and $q = 0$ if $s = 2t$, and $q = p_{t+1}$ if $s = 2t + 1$. Clearly $q \equiv n$. One verifies the existence of a basis $e_1, \dots, e_n$ of V such that

$$\varphi(e_i, e_j) \neq 0 \Leftrightarrow i + j = n + 1,$$

$$F_j = \sum k e_i, \quad 1 \leq i \leq \dim(F_j).$$

We assume that this is the basis of (4.2). Consider the full isotropic flag $F' = (F'_0, \dots, F'_n)$ given by $F'_j = \sum_{i=1}^j k e_i$. Let P and B be the stabilizers in G of the flags F and F' , respectively, cf. (3.2). The group B is the Borel group of G corresponding to the basis $\alpha_1, \dots, \alpha_l$ of Φ . So P is some standard parabolic group P_I , cf. [6] (30.1).

We say that the flag F is *not admissible* if there is a P -invariant subspace F' and an index j with $F_j \subset F' \subset F_{j+1}$ and $F_j \neq F' \neq F_{j+1}$.

Lemma. (a) *The flag F is admissible if and only if $\varepsilon = 1$ or $q \neq 2$.*

(b) *Assume that F is admissible. Then $P = P_I$ where I consists of the indices i with $1 \leq i \leq l$ and $i \neq \dim(F_j)$ for all j .*

(c) *Every parabolic subgroup of G is the stabilizer of some admissible isotropic flag.*

Proof. (a) and (b) are left to the reader. (c) Every parabolic subgroup is conjugate to some P_I with $I \subset \{1, \dots, l\}$. By Remark (4.2) (a) we may assume that $n \neq \varepsilon$ or that $l - 1 \in I$ or that $l \notin I$. By (a) all such sets I occur in (b).

4.4. By Lemma (4.3)(b) the type $(p_1, \dots, p_s)$ of an admissible isotropic flag with stabilizer P is a well defined invariant of P . It is called the *flag type* of P .

Lemma. *Parabolic subgroups in G are conjugate under the action of H if and only if they have the same flag type. The class of parabolic subgroups of G with flag type $(p_1, \dots, p_s)$ splits into two conjugacy classes under the action of G if and only if $\varepsilon = 0$ and $s = 2t$ and $p_t \geq 2$.*

4.5. Let $F = (F_0, \dots, F_s)$ be an admissible isotropic flag of type $(p_1, \dots, p_s)$. Let P be its stabilizer in G . Let L be a Levi factor of P and let S be the radical of L . So S is a maximal torus of the radical of P and L is the centralizer of S in G , cf. [6] (30.2). Consider the weight space decomposition $V = \sum V(\chi)$ of V with respect to S . For $\omega \in X(S)$ we have

$$V(\omega)^\perp = \sum_{\chi \neq -\omega} V(\chi).$$

The group L consists of the $g \in G$ with $gV(\chi) = V(\chi)$ for all $\chi \in X(S)$. One verifies that $X(S)$ has a basis $\chi_1, \dots, \chi_t$ where $t = [\frac{1}{2}s]$ such that

$$V = V(0) \oplus \sum_{i=1}^t (V(\chi_i) \oplus V(-\chi_i)),$$

$$F_j = \sum_{i=1}^j V(\chi_i) \quad \text{if } j \leq t.$$

We may identify

$$L = G(V(0), \varphi|V(0)) \times \prod_{i=1}^t \text{Gl}(V(\chi_i)).$$

In the notations of (4.3) we have

$$p_i = \dim(V(\chi_i)) \quad \text{if } i \leq t, \\ q = \dim(V(0)).$$

We put $v = \text{ord}(p_1, \dots, p_t)$, cf. Section 2. The pair $(v; q)$ is called the *type* of L , or the *Levi type* of P .

Remark. If $\varepsilon = 0$ then $\dim(V(0)) \neq 2$, since $\text{SO}(2)$ is a non-trivial torus, see Lemma (4.3)(a).

4.6. An integer q is called *admissible* if $q \geq 0$, and $q \equiv n$, and $q \neq 2$ if $\varepsilon = 0$. We define $\text{Lev}(n)$ to be the set of pairs $(v; q)$ such that q is admissible and v is a partition with $n = q + 2|v|$.

Lemma. (a) $\text{Lev}(n)$ is the set of the Levi types of the parabolic subgroups of G .

(b) Levi factors of parabolic subgroups of G are conjugate under the action of H if and only if they have the same type.

(c) The class of Levi factors of a given type $(v; q)$ splits into two conjugacy classes under the action of G if and only if $q = \varepsilon = 0$ and $v^i \equiv 0$ for all i .

(d) The number of conjugacy classes $\mathcal{P}$ of parabolic subgroups of G with given Levi type $(v; q)$ is equal to $N(v)$ if $q + \varepsilon \geq 1$, and to $(1 + v_1^{-1}v_2)N(v)$ if $q = \varepsilon = 0$.

Proof. (a) and (b) are immediate. (c) Choose a Levi factor L of type $(v; q)$. The class splits if and only if $H \neq G$ and the normalizer N of L in H is contained in G . The first condition is equivalent to $\varepsilon = 0$. The normalizer N is generated by L and $H(V(0), \varphi|V(0))$ and some standard mappings interchanging weight spaces $V(\pm\chi_i)$, $1 \leq i \leq t$. We have $H(V(0), \varphi|V(0)) \subset G$ if and only if $q = 0$. All interchanging mappings have determinant 1 if and only if $v^i \equiv 0$ for all i .

(d) $N(v)$ is the number of types $(p_1, \dots, p_s)$ of isotropic flags in V such that $v = \text{ord}(p_1, \dots, p_t)$ where $t = [\frac{1}{2}s]$, cf. Section 2. By Lemma (4.4) this settles the case $q + \varepsilon \geq 1$. Assume $q = \varepsilon = 0$. By (4.4) the required number is equal to $2N(v) - N'$ where N' is the number of sequences $(p_1, \dots, p_t)$ with $p_t = 1$ and $v = \text{ord}(p_1, \dots, p_t)$. If $v_1 = v_2$ then $N' = 0$. If $v_1 > v_2$ then $N' = N(\tau)$ where τ is the partition given by $\tau_1 = v_1 - 1$ and $\tau_j = v_j$ for $j \geq 2$. By (2.2) we have

$$N(\tau) = v_1^{-1}(v_1 - v_2)N(v).$$

5. Nilpotent Elements

5.1. Let $x \in \mathfrak{g}$ be nilpotent. By (3.1) there is a unique partition λ and a *Jordan basis* $e(i, j)$, $(i, j) \in \lambda$, such that $xe(i, j) = e(i-1, j)$ if $i > 1$ and $xe(1, j) = 0$. By [10] page 259, this basis can be *normalized* such that we have $\varphi(e(i, j), e(p, q)) \neq 0$ if and only if $i+p = \lambda_j + 1$ and $q = \beta(j)$, where β is some permutation of $\{1, \dots, \lambda^1\}$. By the symmetry relations of (4.1), it follows that $\beta^2 = \text{id}$ and $\lambda_{\beta(j)} = \lambda_j$, and that $\beta(j) \neq j$ whenever $\lambda_j \equiv \varepsilon$. Using Remark (4.1)(b) one verifies that any permutation β satisfying these conditions can be used for x .

Let $\text{Pan}(n, \varepsilon)$ denote the set of the partitions λ such that $|\lambda| = n$ and that the number $\#\{j | \lambda_j = m\}$ is even for every $m \in \mathbb{N}$ with $m \equiv \varepsilon$. Again using Remark (4.1)(b) we obtain

Lemma. (a) *A partition λ is the partition of some nilpotent element $x \in \mathfrak{g}$ if and only if $\lambda \in \text{Pan}(n, \varepsilon)$.*

(b) *Nilpotent elements x and y of $\mathfrak{g}$ are conjugate under H if and only if they have the same partition.*

5.2. Let $x \in \mathfrak{g}$ be nilpotent with partition λ and normalized Jordan basis as in (5.1). We put $M_i = \text{Ker}(x) \cap \text{Im}(x^{i-1})$. The vectors $e(1, j)$ with $1 \leq j \leq \lambda^i$ form a basis of M_i , so that $\dim(M_i) = \lambda^i$. One verifies the existence of a unique bilinear form φ_i on M_i such that $\varphi_i(v, v') = \varphi(v, v'')$ if $v, v' \in M_i$ and $v' = x^{i-1}v''$. This form satisfies

$$\varphi_i(v_1, v_2) = (-1)^{i-1+\varepsilon} \varphi_i(v_2, v_1) \quad \text{if } v_1, v_2 \in M_i.$$

One verifies that M_{i+1} is the kernel of the form φ_i on M_i . So there is an induced non-degenerate bilinear form φ'_i on M_i/M_{i+1} . Let $C_i = H(M_i/M_{i+1}, \varphi'_i)$ be the corresponding orthogonal or symplectic group, cf. (4.1). The canonical morphism

$$Z_H(x) \rightarrow \prod_i C_i$$

is surjective. Its kernel is connected. If $\lambda^i = \lambda^{i+1}$ or $i \equiv \varepsilon$, the group C_i is connected. If $\lambda^i > \lambda^{i+1}$ and $i \not\equiv \varepsilon$ then C_i is an orthogonal group with two connected components. Compare [10] page 261.

5.3. Consider the finite groups $A(x) = Z_G(x)/Z_G(x)^0$ and $A'(x) = Z_H(x)/Z_H(x)^0$, cf. (1.3). We have $A(x) \subset A'(x)$. By (5.2) the group $A'(x)$ is abelian and all its elements have order 2. We may consider $A'(x)$ as a vector space over $\mathbb{F}_2$ with a basis indexed by the pairs (i, λ^i) such that $\lambda^i > \lambda^{i+1}$ and $i \not\equiv \varepsilon$. This index set is equal to the set of pairs (λ_j, j) such that

$$j \in B(\lambda) = \{j \in \mathbb{N} | \lambda_j > \lambda_{j+1}, \lambda_j \not\equiv \varepsilon\}.$$

So we may identify

$$A'(x) = \{a \in \mathbb{F}_2^{\mathbb{N}} | j \notin B(\lambda) \Rightarrow a_j = 0\}.$$

If $\varepsilon=1$ then $G=H$ so that $A(x)=A'(x)$. If $\varepsilon=0$ the group G consists of the elements $g \in H$ with determinant 1. In general we obtain

$$A(x) = \{a \in \mathbb{F}_2^{\mathbb{N}} \mid j \notin B(\lambda) \Rightarrow a_j = 0; \varepsilon = 0 \Rightarrow \sum a_j = 0\}.$$

Remark. Assume, as we may, that the normalization of (5.1) satisfies $\beta(j)=j$ if and only if $\lambda_j \not\equiv \varepsilon$. For $m \in \mathbb{N}$ with $\lambda_m \not\equiv \varepsilon$ we define the reflexion $r(m) \in Z_H(x)$ by

$$\begin{aligned} r(m)e(i, j) &= e(i, j) & \text{if } m \neq j, \\ r(m)e(i, j) &= -e(i, j) & \text{if } m = j. \end{aligned}$$

The image of $r(m)$ in $A'(x)$ is the vector $r'(m)$ with $r'(m)_j = 1$ if and only if $\lambda_m = \lambda_j > \lambda_{j+1}$. So the vectors $r'(j)$ with $j \in B(\lambda)$ form a basis of $A'(x)$. Compare [9] (3.2), (3.9), (3.14).

5.4. Corollary. *Let $x \in \mathfrak{g}$ be nilpotent with partition λ . The G -orbit of x is unequal to the H -orbit of x if and only if $\varepsilon=0$ and $B(\lambda)$ is empty. In that case the H -orbit of x consists of two G -orbits.*

Proof. This is the only case where $G \neq H$ and $A(x)=A'(x)$. Compare [10] page 264.

6. Combinatorics

6.1. Let $\text{Pai}(n, q)$ denote the set of partitions π such that $\pi_j \equiv 1$ if $j \leq q$, and $\pi_j \equiv 0$ if $j > q$. We define the set $\text{Pai}(n)$ to be the (clearly disjoint) union of the sets $\text{Pai}(n, q)$ with q admissible, cf. (4.6). We identify the set $\text{Lev}(n)$ of (4.6) with $\text{Pai}(n)$ using the bijection $(v; q) \mapsto \pi$ given by

$$\pi_j = 2v_j + 1 \quad \text{if } j \leq q, \quad \pi_j = 2v_j \quad \text{if } j > q. \quad (*)$$

So we write $(v; q) = \pi$ if $(*)$ holds.

Remark. If P is a parabolic group with flag type $(p_1, \dots, p_s)$, cf. (4.4), we have thus identified its Levi type $(v; q)$ with the partition $\pi = \text{ord}(p_1, \dots, p_s)$.

6.2. The set $\text{Pan}(n, \varepsilon)$ is defined in (5.1). We define the Spaltenstein mapping $S: \text{Pai}(n) \rightarrow \text{Pan}(n, \varepsilon)$ as follows. If $\pi \in \text{Pai}(n)$, put

$$I(\pi) = \{j \in \mathbb{N} \mid j \not\equiv n, \pi_j \equiv \varepsilon, \pi_j \geq \pi_{j+1} + 2\}$$

and let the partition $\lambda = S(\pi)$ be given by

$$\begin{aligned} \lambda_j &= \pi_j - 1 & \text{if } j \in I(\pi), \\ \lambda_j &= \pi_j + 1 & \text{if } j - 1 \in I(\pi), \\ \lambda_j &= \pi_j & \text{otherwise.} \end{aligned}$$

It is clear that λ is a partition with $|\lambda| = n$. In (6.4) we verify that $\lambda \in \text{Pan}(n, \varepsilon)$.

6.3. Lemma. *Let $\pi \in \text{Pai}(n, q)$, q admissible, and $\lambda = S(\pi)$.*

(a) *Assume $\lambda_j \neq \pi_j$. Then $\lambda_j = \pi_j + (-1)^{n-j}$. We have $j \leq q$ if $\varepsilon = 1$, and $j > q$ if $\varepsilon = 0$. In either case $\pi_j \equiv \varepsilon \neq \lambda_j$.*

(b) *$\# \{j | \lambda_j \equiv 1\} = q + 2(-1)^{\varepsilon} \# I(\pi)$.*

(c) *If $j \neq n$ and $\lambda_j > \lambda_{j+1}$ then $\lambda_j \equiv \lambda_{j+1} \neq \varepsilon$; if $n \equiv 1$ then $\lambda_1 \neq \varepsilon$.*

Proof. (a) Since $q \equiv n$ we have $q \notin I(\pi)$. (b) follows from (a).

(c) Let $j \neq n$ and $\lambda_j > \lambda_{j+1}$. It follows from (a) that $\pi_j > \pi_{j+1}$. Since $j \neq q$ we have $\pi_j \equiv \pi_{j+1}$, so that $\pi_j \geq \pi_{j+1} + 2$. If $\pi_j \equiv \varepsilon$ then $j \in I(\pi)$ and the assertion follows. If $\pi_j \neq \varepsilon$ then $j \notin I(\pi)$ and again the assertion follows. Assume $n \equiv 1$. Then $\varepsilon = 0$ by Remark (4.1)(a), and $q \neq 0$. It follows that $\pi_1 \equiv 1$ and that $\lambda_1 = \pi_1$.

6.4. For a partition λ with $|\lambda| = n$ we consider the condition $C(\lambda)$: if $j \neq n$ then $\lambda_j \equiv \lambda_{j+1}$.

Lemma. *The assertion (6.3)(c) holds if and only if $C(\lambda)$ holds and $\lambda \in \text{Pan}(n, \varepsilon)$.*

The proof may be left to the reader.

6.5. For a partition λ with $|\lambda| = n$ we define

$$J(\lambda) = \{j | \lambda_j \equiv \varepsilon\} \cup \{j, j+1 | j \equiv n, \lambda_j = \lambda_{j+1}\},$$

$$j_1(\lambda) = \sup \{j \in J(\lambda) | \lambda_j \equiv 1\} \quad (\text{possibly } -\infty),$$

$$j_0(\lambda) = \min \{j \in J(\lambda) | \lambda_j \equiv 0\}.$$

Proposition. *Let q be an admissible integer and $\lambda \in \text{Pan}(n, \varepsilon)$.*

(a) *The restriction $S: \text{Pai}(n, q) \rightarrow \text{Pan}(n, \varepsilon)$ is injective.*

(b) *We have $\lambda \in S(\text{Pai}(n, q))$ if and only if $C(\lambda)$ holds and $j_1(\lambda) \leq q < j_0(\lambda)$.*

(c) *We have $\lambda \in S(\text{Pai}(n))$ if and only if $C(\lambda)$ holds and $j_1(\lambda) < j_0(\lambda)$.*

Proof. (a) follows from (6.3)(a).

(b) Assume $\lambda = S(\pi)$ with $\pi \in \text{Pai}(n, q)$. Condition $C(\lambda)$ holds by (6.3)(c) and (6.4). If $j \equiv n$ and $\lambda_j = \lambda_{j+1}$, then $\lambda_j \geq \pi_j \geq \pi_{j+1} \geq \lambda_{j+1}$ so that $\lambda_j = \pi_j$ and $\lambda_{j+1} = \pi_{j+1}$. Together with (6.3)(a) this proves that $\lambda_j = \pi_j$ whenever $j \in J(\lambda)$. This implies $j_1(\lambda) \leq q < j_0(\lambda)$.

Assume that $C(\lambda)$ holds and that $j_1(\lambda) \leq q < j_0(\lambda)$. Let π_* be the unique sequence in $\mathbb{Z}$ with $\pi_j = \lambda_j - (-1)^{n-j} \delta_j$ where $\delta_j \in \{0, 1\}$ such that $\pi_j \equiv 1$ if and only if $j \leq q$. The inequality implies that $\pi_j = \lambda_j$ whenever $j \in J(\lambda)$.

If $\lambda_j \geq \lambda_{j+1} + 2$ then $\pi_j \geq \pi_{j+1}$. If $\lambda_j = \lambda_{j+1} + 1$, then $j \in J(\lambda)$ or $j+1 \in J(\lambda)$, so that $\pi_j \geq \pi_{j+1}$. If $j \equiv n$ and $\lambda_j = \lambda_{j+1}$, then $j \in J(\lambda)$ and $j+1 \in J(\lambda)$, so that $\pi_j = \pi_{j+1}$. If $j \neq n$ then $\pi_j \geq \lambda_j \geq \lambda_{j+1} \geq \pi_{j+1}$. So the sequence π_* is non-increasing. If $\lambda_j = 0$ then $\pi_i = 0$ for all $i > j$. This proves that π_* defines a partition π .

If $j \neq n$ then $\lambda_j \equiv \lambda_{j+1}$ and $\pi_j \equiv \pi_{j+1}$, by $C(\lambda)$ and $q \equiv n$. It follows that $\pi_j = \lambda_j + 1$ if and only if $\pi_{j+1} = \lambda_{j+1} - 1$. Using that $n \equiv 1$ implies $\varepsilon = 0$, one proves that $\pi_1 = \lambda_1 - 1$. It follows that $|\pi| = |\lambda| = n$, so that $\pi \in \text{Pai}(n, q)$.

Moreover it now suffices to prove that $j \in I(\pi)$ if and only if $\pi_j = \lambda_j + 1$. By the last paragraph, $\pi_j = \lambda_j + 1$ implies $j \in I(\pi)$. Assume $j \in I(\pi)$ and $\pi_j \neq \lambda_j + 1$. We have $j \neq n$ and $\pi_j \equiv \varepsilon$ and $\pi_j \geq \pi_{j+1} + 2$ and $\pi_j = \lambda_j$ and $\pi_{j+1} = \lambda_{j+1}$. Condition (6.3)(c) holds by (6.4), so that $\pi_j = \lambda_j \neq \varepsilon$, a contradiction.

(c) Assume that $C(\lambda)$ holds and that $j_1(\lambda) < j_0(\lambda)$. It remains to show the existence of an admissible integer q with $j_1(\lambda) \leq q < j_0(\lambda)$. If $\varepsilon = 1$ then $n \equiv 0$ and either $j_1(\lambda) = -\infty$ or $j_1(\lambda) \equiv n$ by (6.3)(c), so that we may use $q = 0$ or $q = j_1(\lambda)$ respectively. Assume $\varepsilon = 0$. By (6.3)(c) we have $j_0(\lambda) \not\equiv n$. If $j_0(\lambda) \neq 3$ we may use $q = j_0(\lambda) - 1$. If $j_0(\lambda) = 3$ then $n \equiv 0$ and $j_1(\lambda) = -\infty$ so that we may use $q = 0$.

6.6. Lemma. Assume $\varepsilon = 0$. Let $(v; q) = \pi \in \text{Pai}(n)$ and $\lambda = S(\pi)$.

(a) The set $B(\lambda)$ of (5.3) is empty if and only if $q = 0$ and $\lambda = \pi$, if and only if $q = 0$ and $v^i \equiv 0$ for all i .

(b) We have $q > 0$ if and only if there is $j \in B(\lambda)$ such that $j \notin I(\pi)$ and $j - 1 \notin I(\pi)$.

The proof is left to the reader.

7. The Main Theorem

7.1. Let $x \in \mathfrak{g}$ be nilpotent with partition λ , cf. (5.1). We use the description of its disconnected centralizer $A(x)$ given in (5.3). Let $\pi = (v; q)$ be an element of $\text{Pai}(n)$, cf. (6.1). Let $\text{Pol}(x, \pi)$ denote the set of polarizations P of x with Levi type $(v; q)$. The Spaltenstein mapping S is defined in (6.2) and analysed in (6.5). If $\lambda = S(\pi)$, we put

$$u = \# I(\pi) = \frac{1}{2}(-1)^\varepsilon(-q + \# \{j | \lambda_j \equiv 1\}), \quad \text{cf. (6.3)(b).}$$

Theorem. (a) $\text{Pol}(x, \pi)$ is non-empty if and only if $\lambda = S(\pi)$.

(b) If $P \in \text{Pol}(x, \pi)$ its stabilizer $A_P(x)$ in $A(x)$ consists of the elements $a \in A(x)$ with $a_j = a_{j+1}$ whenever $j \in I(\pi)$.

(c) Levi factors of elements of $\text{Pol}(x, \pi)$ are conjugate in G .

(d) If $\lambda = S(\pi)$ then $\text{Pol}(x, \pi)$ consists of N_0 conjugacy classes under $A(x)$ each consisting of N_1 polarizations where

$$N_0 = N(v) \quad \text{and} \quad N_1 = 2^u \quad \text{if} \quad q + \varepsilon \geq 1 \quad \text{or} \quad B(\lambda) \quad \text{is empty.}$$

$$N_0 = (1 + v_1^{-1} v_2) N(v) \quad \text{and} \quad N_1 = 2^{u-1} \quad \text{if} \quad q = \varepsilon = 0 \quad \text{and} \quad B(\lambda) \neq \emptyset.$$

7.2. Lemma. Let P be a parabolic subgroup with Levi type $(v; q) = \pi$. Assume that $\lambda = S(\pi)$. Then $\dim(Gx) = 2 \dim(G/P)$.

Proof. Let L be a Levi factor of P . By (2.3), (2.4) and (4.6) we have $2 \dim(G/P) = 2b(G) - 2b(L)$ where

$$\begin{aligned} 2b(L) &= \sum v^i(v^i - 1) + \frac{1}{2}q(q-1) - (-1)^\varepsilon \left[\frac{1}{2}q \right] \\ &= \sum \frac{1}{2} \pi^i(\pi^i - 1) - (-1)^\varepsilon \left[\frac{1}{2}q \right]. \end{aligned}$$

By [5] (3.8) we have $\dim(Gx) = 2b(G) - \gamma_\varepsilon(\lambda)$ where

$$\gamma_\varepsilon(\lambda) = \sum (j-1)\lambda_j - (-1)^\varepsilon \left[\frac{1}{2} \# \{j | \lambda_j \equiv 1\} \right].$$

Now the result follows from (2.1) and (6.3)(b).

7.3. Lemma. Assume that $\lambda = S(\pi)$. Let $(p_1, \dots, p_s)$ be a sequence in $\mathbb{N}$ with $\pi = \text{ord}(p_1, \dots, p_s)$ and $p_i = p_{s+1-i}$ for $1 \leq i \leq s$. Then x has a polarization P with flag type $(p_1, \dots, p_s)$.

Proof. It suffices to construct an isotropic flag $F = (F_0, \dots, F_s)$ of type $(p_1, \dots, p_s)$ such that $x F_i \subset F_{i-1}$ for $1 \leq i \leq s$. In fact by the assumption $\pi \in \text{Pai}(n)$ the flag F will be admissible. So its stabilizer P in G is parabolic with flag type $(p_1, \dots, p_s)$. The condition $x F_i \subset F_{i-1}$ implies that $x \in \mathfrak{u}(P)$, so P is a polarization of x by (7.2).

Assume $s=1$. Then $\pi_j \leq 1$ for all j . So $I(\pi)$ is empty and $\lambda = \pi$ and $x=0$. We choose $F_0=0$, $F_1=V$.

Assume $s>1$. Put $p=p_1=p_s$. The type $(p_2, \dots, p_{s-1})$ of the required isotropic flag $(F_i/F_1)_{0 < i < s}$ in F_{s-1}/F_1 gives rise to a partition $\rho \in \text{Pai}(n-2p, q)$ given by

$$\rho_j = \# \{i | 2 \leq i \leq s-1, p_i \geq j\},$$

so that $\rho_j = \pi_j - 2$ if $j \leq p$, and $\rho_j = \pi_j$ if $j > p$.

Let $S: \text{Pai}(n-2p) \rightarrow \text{Pan}(n-2p, \varepsilon)$ be the Spaltenstein mapping, cf. (6.2). Put $\mu = S(\rho)$. There are two cases.

(A) $I(\pi) = I(\rho)$. Then $p \equiv n$ or $\pi_p \not\equiv \varepsilon$ or $\pi_p \geq \pi_{p+1} + 4$. We have $\mu_j = \lambda_j - 2$ if $j \leq p$, and $\mu_j = \lambda_j$ if $j > p$. Note that $\lambda_p > \lambda_{p+1}$ since μ is a partition.

(B) $I(\pi) = \{p\} \cup I(\rho)$ and $p \notin I(\rho)$. Then $p \not\equiv n$, $\pi_p \equiv \varepsilon$ and $\pi_p = \pi_{p+1} + 2$. We have $\mu_j = \lambda_j - 2$ if $j < p$, and $\mu_j = \lambda_j$ if $j > p+1$, and $\mu_p = \lambda_p - 1$, $\mu_{p+1} = \lambda_{p+1-1}$. Moreover $\lambda_p = \lambda_{p+1} \not\equiv \varepsilon$ and $\lambda_{p+1} > \lambda_{p+2}$.

We choose a normalized Jordan basis $e(i, j)$, $(i, j) \in \lambda$, of V with respect to x , cf. (5.1). In case (B) we require moreover that $\beta(p) = p+1$. We put $F_0=0$, $F_s=V$. In case (A) we define $F_1 = \sum_{j=1}^p k e(1, j)$. In case (B) we have two possibilities

$$F_1 = \sum_{j=1}^p k e(1, j) \quad \text{or} \quad F_1 = \sum_{j=1, j \neq p}^{p+1} k e(1, j)$$

In both cases we put $F_{s-1} = F_1^\perp$.

One verifies that $x F_1 = 0$ and hence $x F_s \subset F_{s-1}$, that F_1 is contained in F_{s-1} , and that the induced endomorphism x_1 of F_{s-1}/F_1 has partition μ . We proceed by induction.

7.4. Lemma. Let P be the polarization of x constructed in (7.3). The stabilizer $A_P(x)$ consists of the elements $a \in A(x)$ with $a_j = a_{j+1}$ whenever $j \in I(\pi)$.

Proof. Let F be the flag constructed in (7.3). The group $Z_H(x)^0$ is contained in P and therefore it leaves F invariant. So we may define the stabilizer $A'_F(x)$ of F in $A'(x)$. We have $A_P(x) = A(x) \cap A'_F(x)$.

It suffices to prove that $a \in A'(x)$ leaves F invariant if and only if $a_j = a_{j+1}$ whenever $j \in I(\pi)$. The case $s=1$ is trivial since $I(\pi)$ is empty and $A'(x) = A'_F(x)$. We proceed by induction as in (7.3). Clearly any element of $A'(x)$ leaves F_0 and F_s invariant.

Assume $s>1$. Put $p=p_1=p_s$ and $h=\lambda_p$. In case (A) we have $F_1 = M_h$, cf. (5.2). In case (B) the space F_1 is one of the two hyperplanes in M_h with $\varphi_h|_{F_1} = 0$. It follows that $a \in A'(x)$ leaves F_1 and F_{s-1} invariant if and only if case (A) applies or $a_{p+1} = 0$ and case (B) applies.

Assume that $a \in A'(x)$ leaves F_1 and F_{s-1} invariant. Then a has an image $a' \in A'(x_1)$ where x_1 is the induced endomorphism of F_{s-1}/F_1 . The element a leaves F invariant if and only if a' leaves the induced flag $(F_i/F_1)_{0 < i < s}$ in F_{s-1}/F_1 invariant. By induction this is the case if and only if $a'_j = a'_{j+1}$ whenever $j \in I(\rho)$.

Case (A₁). $I(\pi) = I(\rho)$ and $B(\lambda) = B(\mu)$, so that $\lambda_p \equiv \varepsilon$ or $\lambda_p > \lambda_{p+1} + 2$. After the identifications of (5.3) the map $A'(x) \rightarrow A'(x_1)$ is the identity. One verifies this using the basis of Remark (5.3). Now the assertion follows by induction.

Case (A₂). $I(\pi) = I(\rho)$ and $B(\lambda) = \{p\} \cup B(\mu)$ where $p \notin B(\mu)$. So $\lambda_p \not\equiv \varepsilon$ and $\lambda_p = \lambda_{p+1} + 2$. First assume that $\lambda_p > 2$. We have a unique $i > p$ such that $\mu_p = \mu_{p+1} = \mu_i > \mu_{i+1}$. Moreover $i \in B(\mu)$. In the notation of Remark (5.3) we have that the images of $r(p)$ and $r(i)$ in $A'(x_1)$ are both equal to $r'(i)$. It follows that $a' \in A'(x_1)$ is given by $a'_j = a_j$ if $j \notin \{p, i\}$ and $a'_p = 0$ and $a'_i = a_p + a_i$. Now it suffices to verify that $a_j - a_{j+1} = a'_j - a'_{j+1}$ whenever $j \in I(\rho)$. In fact $I(\pi) = I(\rho)$, so we may proceed by induction. Let $j \in I(\rho)$. Then we have $\mu_j \geq \mu_{j+1} > \mu_{j+2}$ and $\mu_{j-1} > \mu_j$ if $j > 1$. It follows that $j \leq p-2$, or $j = p-1$, or $j > i$. Since the coordinates are in $\mathbb{F}_2$ the required verification is trivial. If $\lambda_p = 2$ the proof is similar.

Case (B). $I(\pi) = \{p\} \cup I(\rho)$ and $p \notin I(\rho)$. Then we have $p \notin B(\lambda)$, so $a_p = 0$ and the element $a \in A'(x)$ leaves F_1 and F_{s-1} invariant if and only if $a_p = a_{p+1}$. Assume $a_p = a_{p+1} = 0$. Again using Remark (5.3) one verifies that $a' = a$. If $j \in I(\pi)$ and $j \neq p$ then $j \leq p-2$ or $j \geq p+2$. So the induction is immediate.

7.5. The proof of (7.1) and the verification of (1.2) in our case.

Since H acts transitive on the set of parabolic subgroups of G with a given flag type, cf. Lemma (4.4), Lemma (7.3) implies that every parabolic subgroup of G admits Richardson elements, cf. (1.2)(a). The same argument shows that every polarization of x is obtained by a construction as in (7.3), compare the proof of (1.3). This proves (7.1)(a). Now Lemma (7.4) implies (7.1)(b).

By (4.6)(c), the assertion (7.1)(c) is only then non-trivial if $\lambda = S(\pi)$ and $q = \varepsilon = 0$ and $v^i \equiv 0$ for all i . Then $B(\lambda)$ is empty, cf. (6.6), so that the H -orbit of x splits into two G -orbits, cf. (5.4). Let $(p_1, \dots, p_s)$ be as in (7.3). Put $t = \frac{1}{2}s = v_1$. The construction of (7.3) yields that F_t is generated by the vectors $e(i, j)$ with $i \leq \frac{1}{2}\lambda_j$. So we have

$$F_t = \sum_m x^m \text{Ker}(x^{2m})$$

Since this space is independent of the choice of $(p_1, \dots, p_s)$ the result (7.1)(c) follows easily. Now one may verify (1.2)(b).

Assume that $\lambda = S(\pi)$. By (1.2) the number N_0 of (7.1)(d) can be obtained from (4.6) taking into account the splitting of the Levi type, cf. (4.6) and (6.6)(a). The number N_1 is equal to the index of $A_p(x)$ in $A(x)$. So we have $N_1 = 2^v$ where v is the dimension of the vector space $A(x)/A_p(x)$. If $j \in I(\pi)$ then $j \not\equiv n$ and $j+1 \in B(\lambda)$. This implies that

$$\dim(A'(x)/A'_p(x)) = \# I(\pi) = u$$

in the notations of (7.4). It follows that

$$v \neq u \Leftrightarrow v = u - 1 \Leftrightarrow A'_F(x) \subset A(x) \neq A'(x).$$

We have $A(x) = A'(x)$ if and only if $\varepsilon = 1$ or $B(\lambda)$ is empty. So assume that $\varepsilon = 0$ and $B(\lambda)$ is non-empty. Using (6.6)(b) one verifies that $A'_F(x)$ is not contained in $A(x)$ if and only if $q > 0$.

7.6. Corollary. *For $i = 1, 2$ let $P(i)$ be a polarization of x , say with Levi factor $L(i)$. Let $N_1(i)$ be the number of polarizations of x conjugate to $P(i)$ under $Z_G(x)$. The following conditions are equivalent.*

- (a) $L(1)$ and $L(2)$ are conjugate in G .
- (b) $A_{P(1)}(x) = A_{P(2)}(x)$.
- (c) $N_1(1) = N_1(2)$.

Proof. (a) $\Rightarrow$ (b) by (7.1)(b). The implication (b) $\Rightarrow$ (c) is trivial.

Assume that (c) holds. Write $N_1(1) = N_1(2) = 2^n$. Let $\pi(i) = (v(i); q(i))$ be the Levi type of $P(i)$, so that $\lambda = S(\pi(1)) = S(\pi(2))$. Put $u(i) = \# I(\pi(i))$. By (7.1)(d) we have

$$\begin{aligned} u(i) &= v & \text{if } q(i) + \varepsilon \geq 1 \text{ or } B(\lambda) = \emptyset, \\ u(i) &= v + 1 & \text{if } q(i) = \varepsilon = 0 \text{ and } B(\lambda) \neq \emptyset. \end{aligned}$$

Suppose that $q(1) \neq q(2)$, say $q(1) < q(2)$. By (6.3)(b) we have

$$u(1) - u(2) = \frac{1}{2}(-1)^\varepsilon(q(2) - q(1)) \neq 0.$$

It follows that $\varepsilon = 0$ and $q(1) = 0$ and $u(1) - u(2) = 1$, so that $q(2) = 2$, contradicting the fact that $q(2)$ should be admissible, cf. (6.1). This proves that $q(1) = q(2)$. By (6.5)(a) it follows that $\pi(1) = \pi(2)$ so that (a) follows from (7.1)(c).

7.7. A polarization P of x is called *stable* if P contains $Z_G(x)$. This concept seems to be important in the applications, cf. [2] (6.7)(3).

Corollary. *The element x has stable polarizations if and only if either (i) $\lambda \in \text{Pai}(n)$, or (ii) $n \equiv \varepsilon = 0$ and $\{j | \lambda_j \equiv 1\} = \{r, r+1\}$ for some $r \equiv 1$. Assume (i) or (ii) holds. The stable polarizations of x are the N_0 elements of $\text{Pol}(x, \pi)$ where in case (i) we have $\pi = \lambda$, and in case (ii) we have $\pi_j = \lambda_j$ if $j \notin \{r, r+1\}$ and $\pi_r = \lambda_r + 1$ and $\pi_{r+1} = \lambda_{r+1} - 1$.*

The proof is left to the reader. Note that the conditions (i) and (ii) are mutually exclusive.

Remark. If x is an even nilpotent element, cf. [10] page 241, the parabolic subgroup P considered in loc.cit. is a stable polarization of x . The Levi type of P and hence of all stable polarizations of x , can be deduced from the weighted Dynkin diagram of x , cf. loc.cit. page 243, 263 and 264.

7.8. Corollary. *Assume $n \not\equiv \varepsilon$ (i.e. G is of type B_l or C_l).*

- (a) *The polarizations of x have Levi factors with the same semisimple rank.*
- (b) *If P_1 and P_2 are parabolic groups such that their Levi factors have the same Dynkin diagram, then their Richardson elements are conjugate.*

The proof is left to the reader. The table for D_5 in (7.9) clearly shows that both assertions are false for type D_l .

7.9. Tables. A conjugacy class of Levi factors is characterized by the Dynkin type in terms of A_l, B_l, C_l, D_l . The symbols B_1, C_1, D_2, D_3 are used to distinguish certain parts of the Dynkin diagram of G , though formally $B_1 = C_1 = A_1$, $D_2 = A_1 + A_1$, $D_3 = A_3$. The same information is carried by the Levi type $(v; q)$. The partition v is given by the sequence $v^* = (v^1, \dots, v^t)$ where $v^t > 0 = v^{t+1}$.

The numbers N_0 and N_1 are meant conform (1.4) and (7.1). The partition $\lambda = S(\pi)$ of a Richardson element x is given by the sequence $\lambda_* = (\lambda_1, \dots, \lambda_m)$ where $\lambda_m > 0 = \lambda_{m+1}$. The groups $A(x)$ and $A_p(x)$ are determined by the sets $B(\lambda)$ and $I(\pi)$, cf. (5.3) and (7.1). These sets are given in the tables.

We give the weighted Dynkin diagram of x , in view of remark (7.7). If the H -orbit of x or equivalently the Levi type, splits into two conjugacy classes under the action of G , we write $2 \times N_0$ in the column of N_0 (so that this column adds up to 2^l). In this case we give the Dynkin diagram of one of the G -orbits. We indicate how to get the other diagram by the symbol $\curvearrowright$. A partition λ such that the corresponding nilpotent elements have no polarizations at all, is equipped with a dot and dash line in the column of (v^*, q) .

Tabelle 1

$B_2 = \text{SO}(5)$							
Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*
0	(1 1; 1)	1	1	$\emptyset$	1	5	2 2
B_1	(1; 3)	1	1	$\emptyset \}$	1 3	3 1 1	2 0
A_1	(2; 1)	1	2	2 }			
	— · — · —				3	2 2 1	0 1
B_2	(—; 5)	1	1	$\emptyset$	5	1^5	0 0
$C_2 = \text{Sp}(4)$							
Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*
0	(1 1; 0)	1	1	$\emptyset$	1	4	2 2
A_1	(2; 0)	1	1	$\emptyset \}$	2	2 2	0 2
C_1	(1; 2)	1	2	1 }			
	— · — · —				1	2 1 1	1 0
C_2	(—; 4)	1	1	$\emptyset$	$\emptyset$	1^4	0 0
$B_3 = \text{SO}(7)$							
Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*
0	(1 1 1; 1)	1	1	$\emptyset$	1	7	2 2 2
B_1	(1 1; 3)	1	1	$\emptyset \}$	1 3	5 1 1	2 2 0
A_1	(2 1; 1)	2	2	2 }			
$A_1 + B_1$	(2; 3)	1	1	$\emptyset$	2 3	3 3 1	0 2 0
A_2	(3; 1)	1	1	$\emptyset$	1	3 2 2	1 0 1
B_2	(1; 5)	1	1	$\emptyset$	1 5	3 1^4	2 0 0
	— · — · —				5	2 2 1^3	0 1 0
B_3	(—; 7)	1	1	$\emptyset$	7	1^7	0 0 0

Table 1 (continued)

 $C_3 = \text{Sp}(6)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	$\bigcirc - \bigcirc - \bigcirc - \bigcirc$
0	(1 1 1; 0)	1	1	$\emptyset$	1	6	2 2 2
A_1	(2 1; 0)	2	1	$\emptyset \}$	1 2	4 2	2 0 2
C_1	(1 1; 2)	1	2	1 }			
	-----				1	4 1 1	2 1 0
$A_1 + C_1$	(2; 2)	1	1	$\emptyset$	$\emptyset$	3 3	0 2 0
A_2	(3; 0)	1	1	$\emptyset$	3	2 2 2	0 0 2
C_2	(1; 4)	1	2	1	2	2 2 1 1	0 1 0
	-----				1	2 1 ⁴	1 0 0
C_3	(-; 6)	1	1	$\emptyset$	6	1 ⁶	0 0 0

 $B_4 = \text{SO}(9)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	$\bigcirc - \bigcirc - \bigcirc - \bigcirc - \bigcirc$
0	(1 1 1 1; 1)	1	1	$\emptyset$	1	9	2 2 2 2
B_1	(1 1 1; 3)	1	1	$\emptyset \}$	1 3	7 1 1	2 2 2 0
A_1	(2 1 1; 1)	3	2	2 }			
$A_1 + B_1$	(2 1; 3)	2	1	$\emptyset \}$	1 2 3	5 3 1	2 0 2 0
$A_1 + A_1$	(2 2; 1)	1	2	2 }			
A_2	(3 1; 1)	2	1	$\emptyset$	1	5 2 2	2 1 0 1
B_2	(1 1; 5)	1	1	$\emptyset$	1 5	5 1 ⁴	2 2 0 0
	-----				3	4 4 1	0 2 0 1
$A_2 + B_1$	(3; 3)	1	1	$\emptyset$	3	3 3 3	0 0 2 0
$A_1 + B_2$	(2; 5)	1	1	$\emptyset$	2 5	3 3 1 ³	0 2 0 0
A_3	(4; 1)	1	2	4	1 5	3 2 2 1 1	1 0 1 0
B_3	(1; 7)	1	1	$\emptyset$	1 7	3 1 ⁶	2 0 0 0
	-----				5	2 ⁴ 1	0 0 0 1
	-----				7	2 2 1 ⁵	0 1 0 0
B_4	(-; 9)	1	1	$\emptyset$	9	1 ⁹	0 0 0 0

 $C_4 = \text{Sp}(8)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	$\bigcirc - \bigcirc - \bigcirc - \bigcirc - \bigcirc$
0	(1 1 1 1; 0)	1	1	$\emptyset$	1	8	2 2 2 2
A_1	(2 1 1; 0)	3	1	$\emptyset \}$	1 2	6 2	2 2 0 2
C_1	(1 1 1; 2)	1	2	1 }			
	-----				1	6 1 1	2 2 1 0
$A_1 + A_1$	(2 2; 0)	1	1	$\emptyset \}$	2	4 4	0 2 0 2
$A_1 + C_1$	(2 1; 2)	2	2	1 }			
A_2	(3 1; 0)	2	1	$\emptyset$	1 3	4 2 2	2 0 0 2
C_2	(1 1; 4)	1	2	1	1 2	4 2 1 1	2 0 1 0
	-----				1	4 1 ⁴	2 1 0 0
$A_2 + C_1$	(3; 2)	1	1	$\emptyset$	3	3 3 2	0 1 1 0
$A_1 + C_2$	(2; 4)	1	1	$\emptyset$	$\emptyset$	3 3 1 1	0 2 0 0
A_3	(4; 0)	1	1	$\emptyset$	4	2 ⁴	0 0 0 2
	-----				3	2 ³ 1 1	0 0 1 0
C_3	(1; 6)	1	2	1	2	2 2 1 ⁴	0 1 0 0
	-----				1	2 1 ⁶	1 0 0 0
C_4	(-; 8)	1	1	$\emptyset$	$\emptyset$	1 ⁸	0 0 0 0

Table 1 (continued)

 $D_4 = \text{SO}(8)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*
0	$(1 \ 1 \ 1 \ 1; 0)$	1	1	1	1 2	7 1	2 2 2 2
A_1	$(2 \ 1 \ 1; 0)$	4	1	1	1 2	5 3	2 0 2 2
$A_1 + A_1$	$(2 \ 2; 0)$	2×1	1	$\emptyset$	$\emptyset$	4 4	0 2 $2 \curvearrowright 0$
D_2	$(1 \ 1; 4)$	1	1	$\emptyset$	1 4	5 1 1 1	2 2 0 0
$A_1 + D_2$	$(2; 4)$	1	1	$\emptyset$	2 4	3 3 1 1	0 2 0 0
A_2	$(3 \ 1; 0)$	3	2	1 3			
---	---	---	---	---	1 4	3 2 2 1	1 0 1 1
A_3	$(4; 0)$	2×1	1	$\emptyset$	$\emptyset$	2 2 2 2	0 0 $2 \curvearrowright 0$
D_3	$(1; 6)$	1	1	$\emptyset$	1 6	3 1 ⁵	2 0 0 0
---	---	---	---	---	6	2 2 1 ⁴	0 1 0 0
D_4	$(-; 8)$	1	1	$\emptyset$	8	1 ⁸	0 0 0 0

 $D_5 = \text{SO}(10)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*
0	$(1^5; 0)$	1	1	1	1 2	9 1	2 2 2 2 2
A_1	$(2 \ 1 \ 1 \ 1; 0)$	5	1	1	1 2	7 3	2 2 0 2 2
D_2	$(1 \ 1 \ 1; 4)$	1	1	$\emptyset$	1 4	7 1 1 1	2 2 2 0 0
$A_1 + A_1$	$(2 \ 2 \ 1; 0)$	5	1	1	2	5 5	0 2 0 2 2
$A_1 + D_2$	$(2 \ 1; 4)$	2	1	$\emptyset$	1 2 4	5 3 1 1	2 0 2 0 0
A_2	$(3 \ 1 \ 1; 0)$	4	2	1 3			
---	---	---	---	---	1 4	5 2 2 1	2 1 0 1 1
D_3	$(1 \ 1; 6)$	1	1	$\emptyset$	1 6	5 1 ⁵	2 2 0 0 0
$A_1 + A_2$	$(3 \ 2; 0)$	4	1	3	4	4 4 1 1	0 2 0 1 1
$A_2 + D_2$	$(3; 4)$	1	1	$\emptyset$	3 4	3 3 3 1	0 0 2 0 0
A_3	$(4 \ 1; 0)$	3	1	1	2	3 3 2 2	0 1 0 1 1
$A_1 + D_3$	$(2; 6)$	1	1	$\emptyset$	2 6	3 3 1 ⁴	0 2 0 0 0
---	---	---	---	---	1 6	3 2 2 1 ³	1 0 1 0 0
D_4	$(1; 8)$	1	1	$\emptyset$	1 8	3 1 ⁷	2 0 0 0 0
A_4	$(5; 0)$	2	1	5	6	2 ⁴ 1 1	0 0 0 1 1
---	---	---	---	---	8	2 2 1 ⁶	0 1 0 0 0
D_5	$(-; 10)$	1	1	$\emptyset$	10	1 ¹⁰	0 0 0 0 0

 $D_6 = \text{SO}(12)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*
0	$(1^6; 0)$	1	1	1	1 2	11, 1	2 2 2 2 2 2
A_1	$(2 \ 1^4; 0)$	6	1	1	1 2	9 3	2 2 2 0 2 2
D_2	$(1^4; 4)$	1	1	$\emptyset$	1 4	9 1 1 1	2 2 2 2 0 0
$A_1 + A_1$	$(2 \ 2 \ 1 \ 1; 0)$	9	1	1	1 2	7 5	2 0 2 0 2 2
$A_1 + D_2$	$(2 \ 1 \ 1; 4)$	3	1	$\emptyset$	1 2 4	7 3 1 1	2 2 0 2 0 0
A_2	$(3 \ 1 \ 1 \ 1; 0)$	5	2	1 3			
---	---	---	---	---	1 4	7 2 2 1	2 2 1 0 1 1
D_3	$(1 \ 1 \ 1; 6)$	1	1	$\emptyset$	1 6	7 1 ⁵	2 2 2 0 0 0
$3A_1$	$(2 \ 2 \ 2; 0)$	2×1	1	$\emptyset$	$\emptyset$	6 6	0 2 0 2 $2 \curvearrowright 0$
$2A_1 + D_2$	$(2 \ 2; 4)$	2	1	$\emptyset$	2 4	5 5 1 1	0 2 0 2 0 0
$A_1 + A_2$	$(3 \ 2 \ 1; 0)$	10	2	1 3			

Table 1 (continued)

$A_2 + D_2$	(3 1; 4)	2	1	$\emptyset$	1 3 4	5 3 3 1	2 0 0 2 0 0
A_3	(4 1 1; 0)	4	1	1	1 2	5 3 2 2	2 0 1 0 1 1
$A_1 + D_3$	(2 1; 6)	2	1	$\emptyset$	1 2 6	5 3 1 ⁴	2 0 2 0 0 0
	— — — —				1 6	5 2 2 1 ³	2 1 0 1 0 0
D_4	(1 1; 8)	1	1	$\emptyset$	1 8	5 1 ⁷	2 2 0 0 0 0
$A_2 + A_2$	(3 3; 0)	2	1	3	3 4	4 4 3 1	0 1 1 0 1 1
$A_3 + A_1$	(4 2; 0)	2×2	1	$\emptyset$	$\emptyset$	4 4 2 2	0 2 0 0 2 $\curvearrowright$ 0
	— — — —				6	4 4 1 ⁴	0 2 0 1 0 0
$A_3 + D_2$	(4; 4)	1	1	$\emptyset$	4	3 3 3 3	0 0 0 2 0 0
$A_2 + D_3$	(3; 6)	1	1	$\emptyset$	3 6	3 ³ 1 ³	0 0 2 0 0 0
A_4	(5 1; 0)	3	2	1 5	2 6	3 3 2 2 1 1	0 1 0 1 0 0
$A_1 + D_4$	(2; 8)	1	1	$\emptyset$	2 8	3 3 1 ⁶	0 2 0 0 0 0
	— — — —				1 6	3 2 2 2 2 1	1 0 0 0 1 1
	— — — —				1 8	3 2 2 1 ⁵	1 0 1 0 0 0
D_5	(1; 10)	1	1	$\emptyset$	1, 10	3 1 ⁹	2 0 0 0 0 0
A_5	(6; 0)	2×1	1	$\emptyset$	$\emptyset$	2 ⁶	0 0 0 0 2 $\curvearrowright$ 0
	— — — —				8	2 ⁴ 1 ⁴	0 0 0 1 0 0
	— — — —				10	2 2 1 ⁸	0 1 0 0 0 0
D_6	(—; 12)	1	1	$\emptyset$	12	1 ¹²	0 0 0 0 0 0

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