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Exponential stability for formation control systems with generalized controllers: A unified approach

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ABSTRACT

This paper discusses generalized controllers for distance-based rigid formation shape stabilization and aims to provide a unified approach for the convergence analysis. We consider two types of formation control systems according to different characterizations of target formations: minimally rigid target formation and non-minimally rigid target formation. For the former case, we firstly prove the local *exponential stability* for rigid formation systems when using a general form of shape controllers with certain properties. From this viewpoint, different formation controllers proposed in previous literature can be included in a unified framework. We then extend the result to the case that the target formation is non-minimally rigid, and show that exponential stability of the formation system is still guaranteed with generalized controllers.

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1. Introduction

Formation control of networked multi-agent systems has received considerable attention in recent years due to its extensive applications. One problem of extensive interest is *formation shape control*, i.e. on designing controllers to achieve or maintain a geometrical shape for the formation Oh et al. [1]. As reviewed in [1], the existing results on formation control can be characterized into position-, displacement-, and distance-based control strategies according to different types of sensed and controlled variables. Among these formation control strategies, distance-based formation control receives particular interest as it allows reduced requirement on the sensing capability for individual agent compared to the other two strategies. Thus, this paper focuses on distance-based formation control. In particular, we confine our attention in this paper to *undirected* rigid formations, while relevant discussions on directed formation control can be found in e.g. [2].

With underpinnings from rigid graph theory, it has been established that the formation shape can be achieved by controlling a certain set of inter-agent distances Olfati-Saber and Murray [3]; Anderson et al. [4]; Krick et al. [5]. In the rigid formation stabilization problem, one typical controller that has been studied

extensively in the literature takes the following form (see e.g. [5]):

$$\dot{p}_i = \sum_{j \in \mathcal{N}_i} (p_j - p_i) (\|p_j - p_i\|^2 - d_{ij}^2) \quad (1)$$

where the definitions underpinning the notation will become clear in later sections. The above controller (1), which is derived from a well-defined potential function, serves as a standard control law for stabilizing rigid formations. The dynamics of the formation control system (1) have been investigated in several succeeding papers, e.g. [6–8]. We mention that alternative kinds of formation controllers other than the one in (1) are also available, which have been reported sparsely in the literature (see e.g. [9–12]).

The main objective of this paper is to analyze general forms of formation controllers to stabilize rigid shapes. The main contributions of this paper include a unified approach to discuss the convergence and controller performance of generalized formation controllers, and the associated *exponential stability* of general formation systems when certain properties of the potential function are satisfied. We show that for a large family of formation control systems which generalizes most existing formation controllers in the literature, the exponential stability of the distance error system can be guaranteed (for a list of such controllers, see Section 3.1). As is well-known in the control field, exponential stability has the robust property against small perturbations Khalil [13]. Such robust property has been employed in recent papers Mou et al. [14]; Sun et al. [15]; Garcia de Marina

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et al. [16] to study the behavior of rigid undirected formations when there are mismatches or discrepancies between neighbor agents' distance measurements. Exponential stability enables that the desired equilibrium of the derived distance error system under controller (1) is still exponentially stable with small distance perturbations. Note that in the analysis of the formation robustness issue in [14–16], the formation control system is limited to the case of the controller in (1). To this end, this paper shows that formation distance error systems with generalized controllers, which include many special forms of formation controllers studied in the literature, also inherit this robustness property as a consequence of the exponential stability.

In this paper, we first consider formation stabilization control when the target formation is minimally rigid, which is a common assumption in the literature. We list several requirements of the potential and gradient functions associated with the distributed control for each agent which render an exponential convergence of the formation system. We give an explicit lower bound of the convergence rate, and also discuss several properties of the generalized formation control systems. By deriving a reduced distance error system via the decomposition principle of a rigid framework, we then extend the analysis from the minimally rigid case to the non-minimally rigid case (motivated by the analysis in [14,17]), and further prove that the exponential convergence still holds for non-minimally rigid formation control when generalized formation controllers are applied.

A preliminary version of this paper has been presented in [18]. The extensions of this paper compared to Sun et al. [18] include complete proofs for all the key results which were omitted in [18], detailed discussions on certain properties of generalized formation controllers, new simulation results, and especially, a new section to discuss convergence property for non-minimally rigid formations.

The rest of this paper is organized as follows. In Section 2, preliminary concepts on graph theory and rigidity theory are introduced. In Section 3, we provide detailed analysis on generalized controllers and prove the exponential stability property for minimally rigid formations. Section 4 discusses exponential convergence for formation control systems when the target formation is non-minimally rigid. In Section 5, two sets of simulation examples are provided to demonstrate the controller performance. Finally, Section 6 concludes this paper.

2. Basic concepts on graph and rigidity theory

Consider an undirected graph with m edges and n vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \dots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The matrix relating the nodes to the edges is called the incidence matrix $H = \{h_{ij}\} \in \mathbb{R}^{m \times n}$, whose entries are defined as (with arbitrary edge orientations for the undirected formations considered here)

$$h_{ij} = \begin{cases} 1, & \text{the } i\text{th edge sinks at node } j \\ -1, & \text{the } i\text{th edge leaves node } j \\ 0, & \text{otherwise.} \end{cases}$$

For a connected undirected graph, one has $\ker(H) = \text{span}\{\mathbf{1}_n\}$. Note that for a rigid formation modeled by an undirected graph considered in this paper, the orientation of each edge for writing the incidence matrix can be defined arbitrarily and the stability analysis in the next sections remains unchanged.

Let $p_i \in \mathbb{R}^d$ where $d = \{2, 3\}$ denote a point that is assigned to $i \in \mathcal{V}$. The stacked vector $p = [p_1^T, p_2^T, \dots, p_n^T]^T \in \mathbb{R}^{dn}$ represents the realization of $\mathcal{G}$ in $\mathbb{R}^d$. The pair $(\mathcal{G}, p)$ is said to be a framework of $\mathcal{G}$ in $\mathbb{R}^d$. By introducing the matrix $\bar{H} := H \otimes I_d \in \mathbb{R}^{dm \times dn}$, one

can construct the relative position vector as an image of $\bar{H}$ from the position vector p :

$$z = \bar{H}p \quad (2)$$

where $z = [z_1^T, z_2^T, \dots, z_m^T]^T \in \mathbb{R}^{dm}$, with $z_k \in \mathbb{R}^d$ being the relative position vector for the vertex pair defined by the k th edge.

Using the same ordering of the edge set $\mathcal{E}$ as in the definition of H , the rigidity function $r_{\mathcal{G}}(p) : \mathbb{R}^{dn} \rightarrow \mathbb{R}^m$ associated with the framework $(\mathcal{G}, p)$ is given as:

$$r_{\mathcal{G}}(p) = \frac{1}{2} [\dots, \|p_i - p_j\|^2, \dots]^T, \quad (i, j) \in \mathcal{E} \quad (3)$$

where the norm is the standard Euclidean norm, and the k th component in $r_{\mathcal{G}}(p)$, $\|p_i - p_j\|^2$, corresponds to the squared length of the relative position vector z_k which connects the vertices i and j .

The rigidity of frameworks is then defined as follows.

Definition 1 (Asimow and Roth [19]). A framework $(\mathcal{G}, p)$ is rigid in $\mathbb{R}^d$ if there exists a neighborhood $\mathbb{U}$ of p such that $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) \cap \mathbb{U} = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p)) \cap \mathbb{U}$ where $\mathcal{K}$ is the complete graph with the same vertices as $\mathcal{G}$.

In the following, the set of all frameworks $(\mathcal{G}, p)$ which satisfy the distance constraints is referred to as the set of *target formations*. Let (d_{kij}) denote the desired distance of edge k in the target formation which links agent i and j . We further define

$$e_{kij} = \|p_i - p_j\|^2 - (d_{kij})^2 \quad (4)$$

to denote the squared distance error for edge k . Note we may use e_k and d_k occasionally for notational convenience if no confusion is expected. Also in the context of formation control under discussions, the term *framework* will be occasionally referred to as *formation* realized by a set of n agents. Thus we may also use the two terms, *framework* and *formation* interchangeably if no confusion is caused. Define the distance error vector as $e = [e_1, e_2, \dots, e_m]^T$.

One useful tool to characterize the rigidity property of a framework is the rigidity matrix $R \in \mathbb{R}^{m \times dn}$, which is defined as

$$R(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p}. \quad (5)$$

It is not difficult to see that the row of the rigidity matrix R corresponding to $\{(i, j) \in \mathcal{E}\}$ takes the following form

$$[\mathbf{0}_{1 \times d}, \dots, (p_i - p_j)^T, \dots, \mathbf{0}_{1 \times d}, \dots, (p_j - p_i)^T, \dots, \mathbf{0}_{1 \times d}]. \quad (6)$$

Each edge gives rise to a row of R , and, if an edge links vertices i and j , then the nonzero entries of the corresponding row of R are in the columns from $di - (d - 1)$ to di and from $dj - (d - 1)$ to dj .

Eq. (2) shows that the relative position vector lies in the image of $\bar{H}$. Thus one can redefine the rigidity function, $g_{\mathcal{G}}(z) : \text{Im}(\bar{H}) \rightarrow \mathbb{R}^m$ as $g_{\mathcal{G}}(z) = \frac{1}{2} [\|z_1\|^2, \|z_2\|^2, \dots, \|z_m\|^2]^T$. From (2) and (5), one can obtain the following simple form for the rigidity matrix

$$R(p) = \frac{\partial r_{\mathcal{G}}(p)}{\partial p} = \frac{\partial g_{\mathcal{G}}(z)}{\partial z} \frac{\partial z}{\partial p} = Z^T \bar{H} \quad (7)$$

where $Z = \text{diag}\{z_1, z_2, \dots, z_m\}$. It is obvious that the entries of $R(p)$ are also functions of z , and we will also write it as $R(z)$. The rigidity matrix will be used to determine the infinitesimal rigidity of the framework, as shown in the following theorem.

Theorem 1 (Hendrickson [20]). Consider a framework $(\mathcal{G}, p)$ in d -dimensional space with $n \geq d$ vertices and m edges. It is infinitesimally rigid if and only if

$$\text{rank}(R(p)) = dn - d(d + 1)/2. \quad (8)$$

Specifically, if the framework is infinitesimally rigid in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) and has exactly $2n - 3$ (resp. $3n - 6$) edges, then it is called a minimally and infinitesimally rigid framework. In this paper we focus on the stabilization problem of infinitesimally rigid target formations. From the definition of infinitesimal rigidity, the following result is obvious.

Lemma 1. *If the framework $(\mathcal{G}, p)$ is minimally and infinitesimally rigid in the d -dimensional space, then the matrix $R(z(p))R(z(p))^T$ is positive definite.*

The following result will be useful for later analysis, which shows that there exists a smooth function which maps the distance set of the minimally rigid framework to the distance set of its corresponding framework modeled by a complete graph.

Lemma 2. *Let $r_{\hat{\mathcal{G}}}(q)$ be the rigidity function for a given infinitesimally minimally rigid framework $(\mathcal{G}, q)$ with agents' position vector q . Further let $\hat{r}_{\hat{\mathcal{G}}}(q)$ denote the rigidity function for an associated framework $(\hat{\mathcal{G}}, q)$, in which the vertex set remains the same as $(\mathcal{G}, q)$ but the underlying graph is a complete one (i.e. there exist $n(n-1)/2$ edges which link all vertex pairs). Then there exists a continuously differentiable function $f : r_{\mathcal{G}}(q) \rightarrow \mathbb{R}^{n(n-1)/2}$ for which $\hat{r}_{\hat{\mathcal{G}}}(q) = f(r_{\mathcal{G}}(q))$ holds locally.*

The proof of the above lemma is omitted here and can be found in [14].

3. Exponential stability for minimally rigid formations

In the coordination control of multi-agent systems, the potential-function-based gradient control law is a popular approach to generate distributed controls for each individual agent for achieving a global task Sakurama et al. [21]. We also follow this framework and construct distance potential functions for stabilizing rigid formation shapes. This section aims to provide a unified analysis for the convergence property when generalized controllers are constructed for stabilizing minimally rigid formations. One of the key results in this section is that exponential convergence can be derived for a large family of distributed formation controllers if certain properties are satisfied. To this end, most current results in the literature on formation shape control with different controllers can be included as special cases and studied in a unified framework. In this section, we assume that the target formation is minimally rigid, while this minimal rigidity assumption will be relaxed in the next section.

Let us define a general potential function $V_k : (-d_{kij}^2, \infty) \rightarrow \mathbb{R}_{\geq 0}$ with the following properties:

1. $V_k(\varphi)$ is nonnegative and continuously differentiable on $(-d_{kij}^2, \infty)$;
2. $V_k(\varphi)$ and its gradient $g_k(\varphi) := \frac{\partial V_k(\varphi)}{\partial \varphi}$ are zero if and only if $\varphi = 0$;
3. $g_k(\varphi)$ is strictly monotone increasing and continuously differentiable over its domain.

For each agent i , the local potential is constructed as

$$\Theta_i = \frac{1}{2} \sum_{j \in \mathcal{N}_i} V_k(e_{kij}) \quad (9)$$

where the distance error e_{kij} is defined in (4). We assume each agent is modeled by a single integrator, in the form of $\dot{p}_i = u_i$ where u_i is the control input to be designed. According to the gradient control law, each agent moves via the following gradient flow to minimize its local potential function:

$$\dot{p}_i = u_i = -\nabla_{p_i} \Theta_i = -\sum_{j \in \mathcal{N}_i} (p_i - p_j) g_k(e_{kij}) \quad (10)$$

we note that for stabilizing different inter-agent distances, the function g_k in the controller can take different forms. This implies one can use a group of *heterogeneous agents* for implementing formation control strategy in the sense that each agent may have different sensing/controlling performances for controlling different inter-agent distances as reflected by different g_k .

3.1. Review of special controllers in the literature

The general potential function and the gradient-based controller (10) defined above include many formation controllers proposed in the literature. For example, in [9] the controller is a special case of (10) which takes the following form

$$\dot{p}_i = -\sum_{j \in \mathcal{N}_i} \frac{p_i - p_j}{\|p_i - p_j\|} (\|p_i - p_j\| - d_k) \quad (11)$$

where the gradient function $g_k(e_k)$ is described as

$$g_k(e_k) = 1 - \frac{d_k}{\sqrt{(e_k + d_k^2)}}. \quad (12)$$

In [10], the designed controller takes the following form

$$\dot{p}_i = -\sum_{j \in \mathcal{N}_i} \frac{p_i - p_j}{\|p_i - p_j\|^2} (\|p_i - p_j\|^2 - d_k^2) \quad (13)$$

where $g_k(e_k)$ is

$$g_k(e_k) = 1 - \frac{d_k^2}{(e_k + d_k^2)}. \quad (14)$$

In [22,11,12], the following formation system was studied

$$\dot{p}_i = -\sum_{j \in \mathcal{N}_i} \frac{p_i - p_j}{\|p_i - p_j\|^4} (\|p_i - p_j\|^4 - d_k^4) \quad (15)$$

where $g_k(e_k)$ in this case is described by

$$g_k(e_k) = 1 - \frac{d_k^4}{(e_k + d_k^2)^2}. \quad (16)$$

Note that the above control functions (12), (14) and (16) have the attractive property of ensuring collision avoidance for the formation system, provided the agents are not collocated initially.

The most commonly-used gradient control law for stabilizing rigid formations involves the potential function in the form of $V_k = 1/2 e_k^2$, with the derived g_k in a simple form:

$$g_k(e_k) = e_k. \quad (17)$$

This controller was proposed in [5] and was further studied in e.g. [6–8,23,24].

In this section we will show that all such control functions including those mentioned above guarantee an exponential convergence of the formation control system.

3.2. Convergence analysis

Define a vector Φ to denote the gradients of the potential functions in each edge:

$$\Phi(e) = [g_1(e_1), g_2(e_2), \dots, g_m(e_m)]^T.$$

By noting that $\frac{\partial V_k(e_k)}{\partial p_i} = g_k(e_k) \frac{\partial e_k}{\partial p_i}$ and comparing the expression of the single agent system (10) with the rigidity matrix (6), one can derive the following compact form of the overall formation system:

$$\dot{p} = -R^T(z) \Phi(e) \quad (18)$$

where $R(z)$ is the rigidity matrix. We first note that because the right-hand side of the formation control system (18) with generalized controllers is Lipschitz continuous due to the continuous differentiability of both $(p_i - p_j)$ for all $(i, j) \in \mathcal{E}$ and $g_k(e_k(p))$ for all k , the solution of (18) exists globally.

According to the definition of the error vector e and the derivation of R , one can obtain the following compact equation for the distance error vector

$$\dot{e} = \frac{\partial e}{\partial p} \dot{p} = 2R(z)\dot{p} = -2R(z)R^T(z)\Phi(e). \quad (19)$$

Remark 1. It has been proved in [14] that when the formation shape is close to the desired one, the entries of the matrix $R(z)R^T(z)$ are continuously differentiable functions of e . We also denote $Q(e) := R(z)R^T(z)$ to reflect this fact. So the system described in (19) is a *self-contained* system, and we will call it the *distance error system* in the sequel.

The main result in this section is stated in the following theorem which concerns an exponential stability of the distance error system (19).

Theorem 2. Consider the gradient control (10) derived from the general potential functions with the listed properties. Then the distance error system (19) has an **exponentially stable** equilibrium at the origin.

Proof. We divide the whole proof into three main steps.

(I) Proof of asymptotic stability. Define the overall potential function: $V = \frac{1}{2} \sum_{k=1}^m V_k(\|z_k\|^2 - d_k^2)$ which also serves as the candidate Lyapunov function. Further define a sub-level set $\Psi(\rho) = \{e : V(e) \leq \rho\}$ for some sufficiently small ρ , such that for all the points in the set $\Psi(\rho)$ the formation is infinitesimally minimally rigid and close to the target formation. Thus the rigidity matrix is of full row rank, and further $Q(e) = R(z)R^T(z)$ is positive definite. By calculating the derivative of V along the trajectories of the distance error system (19), one can show

$$\dot{V}(e) = -\Phi^T(e)Q(e)\Phi(e). \quad (20)$$

Further let λ denote the minimum singular value of the rigidity matrix when e is in the set Ψ : $\lambda = \min_{e \in \Psi} \text{eig}(Q(e)) > 0$. Note that the set $\Psi(\rho)$ is compact and the existence of such λ is guaranteed. Then one has

$$\dot{V}(e) \leq -\lambda \|\Phi(e)\|^2 \quad (21)$$

which indicates that $\dot{V}(e)$ is negative definite for $e \in \Psi \setminus \{0\}$. Thus the asymptotic stability of the equilibrium $e = 0$ in the error system (19) is proved.

(II) Proof of two inequalities. To show the exponential convergence of the error system (19), it suffices to prove the following comparison inequalities (see e.g. Theorem 4.10 of [13]):

$$\|\Phi(e)\|^2 \geq \varrho \|e\|^2 \quad (22)$$

$$c_1 \|e\|^2 \leq V(e) \leq c_2 \|e\|^2 \quad (23)$$

where ϱ , c_1 and c_2 are positive constants which need to be determined. The key properties to derive the above inequalities are that the gradient function $g_k(e_k)$ is strictly monotone increasing and locally Lipschitz continuous (due to its continuous differentiability) on $(-d_k^2, \infty)$. We follow a similar approach as that in the previous paper of Cao et al. [25] to show the above inequalities (22) and (23).

Recall the definition of the Lyapunov function candidate:

$$V = \frac{1}{2} \sum_{k=1}^m V_k(\|z_k\|^2 - d_k^2) = \frac{1}{2} \sum_{k=1}^m \int_0^{e_k} g_k(s) ds.$$

Let ϵ be a positive number, which should satisfy

$$\epsilon \leq \min\{d_1^2, d_2^2, \dots, d_m^2\}. \quad (24)$$

We note that for all e which satisfies $e \in \Psi(\rho) \setminus \{0\}$, the existence for such ϵ can be guaranteed.

Let

$$\zeta_k = \inf_{|s| \leq \sqrt{\epsilon}} \frac{dg_k(s)}{ds}, \quad k \in \{1, 2, \dots, m\}. \quad (25)$$

Each ζ_k is positive because each g_k is strictly monotone increasing on $(-d_k^2, \infty)$. From this and the fact that $g_k(0) = 0$ it follows that

$$|g_k(s)| \geq \zeta_k |s|, \quad |s| \leq \sqrt{\epsilon}, \quad k \in \{1, 2, \dots, m\}. \quad (26)$$

Therefore

$$\int_0^{e_k} g_k(s) ds \geq \frac{\zeta_k}{2} e_k^2, \quad |e_k| \leq \sqrt{\epsilon}. \quad (27)$$

Choose

$$2c_1 = \min \left\{ \frac{\zeta_1}{2}, \frac{\zeta_2}{2}, \dots, \frac{\zeta_m}{2} \right\}. \quad (28)$$

One has $V(e) \geq c_1 \|e\|^2$. Then the left inequality of (23) is true.

Since each g_k is Lipschitz continuous and satisfies $g_k(0) = 0$, there exist positive constants δ_k such that $|g_k(s)| \leq \delta_k |s|$ for $|s| \leq \sqrt{\epsilon}$. Then it follows that

$$\int_0^{e_k} g_k(s) ds \leq \frac{\delta_k}{2} e_k^2, \quad |e_k| \leq \sqrt{\epsilon}. \quad (29)$$

Choose

$$2c_2 = \max \left\{ \frac{\delta_1}{2}, \frac{\delta_2}{2}, \dots, \frac{\delta_m}{2} \right\}. \quad (30)$$

One has $V(e) \leq c_2 \|e\|^2$. Then the right part of (23) is proved.

From (26) it is obvious that $|g_k(e_k)|^2 \geq \zeta_k^2 |e_k|^2$. By choosing

$$\zeta^2 = \min\{\zeta_1^2, \zeta_2^2, \dots, \zeta_m^2\} \quad (31)$$

one has $\|\Phi(e)\|^2 \geq \zeta^2 \|e\|^2$. Let $\varrho = \zeta^2$, which proves the inequality in (22).

(III) Proof of exponential stability. This step is standard from the results of the above two steps. By combining the inequalities in (21)–(23), one obtains that

$$\dot{V}(e) \leq -\lambda \varrho \|e\|^2 \leq -\frac{\lambda \varrho}{c_2} V(e). \quad (32)$$

Thus, it follows that

$$\begin{aligned} \|e(t)\| &\leq \left(\frac{V(e(t))}{c_1} \right)^{\frac{1}{2}} \leq \left(\frac{V(e(0))}{c_1} \exp \left(-\frac{\lambda \varrho}{c_2} t \right) \right)^{\frac{1}{2}} \\ &\leq \left(\frac{c_2 \|e(0)\|^2}{c_1} \exp \left(-\frac{\lambda \varrho}{c_2} t \right) \right)^{\frac{1}{2}} \end{aligned} \quad (33)$$

which indicates the following

$$\|e(t)\| \leq \left(\frac{c_2}{c_1} \right)^{\frac{1}{2}} \exp(-\gamma t) \|e(0)\| \quad (34)$$

with the exponential convergence rate not less than $\gamma = \frac{\lambda \varrho}{2c_2}$. $\square$

Remark 2. A recent paper Oh and Ahn [26] also discussed general controllers to stabilize rigid formation shapes. In [26], the potential function is assumed to be positive definite and analytic (see

Assumption 3.1 in [26]), from which the authors proved the local asymptotic convergence of the formation system by exploring the Łojasiewicz inequality Chill [27] for gradient systems. We note these two properties (positive definiteness and analyticity) are not enough to render exponential convergence of formation control systems. From the above proof of Theorem 2, one can see that the properties of the potential function and gradient function listed in the beginning of this section are minimal requirements to guarantee the exponential convergence. We also note finite-time convergence can be obtained for formation control systems with another set of potential functions and gradient controllers. Relevant discussions of this point can be found in [28].

Note that the convergence of inter-agent distance errors of itself does not directly guarantee the convergence of agents' positions $p(t)$ to some fixed points, even though it does guarantee convergence to a correct formation shape. A sufficient condition for this strong convergence of the position system (18) to a stationary formation is guaranteed by the boundedness of the control function and the exponential convergence as proved above, which indicates that the right hand of (18) is exponentially decaying. This ensures that the solution of (18) exists and is finite when $t \rightarrow \infty$. To sum up, one has the following lemma on the convergence of (18) as a consequence of Theorem 2.

Lemma 3. *The generalized control law (10) guarantees the convergence of $p(t)$ to a fixed point.*

Theorem 2 and Lemma 3 show local convergence to the target formation shape, by assuming that the initial formation shape is close to the target formation shape. In general, the formation control system (18) exhibits multiple equilibria. The set for all equilibria is described as $\mathcal{M} = \{p \in \mathbb{R}^{dn} | R(p)^T g(e(p)) = 0\}$.

We call the set of equilibria at which the correct shape is attained the correct equilibrium set,¹ and we denote it by $\mathcal{M}_c = \{p \in \mathbb{R}^{dn} | g(e(p)) = 0\}$.

Correspondingly, the set of equilibria at which the correct shape is not attained is referred to as incorrect equilibrium set, and it is denoted by $\mathcal{M}_i = \{p \in \mathbb{R}^{dn} | R(p)^T g(e(p)) = 0, g(e(p)) \neq 0\}$.

It is still an open problem to confirm the existence of stable incorrect equilibria for rigid formation control systems (10) with general shapes (see discussions in [29]). In the case that the target formation shape is minimally and infinitesimally rigid, one can show the following result which partly characterizes a converged formation at an equilibrium.

Lemma 4. *Suppose the target formation is minimally and infinitesimally rigid. Then the formation system (10) with generalized controllers converges to either a correct equilibrium in $\mathcal{M}_c$, or an incorrect equilibrium in $\mathcal{M}_i$ at which the converged formation shape is not infinitesimally rigid.*

Proof. Note that $\mathcal{M} = \mathcal{M}_c \cup \mathcal{M}_i$. Theorem 2 actually proves that an equilibrium point in the set of correct equilibrium $\mathcal{M}_c$ is locally exponentially convergent. Note that according to the property of the gradient control function $g(e)$, the condition $g(e(p)) = 0$ (resp. $g(e(p)) \neq 0$) to characterize a correct equilibrium (resp. an incorrect equilibrium) is equivalent to the condition $e(p) = 0$ (resp. $e(p) \neq 0$). At an incorrect equilibrium in $\mathcal{M}_i$, the transpose of the rigidity matrix, R^T , has a non-trivial null vector $g(e(p)) \neq 0$. Thus, at such an equilibrium R is not of full row rank, which violates the rank condition shown in Theorem 1 on infinitesimal rigidity. Hence, at an incorrect equilibrium characterized by $e(p) \neq 0$ (or equivalently, $g(e(p)) \neq 0$), the formation shape is not infinitesimally rigid. $\square$

¹ Note that the realization of a target formation may not be unique due to e.g. flip ambiguity Hendrickson [20]. Thus the correct formation shape may not be unique.

3.3. Properties of the formation control system with generalized controllers

We show some intrinsic properties of the formation system with the generalized controllers.

Lemma 5. *The formation stabilization system designed in (10) (and in the compact form (18)) with the generalized controller has the following properties:*

1. The controller is distributed and each agent requires only relative position measurements with respect to its neighboring agents.
2. The center of the mass of the formation is stationary.
3. The measurement and control for each agent are independent of any global coordinates. That is, each agent can use its own coordinate system to measure the necessary relative positions and to implement the control.

The proof is omitted here, as it follows similar ideas to [25] and in particular to the proof of [28, Lemma 4].

Note that Part 3 of Lemma 5 implies the proposed formation system (10) with generalized controllers guarantees the $SE(N)$ invariance (i.e. translational and rotational invariance) of the controller, which is a nice property to enable convenient implementation for networked control systems without coordinate alignment for each individual agent Vasile et al. [30].

Remark 3. The result in this section also provides insights to design other types of formation controllers with guaranteed exponential stability which are not covered in the literature. For example, we can show that by using the following controller

$$\begin{aligned} \dot{p}_i(t) &= u_i(t) \\ &= \sum_{j \in \mathcal{N}_i} \frac{p_j(t) - p_i(t)}{\|p_j(t) - p_i(t)\|} (\|p_j(t) - p_i(t)\|^2 - d_k^2) \end{aligned} \quad (35)$$

the distance error system is (locally) exponentially stable. The formation control system (35) has the property that the bearing and range information is decoupled in the controller. In this case, the function g_k is constructed as

$$g_k(e_k) = \frac{e_k}{(e_k + d_k^2)^{\frac{1}{2}}} = (e_k + d_k^2)^{\frac{1}{2}} - \frac{d_k^2}{(e_k + d_k^2)^{\frac{1}{2}}} \quad (36)$$

which is continuously differentiable and strictly monotone increasing on $(-d_k^2, \infty)$. Thus all the conditions stated in the beginning of Section 3 are satisfied, and the exponential convergence then follows.

As another example, one can consider the following distributed formation controller

$$\dot{p}_i(t) = u_i(t) = \sum_{j \in \mathcal{N}_i} (p_j(t) - p_i(t)) (\|p_j(t) - p_i(t)\| - d_k) \quad (37)$$

which also guarantees that the distance error system is (locally) exponentially stable. In this case, the function g_k is constructed as

$$g_k(e_k) = \frac{e_k}{\|p_j(t) - p_i(t)\| + d_k} = \frac{e_k}{\sqrt{(e_k + d_k^2)} + d_k} \quad (38)$$

which satisfies all the conditions stated above. The curves of these two novel control functions are depicted in Fig. 1.

4. Exponential stability for non-minimally rigid formations

In this section we aim to extend the results from minimally rigid case to non-minimally rigid case, and prove that the exponential convergence of the distance error system still holds. The analysis in this section is partly motivated by Mou et al. [14] and Park and Ahn [17].

Fig. 1. Two novel controller functions g_k with guaranteed exponential convergence of the distance error system. In both plots, we let $d_k = 4$ and thus $e_k \in (-16, \infty)$. The left figure shows the proposed function in (36), and the right figure shows the proposed function in (38).

Fig. 2. An example of graph decomposition of a non-minimally rigid framework. The dashed lines in the decomposed subgraphs indicate the removed edges.

Before starting the analysis, we recall the following result which is rephrased from Theorem 2.3 of [20].

Lemma 6 ([20]). *Every rigid framework $(\mathcal{G}, p)$ with n vertices in $\mathbb{R}^d$ has a rigid subframework with exactly $dn - d(d+1)/2$ edges.*

An example for the graph decomposition of a 4-agent rigid framework is shown in Fig. 2.

Suppose the framework $(\mathcal{G}, p)$ is infinitesimally rigid but not minimally rigid. According to Lemma 6, there exists a subframework which is minimally rigid with exactly $dn - d(d+1)/2$ edges. We choose one such minimally rigid subframeworks and denote it by $(\bar{\mathcal{G}}, p)$, where $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}})$ with $|\bar{\mathcal{E}}| = dn - d(d+1)/2$ is a subgraph of $\mathcal{G}$. The remaining part of the framework with $m - (dn - d(d+1)/2)$ edges is denoted as $(\tilde{\mathcal{G}}, p)$ where $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ with $|\tilde{\mathcal{E}}| = m - (dn - d(d+1)/2)$. Note that $\mathcal{E} = \bar{\mathcal{E}} \cup \tilde{\mathcal{E}}$.

We denote $\bar{e} \in \mathbb{R}^{|\bar{\mathcal{E}}|}$ as the sub-vector of e whose $|\bar{\mathcal{E}}|$ entries are the entries in e corresponding to the edges in $\bar{\mathcal{G}}$. Similarly, we write $\tilde{e} \in \mathbb{R}^{|\tilde{\mathcal{E}}|}$ as the sub-vector of e whose $|\tilde{\mathcal{E}}|$ entries are the entries in e corresponding to the edges in $\tilde{\mathcal{G}}$. There exist two selection matrices, $\bar{P} \in \mathbb{R}^{|\bar{\mathcal{E}}| \times m}$ and $\tilde{P} \in \mathbb{R}^{|\tilde{\mathcal{E}}| \times m}$, such that the partition can be described as

$$\bar{e} = \bar{P}e, \quad \tilde{e} = \tilde{P}e. \quad (39)$$

Note that according to the structure of $\bar{P}$ and $\tilde{P}$, the matrix $P := [\bar{P}^T \tilde{P}^T]^T$ is a permutation matrix. Therefore, there hold $\bar{P}\tilde{P}^T = I_{|\bar{\mathcal{E}}| \times |\tilde{\mathcal{E}}|}$, $\tilde{P}\bar{P}^T = 0_{|\tilde{\mathcal{E}}| \times |\bar{\mathcal{E}}|}$ and $e = \bar{P}^T\bar{e} + \tilde{P}^T\tilde{e}$ because $\bar{P}^T\bar{P} + \tilde{P}^T\tilde{P} = I_{m \times m}$. Following a similar procedure, we can also partition the gradient vector Φ into two sub-vectors $\bar{\Phi}(\bar{e}) := \bar{P}\Phi$ and $\tilde{\Phi}(\tilde{e}) := \tilde{P}\Phi$, corresponding to the subframeworks $(\bar{\mathcal{G}}, p)$ and $(\tilde{\mathcal{G}}, p)$, respectively.

We also denote the rigidity matrix for the subframework $(\bar{\mathcal{G}}, p)$ as $\bar{R} \in \mathbb{R}^{\bar{\mathcal{E}} \times dn}$, and the rigidity matrix for $(\tilde{\mathcal{G}}, p)$ as $\tilde{R} \in \mathbb{R}^{\tilde{\mathcal{E}} \times dn}$. By following similar arguments as above, there hold $\bar{R} = \bar{P}R$ and $\tilde{R} = \tilde{P}R$. Then one can derive the dynamical equation for $\bar{e}$:

$$\begin{aligned} \dot{\bar{e}} &= \bar{P}\dot{e} = -2\bar{P}RR^T(\bar{P}^T\bar{\Phi}(\bar{e}) + \tilde{P}^T\tilde{\Phi}(\tilde{e})) \\ &= -2\bar{R}\bar{R}^T\bar{\Phi}(\bar{e}) - 2\bar{R}\tilde{R}^T\tilde{\Phi}(\tilde{e}). \end{aligned} \quad (40)$$

According to Lemma 2, there exists a smooth function ψ such that $\tilde{e} = \psi(\bar{e})$ holds locally around $\bar{e} = 0$. Note that $\psi(\bar{e}) = 0$ if and only if $\bar{e} = 0$. Also $\tilde{R} = \frac{\partial \tilde{e}}{\partial \bar{e}} = \frac{\partial \psi}{\partial \bar{e}} \frac{\partial \bar{e}}{\partial p} = \bar{F}\bar{R}$ where $\bar{F} := \frac{\partial \psi}{\partial \bar{e}}$. Furthermore, Lemma 2 implies that all entries of $\bar{R}\bar{R}^T$ and $\bar{R}\tilde{R}^T = \bar{R}\bar{R}^T\bar{F}^T$ are functions of $\bar{e}$. Thus, the reduced-order distance error system (40) is self-contained, and can be rewritten as

$$\dot{\bar{e}} = -2\bar{Q}(\bar{e})\bar{\Phi}(\bar{e}) - 2\bar{Q}(\bar{e})\bar{F}^T\tilde{\Phi}(\psi(\bar{e})) \quad (41)$$

where $\bar{Q}(\bar{e}) = \bar{R}\bar{R}^T$. Note that these dynamics hold only locally about the target rigid formation.

The main result in this section concerns the following exponential stability statement.

Theorem 3. *Consider the gradient control (10) derived from the general potential functions with the listed properties. The distance error system (40) for the non-minimally rigid formation has an exponential stable equilibrium at the origin.*

Proof. The proof for the local exponential stability of the equilibrium point $\bar{e} = 0$ is through the linearization approach, which follows similarly to the proof of [14, Theorem 2].

The calculation of the Jacobian matrix of the reduced-order error system (40) at the equilibrium $\bar{e} = 0$ reveals

$$\begin{aligned} J(\bar{e}) &:= \frac{\partial (-2\bar{Q}(\bar{e})\bar{\Phi}(\bar{e}) - 2\bar{Q}(\bar{e})F^T\tilde{\Phi}(\psi(\bar{e})))}{\partial \bar{e}} \bigg|_{\bar{e}=0} \\ &= -2 \left(\frac{\partial \bar{Q}(\bar{e})}{\partial \bar{e}} \bar{\Phi}(\bar{e}) + \frac{\partial (\bar{Q}(\bar{e})F^T)}{\partial \bar{e}} \tilde{\Phi}(\psi(\bar{e})) \right) \bigg|_{\bar{e}=0} \\ &\quad - 2 \left(\bar{Q}(\bar{e}) \frac{\partial \bar{\Phi}(\bar{e})}{\partial \bar{e}} + \bar{Q}(\bar{e})F^T \frac{\partial \tilde{\Phi}(\psi(\bar{e}))}{\partial \bar{e}} \right) \bigg|_{\bar{e}=0}. \end{aligned} \quad (42)$$

Let $\Psi = \{\bar{e} \in \mathbb{R}^{|\bar{e}|} \mid \|\bar{e}\|^2 < r\}$ with a small positive $r > 0$ denote a small neighborhood set around the desired equilibrium $\bar{e} = 0$. We first note in Ψ the solution to the system (40) is bounded. Furthermore, the Jacobian matrix $J(\bar{e})$ is bounded and Lipschitz continuous on the same set Ψ (due to the smoothness of the function ψ , and continuous differentiability of the gradient function g_k). These system properties allow us to employ the Linearization Theorem [13, Theorem 4.13] to prove the local exponential convergence of (40).

By noting that $\bar{\Phi}(\bar{e})|_{\bar{e}=0} = 0$ and $\tilde{\Phi}(\psi(\bar{e}))|_{\bar{e}=0} = 0$, the expression of the above Jacobian matrix can be simplified to

$$J(\bar{e}) := -2\bar{Q}(\bar{e}) \left(\frac{\partial \bar{\Phi}(\bar{e})}{\partial \bar{e}} + F^T \frac{\partial \tilde{\Phi}(\psi(\bar{e}))}{\partial \psi} F \right) \bigg|_{\bar{e}=0}. \quad (43)$$

Note that $\frac{\partial \bar{\Phi}(\bar{e})}{\partial \bar{e}}|_{\bar{e}=0}$ is a diagonal matrix, with the k th diagonal entry being $\frac{dg_k(s)}{ds}$. Because each g_k is strictly monotone increasing on its domain, the derivative $\frac{dg_k(s)}{ds}$ evaluated at $s = 0$ is positive. Thus $\frac{\partial \bar{\Phi}(\bar{e})}{\partial \bar{e}}|_{\bar{e}=0}$ is a positive definite diagonal matrix. The same reasoning implies that $\frac{\partial \tilde{\Phi}(\psi(\bar{e}))}{\partial \psi}|_{\bar{e}=0}$ is also a positive definite diagonal matrix. Thus $\frac{\partial \bar{\Phi}(\bar{e})}{\partial \bar{e}} + F^T \frac{\partial \tilde{\Phi}(\psi(\bar{e}))}{\partial \psi} F$ is positive definite evaluated at $\bar{e} = 0$. Note that $\bar{Q}(0)$ is positive definite due to the infinitesimal and minimal rigidity of the subframework $(\bar{g}, p)$ of the target formation. According to [31, Corollary 7.6.2, Page 486], all eigenvalues of the product matrix $\bar{Q}(\bar{e}) \left(\frac{\partial \bar{\Phi}(\bar{e})}{\partial \bar{e}} + F^T \frac{\partial \tilde{\Phi}(\psi(\bar{e}))}{\partial \psi} F \right) \bigg|_{\bar{e}=0}$ are real and positive. Thus, the Jacobian matrix $J(\bar{e})|_{\bar{e}=0}$ is a stability matrix with all of its eigenvalues being negative real, which implies that $\bar{e} = 0$ is an exponentially stable equilibrium for the linearized system $\dot{\bar{e}} = J(\bar{e})\bar{e}$ of (41). According to [13, Theorem 4.13], the equilibrium point $\bar{e} = 0$ is locally exponentially stable for the nonlinear system (41). $\square$

5. Simulation examples

In this section we provide two simulations to show the behavior of certain formation systems with generalized controllers. Firstly consider a 4-agent formation shape in the 2-D space modeled by a complete graph. The desired distances are given by $d_{12} = d_{34} = 3$, $d_{23} = d_{14} = 4$, $d_{13} = d_{24} = 5$, which correspond to a rectangular target shape. Note that in this case the underlying graph for the target formation has 6 edges, while the maximum rank of the associated rigidity matrix is 5. Thus the formation graph is not minimally rigid, but it contains a minimally rigid subgraph. By the analysis in Section 4, the local exponential convergence of the formation system with generalized controllers can be shown.

In the simulation, the initial conditions for each agent are chosen as $p_1(0) = [0, 0]^T$, $p_2(0) = [-1, 4]^T$, $p_3(0) = [5, 3]^T$ and $p_4(0) = [3, 0]^T$, so that the initial formation is close to the target one. We assume that for different edges the agents are using different controller forms. Specifically, for the edges (1, 3) and (2, 4) the controller contributions in the related agents are based on

Fig. 3. Simulation on stabilization control of an 4-agent rectangular formation shape. The initial and final positions are denoted by circles and squares, respectively. The initial formation is denoted by dotted blue lines, and the final formation is denoted by red solid lines. The black star denotes the formation centroid, which is stationary. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 4. Exponential convergence of the distance errors for 2-D rectangular formation control.

(12), for the edges (1, 2) and (3, 4) the controller contributions are based on (14), and for edges (2, 3) and (1, 4) the agents use the control function of (17). The trajectories of each agent and the final shape are depicted in Fig. 3. The trajectories of each distance error are depicted in Fig. 4, which shows an exponential convergence to the origin.

We then consider a formation control example in 3-D space. The target formation shape is supposed to be a double tetrahedron formation with 9 edges, with the desired distances for each edge being 2. Note that in this case the underlying graph for the target formation shape is minimally rigid, since the maximum rank of the associated rigidity matrix, which equals the number of edges, can be attained by a realization of the target formation. For the controller design, we assume that all the agents adopt the novel control function of (37) proposed in Remark 3. In the simulation, the initial conditions for each agent are chosen as $p_1(0) = [0, -1.0, 0.5]^T$, $p_2(0) = [1.8, 1.6, -0.1]^T$, $p_3(0) = [-0.2, 1.8, 0.05]^T$, $p_4(0) = [1.2, 1.9, 1.7]^T$ and $p_5(0) = [-1.0, -1.5, -1.2]^T$, so that the initial formation is close to the target one. Fig. 5 illustrates the trajectories of each agent, together with the initial shape and final shape. The trajectories of each distance error are depicted in Fig. 6, which also shows an exponential convergence to the origin.

Fig. 5. Simulation on stabilization control of a double tetrahedron formation in 3-D space. The initial and final positions are denoted by circles and squares, respectively. The initial formation is denoted by dashed lines, and the final formation is denoted by red solid lines. The black star denotes the formation centroid, which is stationary. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 6. Exponential convergence of the distance errors for 3-D tetrahedron formation control.

6. Conclusions

In this paper we show a unified approach to analyze the convergence of distance-based rigid formation control systems with distributed formation controllers. The general forms of the formation controllers encompass most controllers with special forms proposed in the literature for stabilizing formation shapes. Furthermore, in the framework studied in this paper, the stabilization control of each distance can be assigned with different distance potentials which also enables an implementation with heterogeneous agents in formation control. Two cases of target formations are discussed in detail: minimally rigid formations and non-minimally rigid formations. For both cases, we prove the *exponential convergence* property of the formation system with the generalized controller. One significant consequence of the exponential stability is that the robustness properties of formation control analyzed in [14,15] can be extended to a large family of rigid formation systems with generalized controllers obtained in this paper.

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References

- [1] K.-K. Oh, M.-C. Park, H.-S. Ahn, A survey of multi-agent formation control, *Automatica* 53 (2015) 424–440.
- [2] S. Mou, M. Cao, A.S. Morse, Target-point formation control, *Automatica* 61 (2015) 113–118.
- [3] R. Olfati-Saber, R.M. Murray, Graph rigidity and distributed formation stabilization of multi-vehicle systems, in: *Proc. of the 41st Conference on Decision and Control, IEEE*, 2002, pp. 2965–2971.
- [4] B.D.O. Anderson, C. Yu, B. Fidan, J.M. Hendrickx, Rigid graph control architectures for autonomous formations, *IEEE Control Syst. Mag.* 28 (6) (2008) 48–63.
- [5] L. Krick, M.E. Broucke, B.A. Francis, Stabilisation of infinitesimally rigid formations of multi-robot networks, *Internat. J. Control* 82 (3) (2009) 423–439.
- [6] F. Dorfler, B. Francis, Geometric analysis of the formation problem for autonomous robots, *IEEE Trans. Automat. Control* 55 (10) (2010) 2379–2384.
- [7] A. Belabbas, S. Mou, A.S. Morse, B.D.O. Anderson, Robustness issues with undirected formations, in: *Proc. of the 51st Conference on Decision and Control*, 2012, pp. 1445–1450.
- [8] B.D.O. Anderson, U. Helmke, Counting critical formations on a line, *SIAM J. Control Optim.* 52 (1) (2014) 219–242.
- [9] B.D.O. Anderson, C. Yu, S. Dasgupta, A.S. Morse, Control of a three-coleader formation in the plane, *Systems Control Lett.* 56 (9) (2007) 573–578.
- [10] S.L. Smith, M.E. Broucke, B.A. Francis, Stabilizing a multi-agent system to an equilateral polygon formation, in: *Proc. of the 17th International Symposium on Mathematical Theory of Networks and Systems*, 2006, pp. 2415–2424.
- [11] D.V. Dimarogonas, K.H. Johansson, Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control, *Automatica* 46 (4) (2010) 695–700.
- [12] Y. Tian, Q. Wang, Global stabilization of rigid formations in the plane, *Automatica* 49 (5) (2013) 1436–1441.
- [13] H.K. Khalil, *Nonlinear Systems*, third ed., Prentice Hall, New Jersey, 2002.
- [14] S. Mou, A.S. Morse, M.A. Belabbas, Z. Sun, B.D.O. Anderson, Undirected rigid formations are problematic, *IEEE Trans. Automat. Control* (2016) <http://dx.doi.org/10.1109/TAC.2015.2504479>, in press.
- [15] Z. Sun, S. Mou, B.D.O. Anderson, A.S. Morse, Rigid motions of 3-d undirected formations with distance mismatch, *IEEE Trans. Automat. Control* (2015) submitted for publication.
- [16] H. Garcia de Marina, M. Cao, B. Jayawardhana, Controlling rigid formations of mobile agents under inconsistent measurements, *IEEE Trans. Robot.* 31 (1) (2015) 31–39.
- [17] M.-C. Park, H.-S. Ahn, Exponential stabilization of infinitesimally rigid formations, in: *Proc. of the 2014 International Conference on Control, Automation and Information Sciences*, 2014, pp. 36–40.
- [18] Z. Sun, Q. Liu, C. Yu, B.D.O. Anderson, Generalized controllers for rigid formation stabilization with application to event-based controller design, in: *Proc. of the 2015 European Control Conference, ECC'15, IEEE*, 2015, pp. 217–222.
- [19] L. Asimow, B. Roth, The rigidity of graphs, II, *J. Math. Anal. Appl.* 68 (1) (1979) 171–190.
- [20] B. Hendrickson, Conditions for unique graph realizations, *SIAM J. Comput.* 21 (1) (1992) 65–84.
- [21] K. Sakurama, S.-I. Azuma, T. Sugie, Distributed controllers for multi-agent coordination via gradient-flow approach, *IEEE Trans. Automat. Control* 60 (6) (2015) 1471–1485.
- [22] D.V. Dimarogonas, K.H. Johansson, On the stability of distance-based formation control, in: *Proc. of the 47th Conference on Decision and Control, IEEE*, 2008, pp. 1200–1205.
- [23] X. Cai, M. de Queiroz, Rigidity-based stabilization of multi-agent formations, *J. Dyn. Syst. Meas. Control* 136 (1) (2014) 014502.
- [24] X. Cai, M. De Queiroz, Adaptive rigidity-based formation control for multirobotic vehicles with dynamics, *IEEE Trans. Control Syst. Technol.* 23 (1) (2015) 389–396.
- [25] M. Cao, A.S. Morse, C. Yu, B.D.O. Anderson, S. Dasgupta, Maintaining a directed, triangular formation of mobile autonomous agents, *Commun. Inf. Syst.* 11 (1) (2011) 1–16.
- [26] K.-K. Oh, H.-S. Ahn, Distance-based undirected formations of single-integrator and double-integrator modeled agents in n -dimensional space, *Internat. J. Robust Nonlinear Control* 24 (12) (2014) 1809–1820.
- [27] R. Chill, On the Łojasiewicz–Simon gradient inequality, *J. Funct. Anal.* 201 (2) (2003) 572–601.
- [28] Z. Sun, S. Mou, M. Deghat, B.D.O. Anderson, Finite time distributed distance-constrained shape stabilization and flocking control for d -dimensional undirected rigid formations, *Internat. J. Robust Nonlinear Control* (2016) <http://dx.doi.org/10.1002/rnc.3477>, in press.
- [29] Z. Sun, U. Helmke, B.D.O. Anderson, Rigid formation shape control in general dimensions: an invariance principle and open problems, in: *Proc. of the 54th Conference on Decision and Control, IEEE, Osaka, Japan*, 2015, pp. 6095–6100.
- [30] C.-I. Vasile, M. Schwager, C. Belta, SE(N) Invariance in Networked Systems, in: *Proc. of the 2015 European Control Conference, ECC'15, IEEE*, 2015, pp. 186–191.
- [31] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 2012.