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Linden, W. von der; Morgenstern, I.; Raedt, H. de

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Quantum Monte Carlo study of quasiparticles in the Hubbard model

W. von der Linden and I. Morgenstern*

IBM Research Division, Zurich Research Laboratory, 8803 Rüschlikon, Switzerland

H. de Raedt

Physics Department, University of Antwerpen, Universiteitsplein 1, B-2610 Wilrijk, Belgium

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We present an improved version of the projector quantum Monte Carlo method, which has recently been proposed by Sorella *et al.* This scheme allows a very precise computation of the ground-state energy of fermionic models. The “minus sign” has been treated without further approximations and does not influence the accuracy. In this paper we focus on the question of the existence of quasiparticles in the Hubbard model in one dimension and resolve the controversy concerning the discontinuity in the momentum distribution function between recent quantum Monte Carlo and weak-coupling renormalization-group techniques.

I. INTRODUCTION

It is generally understood that strong electronic correlations play a decisive role in high- T_c superconductors.^{1–3} This is a general feature of the 3d transition-metal oxide as well as of 4f systems. The importance of electronic correlations in the high- T_c materials is manifested by the failure of band-structure calculations, where the electron-electron interaction is treated on a mean-field level. It is a long-standing challenge to treat the electron-electron interaction properly. For moderate correlation strength, as in the case of normal metals and covalent semiconductors and insulators, the influence of correlations on the quasiparticle energies is qualitatively understood⁴ and accurate *ab initio* band-structure calculations are possible.^{4–7} These schemes work very well as long as the local-density wave functions are in close agreement with the exact quasiparticle orbitals and as long as the correlation does not introduce new poles in the one-particle Green's function. This, however, is not the case for the high- T_c materials, which are Mott-Hubbard insulators that exhibit an additional “impurity” band at the Fermi level⁸ and where the charge fluctuations are strongly reduced on the copper sites even in the metallic regime of doping.^{8–10} These effects are qualitatively understood on the basis of strong on-site Coulomb repulsion.^{1,11,12} It is to date not possible to treat these properties on an *ab initio* level and one must resort to model calculations.

At the outset it was Anderson¹³ who pointed out the importance of the two-dimensional (2D) Hubbard Hamiltonian as the most simple model to mimic the crucial features of the high- T_c materials. Although the question whether the pairing mechanism is mainly driven by the strong correlations on the copper sites is still hotly debated, there is general agreement that the Hubbard model is well suited to explain the electronic structure of these materials^{3,14–17} and their magnetic properties.^{16,18–20}

Most of the theoretical results for the Hubbard Hamiltonian or related models are currently obtained by exact

diagonalization.^{11,21–24} These results are very important in the process of developing simple physical pictures and approximate schemes, as they may be more rigorously tested. On the other hand, this technique is restricted to very small cluster sizes. Other alternatives are the quantum Monte Carlo (QMC) calculations, which are applicable to much larger clusters for which finite-size errors are very small or even negligible. The QMC method has become a very powerful tool in dealing with models for strongly correlated particles.²⁵ Most of these schemes are based on the partition function approach²⁶ and are very useful to obtain thermodynamic properties. The results, so far obtained in the context of the high- T_c superconductors, are for temperatures much higher than the transition temperature. Especially the question whether the pure one-band Hubbard model can explain superconductivity is still an open issue. Therefore, as far as very low temperature properties are concerned, another scheme is desirable. Koonin, Sugiyama, and Friederich²⁷ have proposed the projector QMC method (PQMC) to compute directly the ground state of an interacting fermionic system. This scheme has recently been elaborated upon by Sorella *et al.*,²⁸ who proposed a way to overcome the problem of ill-conditioned determinants, which could also be used in the partition function approach.²⁹ In this paper we present an improved version of the PQMC scheme based on the discrete Hubbard-Stratanovich transformation and the Metropolis Monte Carlo (MC) sampling technique.³⁰ Although there is a controversy over the cases, if any, in which the Langevin dynamics is superior to the MC scheme, the latter algorithm is preferable for the fermionic problem since the nodes in the wave function do not produce potential barriers and the minus sign can be treated in the standard way. In this paper we mainly concentrate on the one-dimensional (1D) Hubbard model and address the question of the jump in the momentum distribution function $n(k)$. This quantity has led to a controversial discussion and has raised doubts as to the accuracy of the PQMC scheme proposed by Sorella *et al.* The PQMC algorithm presented here

works equally well in the 2D Hubbard case and the results will be published elsewhere.

II. PQMC ALGORITHM

The projector QMC scheme, as proposed by Koonin, Sugiyama, and Friederich²⁷ and recently further developed by Sorella,²⁸ is based on the property that the operator $e^{-\Theta\mathcal{H}}$ projects a trial wave function $|\Psi_T\rangle$ onto the state lowest in energy, which is not orthogonal to the trial wave function³¹

$$|\Psi\rangle = e^{-\Theta\mathcal{H}}|\Psi_T\rangle \xrightarrow{\Theta \rightarrow \infty} \propto |\Psi_0^{\text{ex}}\rangle. \quad (1)$$

Therefore this scheme allows, in principle, the determination not only of the ground state, but also of the lowest state for a given symmetry. A similar idea has recently been used to determine numerically the des Cloiseaux and Pearson spin-wave spectrum for the 1D Heisenberg chain.³² The one-band Hubbard model reads

$$\mathcal{H} = \mathcal{H}_{\text{kin}} + \mathcal{H}_{\text{int}} = t \sum_{\langle ij \rangle \rho} a_{i\rho}^\dagger a_{j\rho} + U \sum_i n_{i\uparrow} n_{i\downarrow}, \quad (2)$$

where $a_{i\rho}^{(\dagger)}$ is the annihilation (creation) operator with spin ρ at site i . The kinetic term describes the nearest-neighbor hopping while the on-site Coulomb repulsion is contained in the last term. The bottleneck, restricting the exact diagonalization calculations to very small cluster sizes, is the combinatorial increase of basis states in a many-body problem. To circumvent this problem, the interaction term of the Hamiltonian $\mathcal{H}_{\text{int}}$ in Eq. (1) is replaced by a one-particle potential term with stochastic fields σ .³³ To this end the exponential in Eq. (1) is written as a product of m identical exponentials to allow a decoupling of $\mathcal{H}_{\text{kin}}$ and $\mathcal{H}_{\text{int}}$.³⁰ In contrast to Ref. 28, we use the discrete Hubbard-Stratanovich transformation,³⁰ which yields

$$e^{-\Theta\mathcal{H}}|\Psi_T\rangle = \sum_{\{\sigma_{i,\tau}\}} \prod_{\tau=1}^m \left[e^{i\tau\mathcal{H}_{\text{kin}}} e^{-\lambda} \times \sum_{i=1}^N \sigma_{i,\tau} (n_{i\uparrow} - n_{i\downarrow}) \times e^{-\tau H_{\text{kin}}} \right] |\Psi_T\rangle, \quad (3)$$

with $\lambda = \text{arccosh}(U\Theta/2m)$ and $\tau = t\Theta/2m$, $\sigma_{i,\tau} = \pm 1$. If the trial wave function is a single Slater determinant, the one-particle operators entering Eq. (3) retain this determinantal structure and merely modify the occupied orbitals

$$e^{\hat{h}} \prod_{\alpha=1}^{N_\rho} \left[\sum_{i=1}^N C_{\alpha i}^\rho a_{i\rho}^\dagger |0\rangle \right] = \prod_{\alpha=1}^{N_\rho} \left[\sum_{i=1}^N \sum_{l=1}^N (e^h)_{il} C_{\alpha l}^\rho a_{i\rho}^\dagger \right] |0\rangle, \quad (4)$$

where the one-particle operator is $\hat{h} = \sum_{ij} h_{ij} a_{i\rho}^\dagger a_{j\rho}$ and N_ρ is the number of electrons with spin ρ . The transformation matrix $(e^h)_{il}$ is the exponential of the matrix h_{ij} defining the one-particle operator $\hat{h}$. Thus representing the orbitals by their coefficient vectors $\mathbf{C}_\alpha^\rho$, the operator $e^{\hat{h}}$ acts as matrix $(e^h)_{ij}$. The imaginary time propagation

[index τ in Eq. (3)] of one-particle orbitals in stochastic fields has been studied intensively, and optimal algorithms to treat kinetic and potential terms have been developed.³¹ Here we make use of the product algorithm RS_2 .³¹ Following Ref. 28, we orthogonalize the one-particle orbitals after a given number of Trotter time steps to assure numerical stability. For every Hubbard-Stratanovich field configuration, $\{\sigma\}$, entering the sum in Eq. (3), a single Slater determinant has to be constructed from the propagated occupied orbitals. The approximate to the exact lowest energy state Eq. (1), has, therefore, the form $|\Psi\rangle = \sum_{\{\sigma\}} |\psi\{\sigma'\}\rangle$. The expectation value of an operator $\hat{O}$ can therefore be written

$$\begin{aligned} \langle \hat{O} \rangle_\omega &= \sum_{\{\sigma\}|\{\sigma'\}} \frac{\langle \psi\{\sigma\} | \hat{O} | \psi\{\sigma'\} \rangle}{\langle \psi\{\sigma\} | \psi\{\sigma'\} \rangle} \times \frac{\langle \psi\{\sigma\} | \psi\{\sigma'\} \rangle}{\sum_{\sigma\sigma'} \langle \psi\{\sigma\} | \psi\{\sigma'\} \rangle} \\ &= \sum_{\{\sigma\}|\{\sigma'\}} O(\sigma, \sigma') \omega(\sigma, \sigma'). \end{aligned} \quad (5)$$

In general, the function $\omega(\sigma, \sigma')$ changes sign and is therefore not a proper probability density. To circumvent this problem the sign of ω is attached to the observable under consideration $O(\sigma, \sigma')$ and the absolute value of ω is interpreted as the probability density. The expectation value of $\hat{O}$ is therefore the ratio $\langle \hat{O} \rangle_\omega = \langle \hat{O} \hat{s} \rangle_{|\omega|} / \langle \hat{s} \rangle_{|\omega|}$. In the partition function approach, the average sign $\langle \hat{s} \rangle_{|\omega|}$ as well as $\langle \hat{O} \hat{s} \rangle_{|\omega|}$ goes to zero with increasing β .³⁴ This agrees with our observation of the PQMC scheme when the parameter Θ is increased. Therefore high accuracy is necessary for the Monte Carlo determination of numerator and denominator. One source of numerical errors was the ill-conditioned determinants of nearly singular matrices entering the weight functions. This problem can be avoided by orthogonalizing the occupied one-particle orbitals after a given number of Trotter time steps. The average sign presents no severe problem for the PQMC scheme presented here, since for a proper trial wave function $|\Psi_T\rangle$ a moderate Θ is sufficient to produce very accurate ground-state energies.³⁵

The weight ω is given by the product of the determinants of overlap matrices for up and down spins $\omega = \det(S_{\alpha\alpha'}^\uparrow) \det(S_{\alpha\alpha'}^\downarrow)$ since the Hamiltonian is additive in the spin degrees of freedom,

$$S_{\alpha\alpha'}^\rho = \sum_{i=1}^N C_{\alpha i}^{\rho*}(\sigma) C_{\alpha' i}^\rho(\sigma p), \quad (6)$$

where $C_{\alpha i}^\rho(\sigma)$ is the expansion coefficient of the one-particle orbital α after the imaginary time propagation through the Hubbard-Stratanovich fields. Due to the expansion in determinants [Eq. (5)] Wick's theorem can be used to express more complicated expectation values of fermion operators in terms of the one-particle density matrix

$$\frac{\langle \psi\{\sigma\} | a_{i\rho}^\dagger a_{j\rho} | \psi\{\sigma'\} \rangle}{\langle \psi\{\sigma\} | \psi\{\sigma'\} \rangle} = \sum_{\alpha\alpha'} C_{\alpha i}^{\rho*}(\sigma p) S_{\alpha\alpha'}^{\rho-1} C_{\alpha' j}^\rho(\sigma). \quad (7)$$

The number m of Trotter slices has been chosen to satisfy $\epsilon = \Theta U / m \leq 0.25$, since ϵ^2 is the error of the RS_2 algorithm.

III. NUMERICAL RESULTS

Despite the simple form of the Hubbard model there are still several open questions. Owing to the exact solution by Lieb and Wu³⁶ the ground-state energy as well as the kinetic and interaction contribution are known exactly in the form of integral equations. Other correlation functions than the double occupancy $\langle n_{i\uparrow}n_{i\downarrow} \rangle$ are not accessible by the exact Lieb-Wu equation. Exact results are known in the Heisenberg limit for strong coupling, showing that long-range order in the spin degrees of freedom is destroyed by fluctuations and that the correlation function decays at a rate of $(-1)^r/r$ with distance r .³⁷ The question of the existence of quasiparticles versus localized states or collective excitations is of special interest in the context of high- T_c materials, where an insulator metal phase transition is observed for a given doping concentration.¹⁸ In this respect the momentum distribution function $n(k)$, the discontinuity of which defines the Fermi surface, is a useful quantity. It contains information on how strongly the electron-electron interaction influences the quasiparticles. $n(k)$ is related to the time-ordered Green's function via a contour integral in the upper complex plane. If the quasiparticle picture is valid, the one-particle Green's function is written as $g(k, \omega) = z(k)/[\omega - \epsilon'(k) - i\epsilon''(k)] + \tilde{g}(k)$, where $\tilde{g}(k)$ is the incoherent background, $z(k)$ the pole strength, and $\epsilon'(k) + i\epsilon''(k)$ the complex quasiparticle energy. The imaginary part of the energy changes sign from positive to negative at the Fermi level [associated with a diverging lifetime of $\tau \propto (\omega - \omega_F)^{-2}$] leading to a jump $\Delta n(k)$ in $n(k)$ at the Fermi level of height $z(k_F)$.

According to the exact solution by Lieb and Wu³⁶ the 1D half-filled Hubbard model is an insulator for any nonzero value of the on-site repulsion U . Weak coupling renormalization-group (RG) results³⁸ show that the half-filled Hubbard model scales to strong coupling in which case $n(k)$ is known to be a smooth function.³⁹ This agrees with the PQMC results presented in Fig. 1. Aside from half-filling, in the incommensurate case, the excitation spectrum is gapless³⁶ for all values of U . The RG theory³⁸ predicts for this case that the Hubbard model scales to the Tomonaga-Luttinger (TL) model.⁴⁰ The exact solution^{40,41} of this model yields a power-law behavior for $n(k)$ close to k_F .

$$n^{\text{TL}}(k) = a + b|k - k_F|^{2\alpha} \text{sgn}(k - k_F), \quad (8)$$

where a, b , and α are constants. Accordingly, no jump is to be expected in the 1D Hubbard model. This is in strict contrast to what was found by Sorella *et al.*²⁸ It has been argued that this might be related to ergodicity problems in the Langevin algorithm or other numerical shortcomings. In the framework of the MC algorithm, presented in this paper, such problems can be ruled out, but nevertheless the results are within the statistical accuracy identical to those of Ref. 28. These results have also been obtained very recently in the framework of the grand-canonical partition function approach.⁴² Figure 1 contains $n(k)$ for various values of U . We define a finite-size approximant for the jump $\tilde{\Delta}$ by linearly extrapolating the points closest to k_F from the left and the right. The U

FIG. 1. U dependence of the momentum distribution function for a 32-site ring filled with 15 spin-up and 15 spin-down electrons (dashed curves) in comparison with the half-filled band case for $U=4$ (solid curve). The crosses indicate the MC results.

dependence of $\tilde{\Delta}$ is depicted in Fig. 2, which shows a sort of crossover from a single-particle (itinerant) behavior to an incoherent (localized) state. Since in the quasiparticle picture $\Delta n(k_F)$ is inversely proportional to the effective mass, the latter diverges at k_F for $U < U^*$. This behavior is qualitatively similar to the Brinkman-Rice transition obtained within the Gutzwiller approximation.⁴³ Increasing the system size to 128 sites does not change the qualitative features of Fig. 2. To test the reliability of the PQMC method we compare in Fig. 3 the interaction part of the energy, obtained by MC means, with the Lieb-Wu result. To this end we have solved the Lieb-Wu integral equation numerically to six-digit accuracy. We find excellent agreement between the two results, which is likewise true for the kinetic energy. This clearly proves that the PQMC method yields the correct ground state. The ground state does not break the symmetry to the charge-density wave (CDW) or spin-density wave (SDW) and therefore a finite-size error—in the sense that an incommensurate CDW or SDW does not fit into the system—is not to be expected. Moreover, it has been shown earlier⁴⁴ that finite-size calculations yield the correct divergences in the spin and charge susceptibilities. These diver-

FIG. 2. Quasijump in $n(k)$, as explained in the text, vs on-site repulsion U .

FIG. 3. Number of double occupancies $1/N \sum_{i=1}^N \langle n_{i\uparrow} n_{i\downarrow} \rangle$. The full curve stands for the exact Lieb-Wu result. The numerical data including the statistical errors are represented by dots.

To resolve the discrepancy between the two seemingly contradictory results we analyze the power-law behavior [Eq. (8)] in more detail. First, in the region of weak coupling ($\alpha < 1$), according to Eq. (8) the derivative $dn(k)/dk$ diverges at k_F (Ref. 41) and yields, so to speak, a residual Fermi surface. Increasing the coupling strength removes this divergence and the residual Fermi surface vanishes. In the Hubbard model all coupling constants g entering the TL model are equal to U . The exponent α is then given by³⁸ $\alpha = [(1 - \bar{u}^2)^{-0.5} - 1]/4$, with $\bar{u} = U/[2\pi t \sin(k_F)]$. Evaluating $n^{TL}(k)$ for a lattice with N sites and k spacing $\Delta k = 2\pi/N$, we can estimate the minimum system size, which is necessary to distinguish a continuous structure in $n^{TL}(k)$ from a jump. Increasing $n(k_F) - n(k_F - \Delta k)$ [Eq. (8)] by 0.5 yields the order of magnitude for the critical system size N^* below which the continuous $n^{TL}(k)$ looks like a jump. N^* is given by $\pi e^{1/2\alpha}$, which is of the order 10^{69} for $U=1$. We evaluated $n^{TL}(k)$ (Ref. 46) numerically for the set of k points of a 32-site ring and find precisely the same scenario as that depicted in Figs. 1 and 2.

Thus we see that the results of QMC simulation and

RG theory are both correct. In a strictly mathematical sense there is no jump in $n^{TL}(k)$. From a practical point of view, however, be it a finite cluster calculation or an experiment with finite resolution and for finite temperature, there is a jump indicating a residual Fermi surface. The TL model is a nice example of how the quasiparticle pole in the one-particle Green's function can vanish very close to k_F and how collective excitations develop.^{38,46} But, again, for weak coupling this feature is more of academic interest since any finite cluster calculation or experiment would make it look like a common quasiparticle.⁴⁷ Within the experimental resolution, therefore, the 1D Hubbard model exhibits a crossover from an itinerant behavior at weak coupling to a localized behavior at strong coupling.

IV. CONCLUSION

We have demonstrated that the projector quantum Monte Carlo scheme works very well in connection with the discrete Hubbard-Stratanovich transformation and the Metropolis importance sampling. Our results corroborate the findings of Sorella *et al.* that the 1D Hubbard model for away from half-filling and for $U=4$ shows a quasijump structure. In this paper we investigated the U dependence of the momentum distribution function and found a quasi-metal-insulator transition and resolved the discrepancy between the numerical and the exact results. How the absence of a true jump in $n(k)$ influences thermodynamic properties and how the quasi-metal-insulator transition is reflected in these quantities is planned to be published elsewhere.⁴⁷

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*Also at Institute of Physics, University of Zurich, Schönberggasse 9, 8001 Zurich, Switzerland.

¹P. Fulde in *Proceedings of the International Conference on High-Temperature Superconductors and Materials of Superconductivity, Interlaken, 1989*, edited by J. Müller and J. L. Olsen (North-Holland, Amsterdam, 1989), p. 1769.

²Y. Ando, M. Sera, S. Yamagata, S. Kondoh, M. Onoda, and M. Sato, *Solid State Commun.* **70**, 303 (1989).

³K. Kitazawa, *IBM J. Res. Dev.* **33**, 201 (1989).

⁴P. Horsch, W. von der Linden, and W. D. Lukas, *Solid State Commun.* **62**, 359 (1987); W. von der Linden, P. Horsch, and W. D. Lukas, *Solid State Commun.* **59**, 485 (1986).

⁵S. Horsch, P. Horsch, and P. Fulde, *Phys. Rev. B* **28**, 5977 (1983); W. von der Linden and P. Horsch, *Phys. Rev. B* **37**, 8351 (1988).

⁶W. Godby, M. Schlüter, and L. J. Sham, *Phys. Rev. B* **35**, 4170

(1987).

⁷M. S. Hybertsen and S. G. Louie, *Phys. Rev. B* **34**, 5390 (1986).

⁸T. Takahashi, H. Matsuyama, H. Katayama-Yoshida, Y. Okabe, S. Hosoya, K. Seki, H. Fujimoto, M. Sato, and H. Inohkuchi, *Nature* **334**, 691 (1988); *Phys. Rev. B* **39**, 6636 (1989).

⁹W. Drube, F. J. Himpsel, G. V. Chandrashekar, and M. W. Shafer, *Phys. Rev. B* **38**, 11946 (1988).

¹⁰J. Fink, N. Nücker, H. Romberg, and J. C. Fuggle, *IBM J. Res. Dev.* **33**, 372 (1989).

¹¹P. Horsch, W. E. Stephan, K. von Szczepanski, M. Ziegler, and W. von der Linden, in *Proceedings of the International Conference on M²S-HTSC, Stanford, 1989*, edited by N. E. Phillips *et al.* [Physica C (to be published)].

¹²H. Eschrig, *Physica C* (to be published).

¹³P. W. Anderson, *Science* **235**, 1196 (1987).

- ¹⁴V. J. Emery, Phys. Rev. Lett. **58**, 2794 (1987).
- ¹⁵C. M. Varma, S. Schmitt-Rink, and E. Abrahams, Solid State Commun. **62**, 681 (1987).
- ¹⁶E. B. Stechel and D. Jennison, Phys. Rev. B **38**, 4632 (1988).
- ¹⁷F. C. Zhang and T. M. Rice, Phys. Rev. B **37**, 3759 (1988).
- ¹⁸J. Birgeneau and G. Shirane, in *Physical Properties of High Temperature Superconductors*, edited by D. M. Ginsberg (World Scientific, Singapore, 1989).
- ¹⁹P. Horsch and W. von der Linden, Z. Phys. B **72**, 181 (1988), and references therein.
- ²⁰C. Gross, R. Joynt, and T. M. Rice, Z. Phys. B **68**, 425 (1987).
- ²¹W. H. Stephan, W. von der Linden, and P. Horsch, Phys. Rev. B **39**, 2924 (1989).
- ²²J. Bonca, P. Prelovsek, and I. Sege, Phys. Rev. B **39**, 7074 (1989).
- ²³D. Poilblanc, Phys. Rev. B **39**, 140 (1989).
- ²⁴J. E. Hirsch, S. Tand, E. Loh, and D. J. Scalapino, Phys. Rev. Lett. **60**, 1668 (1988); Phys. Rev. B **39**, 243 (1989).
- ²⁵M. Suzuki, in *Proceedings of the Ninth Taniguchi International Symposium on Quantum Monte Carlo Methods, Susono, 1986*, edited by M. Suzuki (Springer-Verlag, Heidelberg, 1986).
- ²⁶H. De Raedt and A. Lagendijk, Phys. Rep. **127**, 235 (1985), and references therein.
- ²⁷S. E. Koonin, G. Sugiyama, and H. Friederich, in *Proceedings of the International Symposium, Bad Honnef, 1982* (Springer-Verlag, Heidelberg, 1982), p. 214.
- ²⁸S. Sorella, E. Tosatti, S. Baroni, R. Car, and M. Parrinello, in *Proceedings of the Adriatic Research Conference Towards the Theoretical Understanding of the High- T_c Superconductors, Trieste, 1988*, edited by S. Lundqvist (World Scientific, Singapore, 1988); S. Sorella, S. Baroni, R. Car, and M. Parrinello, Europhys. Lett. **8**, 663 (1989).
- ²⁹S. White, D. J. Scalapino, L. Sugar, E. Y. Loh, J. E. Gubernatis, and T. Scalettar, Phys. Rev. B **40**, 506 (1989).
- ³⁰J. E. Hirsch, Phys. Rev. B **28**, 4059 (1983).
- ³¹H. De Raedt, Comp. Phys. Rep. **7**, 1 (1987).
- ³²M. Takahashi, Phys. Rev. Lett. **62**, 2313 (1989).
- ³³L. Stratanovich, Dokl. Akad. Nauk. SSSR **115**, 1907 (1957) [Sov. Phys.—Dokl. **2**, 416 (1958)].
- ³⁴I. Morgenstern and D. Würz, Z. Phys. B **70**, 115 (1988); I. Morgenstern *ibid.* **70**, 291 (1988); Z. Phys. (to be published).
- ³⁵I. Morgenstern, W. von der Linden, and H. de Raedt (unpublished).
- ³⁶E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).
- ³⁷A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1975).
- ³⁸J. Solyom, Adv. Phys. **28**, 201 (1979).
- ³⁹J. Carmelo and D. Baeriswyl, Phys. Rev. B **37**, 7541 (1988).
- ⁴⁰J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963).
- ⁴¹D. C. Mattis and E. H. Lieb, J. Math. Phys. **6**, 375 (1965).
- ⁴²C. Bourbonnais, H. Nelisse, A. Reid, and A.-M. S. Tremblay, Phys. Rev. B **40**, 2297 (1989).
- ⁴³W. F. Brinkman and T. M. Rice, Phys. Rev. B **2**, 4302 (1970).
- ⁴⁴J. E. Hirsch and D. J. Scalapino, Phys. Rev. B **27**, 7169 (1983).
- ⁴⁵D. Baeriswyl and J. J. Forney, J. Phys. C **13**, 3203 (1980).
- ⁴⁶A. Luther and I. Peschel, Phys. Rev. B **9**, 2911 (1974).
- ⁴⁷D. Baeriswyl, W. von der Linden, and I. Morgenstern (unpublished).