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Published in:
IEEE Transactions on Automatic Control

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1982

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
Nijmeijer, H., & Schaft, A. V. D. (1982). Controlled Invariance for Nonlinear Systems. *IEEE Transactions on Automatic Control*.

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Controlled Invariance for Nonlinear Systems

HENK NIJMEIJER AND ARJAN VAN DER SCHAFT

Abstract—Necessary and sufficient conditions are derived for “ (A, B) -invariance,” called here *controlled invariance*, for nonlinear systems $\dot{x} = f(x, u)$. The obtained results generalize and elucidate already known results about systems $\dot{x} = A(x) - \sum_{i=1}^m u_i B_i(x)$. A new and direct differential geometric interpretation of the concept of controlled invariance and the derived conditions is given.

I. INTRODUCTION

BASIC to the solution of various problems in linear systems theory is the notion of (A, B) -invariance, also called *controlled invariance* (cf. [1], [13]). Recently, several people studied the problem of generalizing this notion to nonlinear systems of the form

$$\dot{x} = A(x) + \sum_{i=1}^m u_i B_i(x) \quad (1.1)$$

(cf. [4]–[9]). Also, a related but different notion can be found in [14]. Actually, very recently conditions have been found which seem very conclusive for this class of systems (cf. [6], [9]).

The aim of this paper is to generalize the concept further to general nonlinear systems

$$\dot{x} = f(x, u) \quad (1.2)$$

and to derive conditions similar to those derived for systems of the form (1.1). In the course of doing this, it became clear that the concept of controlled invariance can be translated, in a natural and clarifying way, into classical differential geometric notions like integrability conditions and connections on fiber bundles. Actually, we will show that this point of view also elucidates the already known results about systems of the form (1.1) (we will call these systems *affine systems*).

Before going on, we will briefly summarize some of the ideas and results about controlled invariance for linear and affine systems (for an introduction see also [4], [5], [8]). First, we define the related notion of *invariance*. Consider a linear system

$$\dot{x} = Ax + Bu, \quad x \in \mathcal{X} := \mathbb{R}^n, \quad u \in \mathcal{U} := \mathbb{R}^m. \quad (1.3)$$

Manuscript received March 24, 1981; revised November 25, 1981 and January 4, 1982. Paper recommended by A. N. Michel, Past Chairman of the Stability, Nonlinear, and Distributed Systems Committee.

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We call a linear subspace $\mathcal{V} \subset \mathcal{X}$ *invariant* if $A\mathcal{V} \subset \mathcal{V}$. We can interpret this condition in the following way. The collection of affine subspaces $x + \mathcal{V}$, $x \in \mathbb{R}^n$, can be regarded as the leaves of a foliation of $\mathbb{R}^n$. Then $A\mathcal{V} \subset \mathcal{V}$ is equivalent to saying that the system (1.3) leaves the foliation invariant; i.e., take two arbitrary points x_1 and x_2 on the same leaf and take an arbitrary input function $\bar{u}(\cdot)$; then the integral curves starting from x_1 and x_2 , generated by $\dot{x} = Ax + B\bar{u}$, intersect at every time t the same leaf.

This idea can be generalized to nonlinear systems

$$\dot{x} = f(x, u), \quad x \in M, \quad M \text{ a manifold.} \quad (1.4)$$

Take instead of a linear subspace $\mathcal{V}$ an involutive *distribution* D on M . The maximal integral manifolds of this distribution are the leaves of a foliation of M . Then we say that the distribution D is *invariant* if again for every input function $\bar{u}(\cdot)$, the system $\dot{x} = f(x, \bar{u})$ leaves the foliation invariant.

Actually, it is a standard fact from differential geometry that this condition is, just as in the linear case, equivalent to an *infinitesimal* condition, namely,

$$[f(\cdot, \bar{u}), D] \subset D.$$

(See the end of this section for notation.) Controlled invariance is defined as follows. An involutive distribution D is called *controlled invariant* if there exists a feedback $u \mapsto v := \alpha(x, u)$ such that after applying this feedback, D is invariant with respect to the modified dynamics

$$\dot{x} = \tilde{f}(x, v).$$

D is called *locally controlled invariant* if we can only find a local feedback (see Section III). Within the “category” of linear systems, feedbacks should have the form

$$u \mapsto v := u - Fx$$

and for affine systems

$$u \mapsto v := M(x)u - v(x).$$

The defect in this definition of controlled invariance is that it requires knowledge of the feedback needed. Therefore, conditions should be sought on the distribution D and the system $\dot{x} = f(x, u)$ which ensure the existence of a feedback which makes D invariant. In fact, for linear systems (1.3) it can be easily proven that

$$A\mathcal{V} \subset \mathcal{V} + \text{Im } B$$

is necessary and sufficient for the existence of a matrix F such that $(A + BF)^{\forall} \subset \mathcal{V}$.

Very recently, in [6] and independently in [9] the following result has been proven for affine systems:

$$\dot{x} = A(x) + \sum_{i=1}^m u_i B_i(x).$$

Define the affine distribution Δ by $\Delta(x) := A(x) + \text{span}\{B_1(x), \dots, B_m(x)\}$ and the distribution Δ_0 by $\Delta_0(x) := \text{span}\{B_1(x), \dots, B_m(x)\}$. Then a distribution D is locally controlled invariant iff

$$[\Delta, D] \subset D + \Delta_0$$

(see the end of this section for the notation), where it is supposed that the dimension of $D \cap \Delta_0$ is constant. This last result includes an earlier result in [4].

Finally, in this paper we will give the conditions for controlled invariance for general systems $\dot{x} = f(x, u)$. (See Section IV.)

The outline of the paper is as follows. Section II contains preliminaries about definitions of nonlinear control systems which will open up the way to the definitions of controlled invariance in Section III. It will be argued that a natural concept for local controlled invariance is the idea of an (integrable) connection, which will be dealt with in Section IV. It will be shown here that for affine systems the vanishing of the torsion and the curvature tensor of an affine connection exactly gives the integrability conditions needed for the construction of a feedback. Furthermore, the condition for controlled invariance for general nonlinear systems is derived. Section V contains the Conclusion.

Some Notation

Our basic reference to differential geometry will be [11]. All our objects like manifolds, maps, etc. are C^∞ . We call Δ an affine distribution on a manifold M if Δ in every $x \in M$ is given by an affine subspace $\Delta(x) \subset T_x M$ (in a smooth way). Given two (affine) distributions D_1, D_2 , then we define the distribution

$$[D_1, D_2] := \{[X, Y] \mid X \in D_1, Y \in D_2\}$$

where $[\cdot, \cdot]$ is the Lie bracket. We will only consider the regular case, so distributions will always have constant dimension (see the Conclusion). Given a k -dimensional distribution D on M (n -dimensional with $n \geq k$), we can construct a $2k$ -dimensional distribution on TM , denoted by $\tilde{D}$, in the following way. Define the codistribution P by

$$P(x) = \{\theta \in T_x^* M \mid \theta(X) = 0 \text{ for every } X \in D(x), x \in M\}.$$

Then P has a basis (over the ring of smooth functions on M) of $n-k$ one-forms $\theta_1, \dots, \theta_{n-k}$. Since $\theta_i \in T_x^* M$ we can also consider θ_i as a real function on TM . Now we define $\dot{\theta}_i \in T^* TM$ by

$$\dot{\theta}_i(X) = X(\theta_i), \quad \text{with } X \text{ vector field on } TM.$$

Denote the natural projection from TM onto M by π . Then

also $\pi^* \theta_i \in T^* TM$. Define the codistribution $\dot{P}$ on TM by

$$\dot{P} = \text{span}\{\pi^* \theta_1, \dots, \pi^* \theta_{n-k}, \dot{\theta}_1, \dots, \dot{\theta}_{n-k}\}.$$

Then $\dot{D}$ is defined by

$$\dot{D}(z) = \{X \in T_z TM \mid \theta(X) = 0 \text{ for every } \theta \in \dot{P}(z), z \in TM\}.$$

If D is an involutive distribution we can give the following simple description of $\dot{D}$ in local coordinates. Take coordinates $(x_1, \dots, x_n)$ for M (from now on we shall always assume M to be an n -dimensional manifold) such that

$$D = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right\}, \quad \text{with } k \leq n.$$

Denote the corresponding coordinates for TM by $(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n)$ ($\dot{x}_j: TM \rightarrow \mathbb{R}$ is defined as $\dot{x}_j(v) := dx_j(v)$, for $v \in TM$). Then

$$\dot{D} = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial \dot{x}_1}, \dots, \frac{\partial}{\partial \dot{x}_k} \right\}.$$

II. PRELIMINARIES

Before going to the problem of controlled invariance for general nonlinear systems, we will first review the definitions of nonlinear control systems we shall use henceforth. This new approach was proposed by Willems [12], and elaborated on in [8], [10], and is related to recent proposals of Brockett [2]. In fact, the problem centers around a coordinate-free way of defining the equations

$$\dot{x} = f(x, u) \quad (2.1)$$

where x is the state of the system and u is the input. Usually this is done by looking at (2.1) as a family of globally defined vector fields $f(\cdot, u)$ on the state space manifold, parameterized by u . However, there are serious objections to this definition (cf. [2], [10], [12]) and, moreover, in many cases it happens that the input space is state dependent.

Therefore the most natural definition seems to be as follows.

Definition 2.1 (Nonlinear Control System) (cf. [2], [12]): A nonlinear control system Σ is a 3-tuple $\Sigma(M, B, f)$ with M a manifold, B a fiber bundle above M with projection $\pi: B \rightarrow M$, and f a smooth map such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & TM \\ & \searrow \pi & \swarrow \pi_M \\ & M & \end{array}$$

commutes (π_M is the natural projection of TM on M).

Remark 1: M is to be considered as the state space while the fibers of B represent the (state-dependent) input spaces. If we denote coordinates for M by x , and coordi-

nates for B by (x, u) , with u coordinates for the fibers, which are assumed to be m -dimensional, then locally this definition comes down to (writing f as $(x, u) \rightarrow (x, f(x, u))$, abuse of notation!)

$$\dot{x} = f(x, u).$$

Remark 2: The usual approach is recovered by taking B a trivial bundle, i.e., $B = M \times U$, with U (most times) $\subset \mathbb{R}^m$.

Remark 3: Note that our definition is also coordinate-free with respect to the inputs, i.e., there are no *a priori* specified coordinates for the input space as in the usual approach where $U \subset \mathbb{R}^m$ and hence already has coordinates.

In this framework *feedback* can be defined in an appealing way (cf. [2], [12]). A system $\Sigma(M, B, f)$ is *feedback equivalent* to a system $\tilde{\Sigma}(M, B, \tilde{f})$ iff there exists a bundle isomorphism $\alpha: B \rightarrow B$ such that the diagram

commutes. With the same abuse of notation as in Remark 1 we shall write α in local coordinates as $(x, u) \rightarrow (x, \alpha(x, u))$.

A special, but important and often studied class of nonlinear systems is given by the following.

Definition 2.2 (Affine Control System): A nonlinear control system $\Sigma(M, B, f)$ is an *affine control system* if B is a *vector bundle* and the map f restricted to the fibers of B is an *affine map* into the fibers of TM . Also we assume, to avoid singularities, that f is an immersion.

Remark 1: Because the fibers of B and TM are vector spaces, "affine" is well defined.

Remark 2: If we take coordinates x for M and affine coordinates $(u_1, \dots, u_m)$ for the fibers of B (i.e., affine maps from the fibers into $\mathbb{R}$), then the system is locally described by

$$\dot{x} = A(x) + \sum_{i=1}^m u_i B_i(x)$$

where $\text{span}\{B_1(x), \dots, B_m(x)\}$ has constant dimension.

Remark 3: Note that the class of feedbacks which preserve the affine structure consist of those $\alpha: B \rightarrow B$ which restricted to the fibers are affine. Hence, in coordinates as above

$$(x, u) \xrightarrow{\alpha} (x, M(x)u - v(x))$$

with $M(x)$ an $m \times m$ matrix (nonsingular).

An equivalent definition is obtained by looking only at the image of the map f in TM . Because f is affine, the image of the fiber of B above a point $x \in M$ under f is an affine subspace of $T_x M$. Hence, we obtain (cf. [8], [9]) the following.

Definition 2.2': An affine system on a manifold M is an affine distribution Δ .

Remark: Define $\Delta_0 := \Delta - \Delta := \{X - Y \mid X, Y \in \Delta\}$. Then Δ_0 is a distribution, given in local coordinates as above by $\text{span}\{B_1(x), \dots, B_m(x)\}$. We denote the affine system by (Δ, Δ_0) .

As already noted, our definition is also coordinate-free with respect to the inputs. A local coordinatization of B is given by a trivializing chart, i.e., an open neighborhood 0 such that $\pi^{-1}(0) \simeq 0 \times F$, where $\simeq$ stands for isomorphic and F is the so-called standard fiber. Notice that a coordinatization of 0 and F immediately gives a coordinatization (x, u) of $\pi^{-1}(0)$ such that x are coordinates for $0 \subset M$. We will call these kind of coordinates *fiber respecting*.

In general, there are many trivializing charts, and hence many fiber respecting coordinatizations of B . In this context it is easy to see that, given a local fiber respecting coordinatization of B , *feedback* $(x, u) \mapsto (x, \alpha(x, u))$ can be interpreted as defining a new fiber respecting coordinatization (x, v) with $v = \alpha(x, u)$. This idea, translating feedback into choice of coordinates, will be used in the sequel.

Finally, we will define the extended system, introduced in [10], which will be important henceforth.

Definition 2.3 (Extended System): Let $\Sigma(M, B, f)$ be a control system (Definition 2.1). The *extended system*, denoted $\Sigma^e(M, B, f)$, is an affine system (Definition 2.2') constructed in the following way. Take as state space the manifold B . Let $(\bar{x}, \bar{v})$ be a point in B . We construct an affine subset $\Delta^e(\bar{x}, \bar{v})$ of $T_{(\bar{x}, \bar{v})} B$ as follows. The map $f: B \rightarrow TM$ gives a vector $f(\bar{x}, \bar{v}) \in T_{\bar{x}} M$. Now define

$$\Delta^e(\bar{x}, \bar{v}) := \{X \in T_{(\bar{x}, \bar{v})} B \mid \pi_* X = f(\bar{x}, \bar{v})\}.$$

Then Δ^e , in every (x, v) defined as above, is an affine distribution on B . It is easy to see that $\Delta_0^e := \Delta^e - \Delta^e = \{X \in TB \mid \pi_* X = 0\}$. Hence (Δ^e, Δ_0^e) is an affine system on B , denoted by $\Sigma^e(M, B, f)$.

III. CONTROLLED INVARIANCE FOR NONLINEAR CONTROL SYSTEMS

As we saw in the Introduction, the underlying idea of (A, B) -invariance or controlled invariance is the following. Let D be a distribution, which is involutive and therefore induces a foliation. Then D is invariant with respect to the dynamics of a system $\dot{x} = f(x, u)$ if for any two points x_1 and x_2 on the same leaf of the foliation and for all input functions $u(\cdot)$ the integral curves starting from x_1 and x_2 with a fixed $\bar{u}(\cdot)$ will be on the same leaf at the same time t . D is controlled invariant if this holds after applying feedback. The infinitesimal translation of this gives the following (preliminary) definition (see [4]–[6]).

Let $\Sigma(M, B, f)$ be a control system. Let (x, u) be fiber respecting coordinates for B , in which the control system has the form $\dot{x} = f(x, u)$. A distribution D (involutive) on M is called *controlled invariant* if there exists a *feedback*, i.e., a bundle isomorphism $\alpha: B \rightarrow B$, in coordinates given by

$$(x, u) \xrightarrow{\alpha} (x, v := \alpha(x, u))$$

such that the control system in these new coordinates (x, v) given by $\dot{x} = \tilde{f}(x, v)$ satisfies

$$[\tilde{f}(\cdot, v), D] \subset D, \quad \text{for every } v \text{ constant.}$$

Remark 1: This readily implies that for every time function $\bar{v}(\cdot)$ also $[\tilde{f}(\cdot, \bar{v}), D] \subset D$.

The defect of this definition is that it already assumes a choice of input coordinates u . By doing this, it obscures the problem because this preliminary definition is easily seen (see Section II) to be equivalent to the following.

Definition 3.1 (Local Controlled Invariance): Let $\Sigma(M, B, f)$ be a control system. An involutive distribution D on M is called local controlled invariant if locally around each point $x_0 \in M$ there exist fiber respecting coordinates (x, u) for B such that for every fixed u $[f(\cdot, u), D] \subset D$.

In fact, this definition can be made totally coordinate-free. For this we need the concept of an (integrable) connection, which will be treated in the next section. The final formulation is given in Theorem 4.19. There is an obvious extension of the notion of local controlled invariance in this framework to (global) controlled invariance. The basic idea is that local solutions given by Definition 3.1 coincide on every intersection of their domains. We will formalize this in the next definition, although we will mainly deal with the local controlled invariance in this paper.

Definition 3.2 (Global Controlled Invariance): Let D be a locally controlled invariant distribution for a control system $\Sigma(M, B, f)$. D is called (globally) controlled invariant if we can find a covering family of open neighborhoods $\{0_i\}_{i \in I}$ for M , consisting of trivializing charts for the bundle B , with the following properties. Let $p_i: \pi^{-1}(0_i) \rightarrow 0_i \times F$ be the trivializing diffeomorphism. Suppose $0_i \cap 0_j \neq \emptyset$; then the following diagram commutes:

$$\begin{array}{ccc} & \pi^{-1}(0_i \cap 0_j) & \\ p_i \swarrow & & \searrow p_j \\ (0_i \cap 0_j) \times F & \xrightarrow{P_{ij}} & (0_i \cap 0_j) \times F \end{array}$$

where P_{ij} has the form

$$P_{ij}(x, u) = (x, r_{ij}(x, u)), \quad (x \in 0_i \cap 0_j, u \in F).$$

Now we demand the following.

1) D is locally controlled invariant on each 0_i ($i \in I$) with the associated fiber respecting coordinates (x^i, u^i) .

2) $r_{ij}(x, u)$ does not depend on x , for every $i, j \in I$.

In applications the concept of controlled invariance is often used to factor out a part of the state space (cf. [4], [5]). Even Definition 3.2 only ensures that locally the controlled invariant distribution can be factored out, and in fact there may be obstructions to do this globally (cf.

[5]). Therefore, we could also go the other way around and see what we mean by globally factoring out. Actually, we will give a definition of a *quotient system* which implies controlled invariance.

Definition 3.3 (Quotient System): Let $\Sigma(M, B, f)$ be a control system. A control system $\tilde{\Sigma}(\tilde{M}, \tilde{B}, \tilde{f})$ is called a *quotient system* of Σ if there exist surjective submersions Φ and ϕ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & TM \\ \Phi \searrow & & \nearrow \phi_* \\ \tilde{B} & \xrightarrow{\tilde{f}} & T\tilde{M} \\ \pi \searrow & & \nearrow \pi_M \\ \tilde{M} & \xrightarrow{\tilde{\phi}} & M \end{array} \quad (3.1)$$

commutes.

Remark: Compare this to the definition of minimality in [10].

In order to see that this definition implies controlled invariance, we have to make the following observations (cf. also [10]). Because Φ and ϕ are surjective submersions they induce the involutive distributions

$$E := \{X \in TB \mid \Phi_* X = 0\} \quad \text{resp.}$$

$$D := \{X \in TM \mid \phi_* X = 0\}$$

Lemma 3.4: Let $\tilde{\Sigma}$ be a quotient system of Σ as in Definition 3.3. Let D be defined as above; then D is controlled invariant with respect to Σ .

Proof: Diagram 3.1 has two commuting subdiagrams which respectively give

$$1) \pi_* E = D, \text{ and}$$

$$2) f_* E \subset D$$

(because it is readily seen that ϕ_* induces the distribution $\tilde{D}$ in Section I).

Now the distribution E in fact defines fiber respecting coordinates above the leaves of the foliation generated by D in the following way. Take a leaf F of the foliation. Restrict the bundle B to this leaf. Denote this new fiber bundle above F by B_F . Because $\pi_* E = D$ and E is involutive, E defines sections in B_F which project onto F . (The sections are the maximal integral manifolds of E .) We can define coordinates u for the fibers of B_F , such that $u^{-1}(c)$, with c constant, are the sections of E in B_F .

Assume for a moment that Φ_* restricted to the fibers of B is bijective. Then one can see that, given an arbitrary fiber respecting coordinatization of $\tilde{B}$, the process above generates in a unique way fiber respecting coordinates for B . When Φ_* restricted to the fibers has a nontrivial null space, then for this part of the fiber we may arbitrarily complete the coordinates.

Finally, take coordinates $x_1, \dots, x_n$

$$D = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right\}, \quad \text{with } k \leq n.$$

Then construct fiber respecting coordinates (x, u) as above. In these coordinates

$$E = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right\}, \quad \text{and } f_*E \subset \dot{D} \text{ implies}$$

$$j\text{th component } \left(\frac{\partial f}{\partial x_i}(x, u) \right) = 0, \\ \text{with } i = 1, \dots, k \quad j = k+1, \dots, n$$

(where $\dot{x} = f(x, u)$ is the representation of Σ in (x, u)). This is equivalent with

$$[f(\cdot, u), D] \subset D.$$

Remark 1: This proof also exactly shows which freedom one has in choosing coordinates (or in constructing feedback) such that in these coordinates D is invariant. In fact, loosely speaking, outside of the distribution D the coordinates for the fibers are arbitrary. Above the distribution D the coordinates for the fibers are uniquely determined except for the part which Φ_* send to zero. This last part consists exactly of the inputs which are factored out in Diagram 3.1 and which do not appear anymore in the quotient system.

Remark 2: An interesting special case of controlled invariance is when $f_*(\pi_*^{-1}(D)) \subset \dot{D}$. The proof above shows that in this situation D is invariant for all fiber respecting coordinatizations (x, u) of B . Also, it is easily seen that the system factored out by D is autonomous (the input space consists of only one point).

Finally, we can also relate controlled invariance in a system $\Sigma(M, B, f)$ with controlled invariance in the extended system $\Sigma^e(M, B, f)$ denoted by (Δ^e, Δ_0^e) (see Definition 2.3). In local coordinates it is easily proven (see also [10]).

Lemma 3.5:

$$f_*E \subset \dot{D}, \text{ with } \pi_*E = D, \text{ is equivalent to} \\ [\Delta^e, E] \subset E + \Delta_0^e.$$

As is known from recent work ([4], [6], [9]; see the Introduction) the last expression $[\Delta^e, E] \subset E + \Delta_0^e$ is equivalent to the local controlled invariance of E with respect to the affine system (Δ^e, Δ_0^e) if $E \cap \Delta_0^e$ has fixed dimension. Therefore, combining conditions 1) and 2) (in the proof of Lemma 3.4) and Lemma 3.5 gives the following.

Proposition 3.6: An involutive distribution D is locally controlled invariant with respect to $\Sigma(M, B, f)$ iff there exists an involutive distribution E , with $\pi_*E = D$, such that E is locally controlled invariant with respect to $\Sigma^e(M, B, f)$.

Remark 3: It is emphasized that the last proposition does not reduce the problem to the study of controlled invariance for affine control systems since the distribution E , satisfying $\pi_*E = D$, is unknown.

Remark 4: We have defined controlled invariance by requiring that after applying feedback, the modified dynamics leave the foliation invariant for all input functions. Of course, this demand might be too strong and we could be content if the foliation is invariant for only a part of the inputs. We will call this *degenerate controlled invariance*. Definitions 3.1–3.3 can be readily adapted to cover this situation. For instance, we require that the needed feedback α is no longer an isomorphism, and in Definition 3.3 we allow Φ to be a *partial* map (cf. [15]; see also [14]). However, finding necessary and sufficient conditions for degenerate controlled invariance seems to be harder than for the (full) controlled invariant case, and we will leave it for the moment. (Note that in the linear case degenerate controlled invariance implies full controlled invariance.)

IV. CONTROLLED INVARIANCE AND CONNECTIONS

In this section we introduce the concept of a connection on a fiber bundle and we will relate this to the controlled invariance as introduced in Section III. For a more detailed treatment of a connection the reader is referred to the literature on differential geometry. (See, e.g., [3].)

Definition 4.1: Let $\pi: B \rightarrow M$ be a smooth (fiber) bundle. A tangent vector $v \in T_p B$, $p \in B$, is said to be *vertical* if $\pi_*(v) = 0$. $V(p)$ denotes the set of all vertical tangent vectors in p . A distribution H on B is said to be *horizontal* if $T_p B = H(p) \oplus V(p)$ for all $p \in B$.

Remark: We see that $H \subset V(B)$ is horizontal implies that for all $p \in M$, $H(p)$ is a linear subspace of $T_p B$ with the following properties:

$$\dim H(p) = \dim M \\ H(p) \cap V(p) = 0.$$

π_* maps $H(p)$ isomorphically onto $T_{\pi(p)} M$.

Now the next definition will be clear.

Definition 4.2: A curve $\sigma: \mathbb{R} \rightarrow B$ is *horizontal* with respect to a horizontal distribution H on B if $\sigma'(t) \in H(\sigma(t))$ for all $t \in \mathbb{R}$, i.e., σ is an integral curve of a vector field which belongs to the horizontal distribution H on B .

We are now able to define a connection as follows.

Definition 4.3: Let $\pi: B \rightarrow M$ be a smooth bundle, and let H be a horizontal distribution on B . H determines a *nonlinear connection* for $\pi: B \rightarrow M$ which is defined by the following *lifting* procedure.

For every curve $\sigma_1: \mathbb{R} \rightarrow M$ and each point $p \in \pi^{-1}(\sigma_1(0))$ there exist $\epsilon > 0$ and a horizontal curve $\sigma: (-\epsilon, \epsilon) \rightarrow B$ such that for $t \in (-\epsilon, \epsilon)$

$$\pi(\sigma(t)) = \sigma_1(t), \quad \sigma(0) = p.$$

Remarks:

1) When every curve in M can be globally lifted to an integral curve of H , we have what is called a horizontally complete nonlinear connection. In general, a nonlinear connection is not horizontally complete; a curve in M can only be locally lifted to an integral curve of H .

2) In the literature there exist a couple of different definitions of a connection (introduced by different people). The above definition, in fact, defines the Ehresmann connection.

The next proposition gives a uniqueness property of the lift σ of σ_1 in Definition 4.3.

Proposition 4.4: *Let H be a horizontal distribution on B which defines a nonlinear connection for $\pi: B \rightarrow M$; then the lift $\sigma: (-\epsilon, \epsilon) \rightarrow B$ of a curve $\sigma_1: \mathbb{R} \rightarrow M$ defined by Definition 4.3 is unique.*

And so we have, as a direct consequence, the following.

Proposition 4.5: *Let H be a horizontal distribution on B which defines a horizontally nonlinear connection for $\pi: B \rightarrow M$. Let σ_1 be a curve between the points m_1 and m_2 on M ; then the connection determines a diffeomorphism, depending on σ_1 between the fibers $\pi^{-1}(m_1)$ and $\pi^{-1}(m_2)$.*

Next we will define an important class of nonlinear connections.

Definition 4.6: Let $\pi: B \rightarrow M$ be a vector bundle, i.e., for all $m \in M$, $\pi^{-1}(m)$ is a real vector space. A horizontally complete connection defined by a horizontal distribution is called an *affine connection* if the fiber diffeomorphisms defined by the connection are affine isomorphisms between the vector space fibers.

Another useful property is given by the following.

Definition 4.7: Let $\pi: B \rightarrow M$ be a smooth bundle. Let H be a horizontal distribution on B which defines a nonlinear connection. The connection is *integrable* if $[H, H] \subset H$, i.e., H is integrable as a vector field system.

The integrability of a connection of a horizontal distribution H implies that through each point $p \in B$ there passes a unique maximal connected integral submanifold M' of H (according to Frobenius' theorem) and this submanifold M' is transverse to the fibers of π , i.e., for all $q \in M'$, we have $T_q B = T_q M' \oplus V(q)$.

For later use we will investigate the integrability of an affine connection in detail.

According to Definition 4.6 we can choose an (affine) coordinate system for B : $(x, v) = (x_1, \dots, x_n, v_1, \dots, v_m)$ where $(x_1, \dots, x_n)$ is a coordinatization of M such that the linear subspace $H(x, v) \subset T_{(x, v)} B$ (Definition 4.1) has a basis $X_1, \dots, X_n$ of the following form (see [3]):

$$X_i(x, v) = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v] \frac{\partial}{\partial v} \quad i=1, \dots, n \quad (4.1)$$

where

$h_i(x)$ is an m vector

$K_i(x)$ is a $m \times m$ -matrix

$$\frac{\partial}{\partial v} = \left(\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_m} \right)^t \quad ((\dots)^t \text{ denotes transposed}).$$

Actually, from (4.1) it follows that an affine connection is horizontally complete (cf. [3, p. 112]).

Now $[H, H] \subset H$ implies

$$\begin{aligned} [X_i, X_j](x, v) &= \left[\frac{\partial}{\partial x_i} + (h_i(x) + K_i(x)v) \frac{\partial}{\partial v}, \right. \\ &\quad \left. \frac{\partial}{\partial x_j} + (h_j(x) + K_j(x)v) \frac{\partial}{\partial v} \right] \\ &= \left[\frac{\partial h_j(x)}{\partial x_i} - \frac{\partial h_i(x)}{\partial x_j} + K_j(x)h_i(x) \right. \\ &\quad \left. - K_i(x)h_j(x) \right] \frac{\partial}{\partial v} \\ &\quad + \left[\frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) \right. \\ &\quad \left. - K_i(x)K_j(x) \right] v \frac{\partial}{\partial v} \\ &= 0 \quad \text{for all } (x, v). \end{aligned} \quad (4.2)$$

Therefore

$$\frac{\partial h_j(x)}{\partial x_i} - \frac{\partial h_i(x)}{\partial x_j} + K_j(x)h_i(x) - K_i(x)h_j(x) = 0 \quad (4.3)$$

and

$$\begin{aligned} \frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) &= 0 \\ \text{for } i, j = 1, \dots, n. \end{aligned} \quad (4.4)$$

We can also work out the integrability condition (4.2) in a dual fashion, dual in the sense that we translate (4.2) to the cotangent space of B . The integrability of H then guarantees that two 2-forms, called the *torsion tensor* and the *curvature tensor*, vanish (see, e.g., [3]). This requirement is exactly equivalent to (4.3) and (4.4), and thus we will call this the *torsion equation*, resp. the *curvature equation*. Conversely, an integrable affine connection will be defined by the vector fields given by (4.1) where $h_i(x)$ and $K_i(x)$ satisfy the torsion and curvature equation.

Let D be an involutive distribution of fixed dimension k on M . Let H be a horizontal distribution on B which induces an integrable affine connection on $\pi: B \rightarrow M$. Then this connection defines a unique lifting procedure for the distribution D (see Definition 4.3). In fact, choose a coordinate system $(x_1, \dots, x_n)$ for M as in the Frobenius' theorem; then D is spanned by the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_k$.

Let $H(x, v) \subset T_{(x, v)} B$ be spanned by [as in (4.1)]

$$X_i = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v] \frac{\partial}{\partial v} \quad i=1, \dots, n.$$

Then the lifting of the distribution D gives a new involutive

distribution D_l which is spanned by

$$X_i = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v] \frac{\partial}{\partial v} \quad i=1, \dots, k. \quad (4.5)$$

Remark: The basis $X_1, \dots, X_k$ for D_l defined by (4.5) satisfies

$$\begin{aligned} \frac{\partial h_j(x)}{\partial x_i} - \frac{\partial h_i(x)}{\partial x_j} + K_j(x)h_i(x) - K_i(x)h_j(x) &= 0. \\ \frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) &= 0 \\ \text{for } i, j = 1, \dots, k. \end{aligned} \quad (4.6)$$

Now assume that we have given an affine control system (Δ, Δ_0) as in Definition 2.2'. We will denote the extended system (see Definition 2.3) by Δ^e with "input space" Δ_0^e .

After these preparations we can state the next theorem, which gives a nice geometric interpretation of the results of [4], [6], [9] and can be useful in understanding the structure of affine control systems.

Theorem 4.8: D is a locally controlled invariant distribution for an affine system (Δ, Δ_0) iff there exists an integrable affine connection for $\pi: B \rightarrow M$ such that $[\Delta^e, D_l] \subset D_l + \Delta_0^e$.

Proof: ($\Rightarrow$) Suppose there exists an integrable affine connection for $\pi: B \rightarrow M$ with $[\Delta^e, D_l] \subset D_l + \Delta_0^e$. The horizontal system on B which defines the affine connection is, according to (4.1), given by

$$X_i(x, v) = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v] \frac{\partial}{\partial v} \quad i=1, \dots, n \quad (4.1)$$

where (x, v) is an affine coordinate system for B . By the integrability it follows that h_i and K_i satisfy the curvature and torsion equation (4.6). Let the control system on M be given by

$$\begin{aligned} \dot{x}(t) &= A(x(t)) + \sum_{i=1}^m v_i(t) B_i(x(t)) \\ &=: A(x(t)) + B(x(t))v(t) \end{aligned} \quad (4.7)$$

where $B(x)$, an (n, m) -matrix with columns $B_i(x)$ and $v(t) = (v_1(t), \dots, v_m(t))^T$. So the extended system has the form

$$\begin{cases} \dot{x}(t) = A(x(t)) + B(x(t))v(t) \\ \dot{v}(t) = u(t). \end{cases} \quad (4.8)$$

From (4.5) we know that D_l is spanned by

$$X_i(x, v) = \frac{\partial}{\partial x_i} + [h_i(x) + K_i(x)v] \frac{\partial}{\partial v} \quad i=1, \dots, k.$$

So from $[\Delta^e, D_l] \subset D_l + \Delta_0^e$ we deduce for all $i=1, \dots, k$

$$\begin{aligned} &\left[(A(x) + B(x)v) \frac{\partial}{\partial x} + u \frac{\partial}{\partial v}, \frac{\partial}{\partial x_i} + (h_i(x) + K_i(x)v) \frac{\partial}{\partial v} \right] \\ &\in \text{span} \left\{ \frac{\partial}{\partial x_i} + (h_i(x) + K_i(x)v) \frac{\partial}{\partial v}, \frac{\partial}{\partial v}, i=1, \dots, k \right\}. \end{aligned} \quad (4.9)$$

Computing the Lie bracket of (4.9) leads to

$$\begin{aligned} &\left[\frac{\partial A(x)}{\partial x_i} + \frac{\partial B(x)}{\partial x_i} v + B(x)h_i(x) + B(x)K_i(x)v \right] \frac{\partial}{\partial x} \\ &\in \text{span} \left\{ \frac{\partial}{\partial x_i} + (h_i(x) + K_i(x)v) \frac{\partial}{\partial v}, \frac{\partial}{\partial v}, i=1, \dots, k \right\} \end{aligned}$$

for all $i=1, \dots, k$.

Therefore

$$\begin{aligned} &j\text{th component of } \left(\frac{\partial A(x)}{\partial x_i} + \frac{\partial B(x)}{\partial x_i} v \right. \\ &\quad \left. + B(x)h_i(x) + B(x)K_i(x)v \right) = 0 \\ &\text{for all } (x, v) \quad j=k+1, \dots, n \quad i=1, \dots, k. \end{aligned} \quad (4.10)$$

Thus we have

$$\begin{aligned} &j\text{th component of } \left(\frac{\partial A(x)}{\partial x_i} + B(x)h_i(x) \right) = 0 \\ &\quad j=k+1, \dots, n \quad i=1, \dots, k \end{aligned} \quad (4.11)$$

$$\begin{aligned} &j\text{th component of } \left(\frac{\partial B(x)}{\partial x_i} + B(x)K_i(x) \right) = 0 \\ &\quad j=k+1, \dots, n \quad i=1, \dots, k \end{aligned} \quad (4.12)$$

where $h_i(x)$ and $K_i(x)$ satisfy (4.3) and (4.4).

Now (4.11), together with the curvature equation (4.4), is an old friend (cf. Nijmeijer [9], Isidori *et al.* [6]). We deduce from [6] and [9] that there exists a nonsingular (m, m) -matrix $M(x)$ such that

$$\begin{aligned} &j\text{th component of } \left(\frac{\partial}{\partial x_i} [B(x) \cdot M(x)] \right) = 0 \\ &\quad j=k+1, \dots, n \quad i=1, \dots, k. \end{aligned} \quad (4.13)$$

Let

$$\tilde{B}(x) := B(x)M(x). \quad (4.14)$$

Furthermore, we see

$$\begin{aligned}
 & j\text{th component of } \left(\frac{\partial}{\partial x_s} [B(x)h_i(x)] \right) \\
 &= j\text{th component of } \left(\frac{\partial B(x)}{\partial x_s} h_i(x) + B(x) \frac{\partial h_i(x)}{\partial x_s} \right) \\
 &= j\text{th component of } \left(-B(x)K_s(x)h_i(x) + B(x) \frac{\partial h_i(x)}{\partial x_s} \right)
 \end{aligned}$$

and by the torsion equation (4.3), the last expression equals

$$\begin{aligned}
 & j\text{th component of } \left(\frac{\partial}{\partial x_i} [B(x)h_s(x)] \right) \\
 & \quad i, s = 1, \dots, k \quad j = k+1, \dots, n.
 \end{aligned}$$

It follows, combining (4.11) and Hirschorn [4]—in fact, Frobenius' theorem—that there exists an m vector $v(x)$ such that

$$\begin{aligned}
 & j\text{th component of } \left(\frac{\partial}{\partial x_i} [A(x) + B(x)v(x)] \right) = 0 \\
 & \quad j = k+1, \dots, n \quad i = 1, \dots, k.
 \end{aligned} \quad (4.15)$$

Thus, if we use a feedback $v(t) = M(x)v(t) + \tilde{v}(x)$ for the system (4.7), we get

$$\dot{x}(t) = A(x) + B(x)\tilde{v}(x) + \tilde{B}(x)v(t) \quad (4.16)$$

and so by using (4.13) and (4.15) we see that the distribution D is controlled invariant for the system (4.16).

($\Rightarrow$) Let D be a controlled invariant distribution for the system given by (4.7), where D is spanned by the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_k$. For the construction of an integrable affine connection we need matrices $h_i(x)$ and $K_i(x)$ which satisfy the torsion and the curvature equation (4.3) and (4.4). From Isidori *et al.* [6] and Nijmeijer [9] we know that there exist (m, m) -matrices $K_i(x)$ such that

$$\frac{\partial B(x)}{\partial x_i} = B(x)K_i(x) \pmod{D} \quad i = 1, \dots, k$$

and these matrices $K_i(x)$ satisfy

$$\begin{aligned}
 & \frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) = 0 \\
 & \quad i, j = 1, \dots, k.
 \end{aligned}$$

According to [6] (see also Remark 1 after Lemma 3.4) we can also define matrices $K_{k+1}(x), \dots, K_n(x)$ such that

$$\begin{aligned}
 & \frac{\partial K_j(x)}{\partial x_i} - \frac{\partial K_i(x)}{\partial x_j} + K_j(x)K_i(x) - K_i(x)K_j(x) = 0 \\
 & \quad i, j = 1, \dots, n, \quad (4.4)
 \end{aligned}$$

i.e., the curvature equation (4.4)!

Furthermore, it follows from the fact that D is controlled invariant that

$$\frac{\partial A(x)}{\partial x_i} = B(x)h_i(x) \pmod{D} \quad i = 1, \dots, k$$

where the vectors $h_i(x)$ satisfy

$$\begin{aligned}
 & \frac{\partial h_i(x)}{\partial x_j} - \frac{\partial h_j(x)}{\partial x_i} - K_i(x)h_j(x) + K_j(x)h_i(x) = 0 \\
 & \quad i, j = 1, \dots, k.
 \end{aligned}$$

In the same way as in [6] we can define vectors $h_{k+1}(x), \dots, h_n(x)$ such that

$$\begin{aligned}
 & \frac{\partial h_i(x)}{\partial x_j} - \frac{\partial h_j(x)}{\partial x_i} - K_i(x)h_j(x) + K_j(x)h_i(x) = 0 \\
 & \quad i, j = 1, \dots, n. \quad (4.3)
 \end{aligned}$$

Thus, the matrices $h_i(x)$ and $K_i(x)$ define an integrable affine connection. $\square$

Remark: Under certain conditions, it is possible to drop the adjective "locally" in this theorem. For example, see [6]; if the state space M is simply connected, the feedback is globally well-defined.

Next, we want to investigate the situation for a general control system $\Sigma(M, B, f)$ as defined in Definition 2.1.

First, we will formulate the integrability of a nonlinear connection in the same way as we have done for an affine connection. Following the notation as used after Definition 4.7 we have that the nonlinear connection is spanned by vector fields $X_1, \dots, X_n$ of the following form:

$$X_i(x, v) = \frac{\partial}{\partial x_i} + h_i(x, v) \frac{\partial}{\partial v} \quad i = 1, \dots, n \quad (4.17)$$

where $h_i(x, v)$ is an m -vector

$$\frac{\partial}{\partial v} = \left(\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_m} \right)^T.$$

From the integrability we derive that

$$\begin{aligned}
 & \left[\frac{\partial}{\partial x_i} + h_i(x, v) \frac{\partial}{\partial v}, \frac{\partial}{\partial x_j} + h_j(x, v) \frac{\partial}{\partial v} \right] \\
 &= \left[\frac{\partial h_j}{\partial x_i}(x, v) - \frac{\partial h_i}{\partial x_j}(x, v) + \frac{\partial h_j}{\partial v}(x, v) \cdot h_i(x, v) \right. \\
 & \quad \left. - \frac{\partial h_i}{\partial v}(x, v) \cdot h_j(x, v) \right] \frac{\partial}{\partial v} = 0.
 \end{aligned}$$

Remark: $\partial h_i/\partial v(x, v)$ is an (m, m) -matrix consisting of the columns $\partial h_i/\partial v_j(x, v)$. Therefore

$$\begin{aligned}
 & \frac{\partial h_j}{\partial x_i}(x, v) - \frac{\partial h_i}{\partial x_j}(x, v) + \frac{\partial h_j}{\partial v}(x, v) \cdot h_i(x, v) \\
 & \quad - \frac{\partial h_i}{\partial v}(x, v) \cdot h_j(x, v) = 0. \quad (4.18)
 \end{aligned}$$

Now the following theorem will be the direct generalization of Theorem 4.8.

Theorem 4.9: D is a controlled invariant distribution for a control system $\Sigma(M, B, f)$ iff there exists an integrable nonlinear connection for $\pi: B \rightarrow M$ such that $[\Delta^e, D_i] \subset D_i + \Delta_0^e$.

Proof: ($\Leftarrow$) Suppose that there exists an integrable nonlinear connection for $\pi: B \rightarrow M$ with $[\Delta^e, D_i] \subset D_i + \Delta_0^e$. The horizontal system on B which defines the connection is, according to (4.17), given by

$$X_i(x, v) = \frac{\partial}{\partial x_i} + h_i(x, v) \frac{\partial}{\partial v} \quad i = 1, \dots, n$$

where the $h_i(x, v)$ satisfy (4.18).

Let the control system on M be given by

$$\dot{x}(t) = f(x(t), v(t)). \quad (4.19)$$

So the extended system has the form

$$\begin{cases} \dot{x}(t) = f(x(t), v(t)) \\ \dot{v}(t) = u(t) \end{cases} \quad (4.20)$$

As in (4.5), the distribution D_i is spanned by

$$X_i(x, v) = \frac{\partial}{\partial x_i} + h_i(x, v) \frac{\partial}{\partial v} \quad i = 1, \dots, k.$$

So from $[\Delta^e, D_i] \subset D_i + \Delta_0^e$ we deduce that

$$j\text{th component of } \left(\frac{\partial f}{\partial x_i}(x, v) + \frac{\partial f}{\partial v}(x, v) \cdot h_i(x, v) \right) = 0 \\ i = 1, \dots, k \quad j = k+1, \dots, n \quad (4.21)$$

where the $h_i(x, v)$ satisfy (4.18).

Now consider the set of partial differential equations

$$\begin{cases} \frac{\partial \alpha}{\partial x_i}(x, \tilde{v}) = h_i(x, \alpha(x, \tilde{v})) & i = 1, \dots, n \\ \alpha(0, 0) = I_{m, m} \end{cases} \quad (4.22)$$

From Frobenius' theorem (see [11]) we know that there exists a unique solution $\alpha(x, \tilde{v})$ of (4.22) iff the integrability condition (4.18) is satisfied. Hence, if we apply a feedback $v(t) = \alpha(x, \tilde{v}(t))$ to the system (4.19) we get

$$\dot{x}(t) = f(x, \alpha(x, \tilde{v}(t))) \quad (4.23)$$

and by using (4.21) we see that the distribution D is controlled invariant for (4.23).

($\Rightarrow$) Let D be a controlled invariant distribution for the system given by (4.19) where D is spanned by $\partial/\partial x_1, \dots, \partial/\partial x_k$. For the construction of an integrable nonlinear connection we need matrices $h_i(x, v)$ which satisfy (4.18). By the fact that D is controlled invariant we know that there exists an (m, m) -matrix $\alpha(x, \tilde{v})$ with $\partial\alpha/\partial\tilde{v}(x, \tilde{v})$ nonsingular, i.e., the map $\tilde{v} \mapsto v: \alpha(x, \tilde{v})$ is invertible. We will denote—abuse of notation!—the inverse of this map by $\alpha^{-1}(x, v)$.

Define

$$h_i(x, v) := \left(\frac{\partial \alpha}{\partial x_i}(x, \tilde{v}) \right) \Big|_{\tilde{v} = \alpha^{-1}(x, v)} \quad i = 1, \dots, n. \quad (4.24)$$

Now from (4.24) we see that

$$\frac{\partial \alpha}{\partial x_i}(x, \tilde{v}) = h_i(x, \alpha(x, \tilde{v})) \quad (4.22)$$

and therefore

$$\begin{aligned} \frac{\partial h_j}{\partial x_i}(x, v) - \frac{\partial h_i}{\partial x_j}(x, v) + \frac{\partial h_j}{\partial v}(x, v) \cdot h_i(x, v) \\ - \frac{\partial h_i}{\partial v}(x, v) \cdot h_j(x, v) = 0 \quad i, j = 1, \dots, n, \end{aligned} \quad (4.18)$$

i.e., the integrability condition for a nonlinear connection defined by

$$X_i(x, v) = \frac{\partial}{\partial x_i} + h_i(x, v) \frac{\partial}{\partial v} \quad i = 1, \dots, n. \quad (4.17)$$

□

To conclude this section we want to give the conditions under which a distribution is locally controlled invariant for a system $\Sigma(M, B, f)$. First, we will solve this problem in a local fashion (coordinate dependent) and afterwards we give the main theorem, Theorem 4.12. Let, as before, the control system be given (locally) by $\dot{x} = f(x, v)$ and let

$$\text{span} \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right\} = D.$$

Suppose that there exist m -vectors $m_i(x, v)$ ($i = 1, \dots, k$) such that

$$s\text{th component of } \left(\frac{\partial f}{\partial x_i}(x, v) + \frac{\partial f}{\partial v}(x, v) \cdot m_i(x, v) \right) = 0 \\ i = 1, \dots, k \quad s = k+1, \dots, n. \quad (4.25)$$

Then it follows that

$$\begin{aligned} s\text{th component of } \left(\frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i}(x, v) + \frac{\partial f}{\partial v}(x, v) \cdot m_i(x, v) \right) \right) \\ = s\text{th component of } \left(\frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j}(x, v) + \frac{\partial f}{\partial v}(x, v) \cdot m_j(x, v) \right) \right) \\ i, j = 1, \dots, k \quad s = k+1, \dots, n. \end{aligned}$$

Hence,

$$\begin{aligned} s\text{th component of } \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(x, v) + \frac{\partial^2 f}{\partial x_j \partial v}(x, v) m_i(x, v) \right) \\ + \frac{\partial f}{\partial v}(x, v) \cdot \frac{\partial m_i}{\partial x_j}(x, v) \\ = s\text{th component of } \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x, v) + \frac{\partial^2 f}{\partial x_i \partial v}(x, v) m_j(x, v) \right) \\ + \frac{\partial f}{\partial v}(x, v) \cdot \frac{\partial m_j}{\partial x_i}(x, v). \end{aligned}$$

Therefore

$$\begin{aligned} & s \text{ th component of } \left(\frac{\partial^2 f}{\partial x_j \partial v}(x, v) \cdot m_i(x, v) + \frac{\partial f}{\partial v}(x, v) \cdot \frac{\partial m_i}{\partial x_j}(x, v) \right) \\ &= s \text{ th component of } \left(\frac{\partial^2 f}{\partial x_i \partial v}(x, v) \cdot m_j(x, v) + \frac{\partial f}{\partial v}(x, v) \cdot \frac{\partial m_j}{\partial x_i}(x, v) \right) \\ & \quad i, j = 1, \dots, k \quad s = k+1, \dots, n. \quad (4.26) \end{aligned}$$

Now

$$\begin{aligned} & s \text{ th component of } \left(\frac{\partial^2 f}{\partial x_j \partial v}(x, v) \cdot m_i(x, v) \right) \\ &= \frac{\partial}{\partial v} \left(\frac{\partial f^s}{\partial x_j}(x, v) \right) \cdot m_i(x, v) \\ &= \frac{\partial}{\partial v} \left(-\frac{\partial f^s}{\partial v}(x, v) \cdot m_j(x, v) \right) \cdot m_i(x, v) \quad [\text{by (4.25)}] \\ &= -m'_i(x, v) \frac{\partial^2 f^s}{\partial v^2}(x, v) m_j(x, v) \\ & \quad - \frac{\partial f^s}{\partial v}(x, v) \cdot \frac{\partial m_j}{\partial v}(x, v) \cdot m_i(x, v). \quad (4.27) \end{aligned}$$

Substituting (4.27), and a similar expression for the left-hand side of (4.26), in (4.26) leads to

$$\begin{aligned} & \frac{\partial f^s}{\partial v}(x, v) \cdot \frac{\partial m_i}{\partial x_j}(x, v) - m'_i(x, v) \frac{\partial^2 f^s}{\partial v^2}(x, v) m_j(x, v) \\ & \quad - \frac{\partial f^s}{\partial v}(x, v) \cdot \frac{\partial m_j}{\partial v}(x, v) \cdot m_i(x, v) \\ &= \frac{\partial f^s}{\partial v}(x, v) \cdot \frac{\partial m_j}{\partial x_i}(x, v) - m'_j(x, v) \frac{\partial^2 f^s}{\partial v^2}(x, v) m_i(x, v) \\ & \quad - \frac{\partial f^s}{\partial v}(x, v) \cdot \frac{\partial m_i}{\partial v}(x, v) \cdot m_j(x, v) \\ &= 0 \quad i, j = 1, \dots, k \quad s = k+1, \dots, n. \end{aligned}$$

So

$$\begin{aligned} & \frac{\partial f^s}{\partial v}(x, v) \left[\frac{\partial m_i}{\partial x_j}(x, v) - \frac{\partial m_j}{\partial v}(x, v) m_i(x, v) \right. \\ & \quad \left. - \frac{\partial m_j}{\partial x_i}(x, v) + \frac{\partial m_i}{\partial v}(x, v) \cdot m_j(x, v) \right] \\ &= 0 \quad i, j = 1, \dots, k \quad s = k+1, \dots, n. \quad (4.28) \end{aligned}$$

Suppose that the matrix

$$\left(\frac{\partial f^s}{\partial v}(x, v) \right)_{s=k+1, \dots, n} \text{ has full rank;} \quad (4.29)$$

then (4.28) leads to

$$\begin{aligned} & \frac{\partial m_i}{\partial x_j}(x, v) - \frac{\partial m_j}{\partial x_i}(x, v) + \frac{\partial m_i}{\partial v}(x, v) m_j(x, v) \\ & \quad - \frac{\partial m_j}{\partial v}(x, v) m_i(x, v) = 0 \quad i, j = 1, \dots, k, \quad (4.30) \end{aligned}$$

i.e., a *partial* integrability condition as in (4.18)!

We need the following simple but crucial lemma.

Lemma 4.10: The set of partial differential equations

$$\begin{cases} \frac{\partial \alpha}{\partial x_i}(x, \tilde{v}) = m_i(x, \alpha(x, \tilde{v})) & i = 1, \dots, k \\ \alpha(0, 0) = I_{m, m} \end{cases} \quad (4.31)$$

has a solution.

Remark: This set of partial differential equations (4.31) is nearly the same as in (4.22). We cannot immediately apply Frobenius' theorem, while not all partial derivatives of α are specified (compare to [9]).

Proof: There exist $m_{k+1}(x, v), \dots, m_n(x, v)$ such that

$$\begin{aligned} & \frac{\partial m_i}{\partial x_j}(x, v) - \frac{\partial m_j}{\partial x_i}(x, v) + \frac{\partial m_i}{\partial v}(x, v) \cdot m_j(x, v) \\ & \quad - \frac{\partial m_j}{\partial v}(x, v) \cdot m_i(x, v) = 0 \quad i, j = 1, \dots, n. \quad (4.18) \end{aligned}$$

(See [9]; see also (4.4); this follows from the fact that the distribution $D = TM$ is controlled invariant).

Finally, apply Frobenius' theorem. $\square$

Corollary 4.11: If there exist $m_i(x, v)$ ($i = 1, \dots, k$) which satisfy (4.25) and condition (4.29) is fulfilled, then the distribution D is locally controlled invariant for the system $\dot{x}(t) = f(x(t), \alpha(x(t), \tilde{v}(t)))$, where $\alpha(x, \tilde{v})$ is defined by Lemma 4.10.

Finally, we will give a coordinate-free way the analog of [6] and [9] for a nonlinear control system $\Sigma(M, B, f)$. Recall the definition of $\dot{D}$ for a given distribution D (see notation at the end of Section I).

Theorem 4.12: Let $\Sigma(M, B, f)$ be a nonlinear control system and let D be an involutive distribution of fixed dimension on M . If $f_*(\Delta_0^e) \cap \dot{D}$ has fixed dimension, then we have the following equivalence.

D is locally controlled invariant iff

$$f_*(\pi_*^{-1}(D)) \subset \dot{D} + f_*(\Delta_0^e). \quad (4.32)$$

Proof: ($\Rightarrow$) Direct ($\Leftarrow$) work out in local coordinates, and suppose $f_*(\Delta_0^e) \cap \dot{D} = 0$. Then the result is given by Corollary 4.12. In a similar way as in Isidori *et al.* [6] and Nijmeijer [9], we derive the same result in the case that $f_*(\Delta_0^e) \cap \dot{D}$ has fixed dimension. $\square$

Remark: The problem of global controlled invariance is directly related to the so-called holonomy group of the integrable connection. However, we will leave it for the moment.

V. CONCLUSION

The main result of this paper is Theorem 4.12 which gives necessary and sufficient conditions for local controlled invariance in general nonlinear systems. With the aid of this theorem, the disturbance decoupling problem (see [13]), for instance, can be readily solved locally, analogous to [4], [5]. Very surprising results are Theorems 4.8 and 4.9 where the concept of controlled invariance is directly related to the well-known differential geometric notion of an integrable connection.

It would be interesting to look for similar results in the case of degenerate controlled invariance, as sketched in Remark 4 of Section III. As already stated in some of the remarks, after solving the local controlled invariance there remain global problems. Essentially we can divide them into two categories, namely

1) we may only *locally* be able to construct a feedback; and

2) the controlled invariant distribution may not be factored out globally (in a smooth fashion).

Both problems seem to involve the whole machinery of algebraic topology.

Finally, in this paper we have treated only the regular case: all our distributions have constant dimension. It seems interesting and useful to extend the results to the nonregular (but, for instance, analytic) case.

ACKNOWLEDGMENT

The authors would like to thank Prof. J. C. Willems for many stimulating discussions. Also, we would like to thank Dr. J. H. van Schuppen for his encouragement.

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