Energy-based Lyapunov functions for forced Hamiltonian systems with dissipation

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Abstract

It is well known that the total energy is a suitable Lyapunov function to study the stability of the trivial equilibrium of an isolated standard Hamiltonian system. In many practical instances, however, the system is in interaction with its environment through some constant forcing terms. This gives rise to what we call forced Hamiltonian systems, for which the equilibria of interest are now different from zero. When the system is linear a Lyapunov function can be immediately obtained by simply shifting the coordinates in the total energy. However, for nonlinear systems there is no guarantee that this incremental energy is, not even locally, a Lyapunov function. In this paper we propose a constructive procedure to modify the total energy function of forced Hamiltonian systems with dissipation in order to generate Lyapunov functions for non-zero equilibria. A key step in the procedure, which is motivated from energy-balance considerations standard in network modeling of physical systems, is to embed the system into a larger Hamiltonian system for which a series of Casimir functions (i.e., first integrals) can be easily constructed. Interestingly enough, for linear systems the resulting Lyapunov function is the incremental energy, thus our derivations provide a physical explanation to it. An easily verifiable necessary and sufficient condition for the applicability of the technique in the general nonlinear case is given. Some examples that illustrate the method are given.

1 Problem formulation

Network modeling of energy-conserving lumped-parameter physical systems [1] [10] is often performed by replacing the independent storage elements by a single state variable, leading to models of the form (called port controlled Hamiltonian systems [5] [16])

\[ \Sigma : \begin{align*}
\dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}(x)
\end{align*} \]

where \( x \in \Xi \), an \( n \)-dimensional manifold, \( u, y \in \mathbb{R}^m \), and \( x = [x_1, \ldots, x_n]^T \) are the energy variables, the smooth function \( H(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R} \) represents the total stored energy,

and \( u, y \) are the port power variables. The interconnection structure is captured in the \( n \times n \) matrix \( J(x) \) and the \( n \times m \) matrix \( g(x) \), both depending smoothly on the state \( x \). Because of the assumption of energy-conservation, the matrix \( J(x) \) is skew-symmetric, that is,

\[ J(x) = -J^T(x), \quad \forall x \in \Xi \]

and defines a generalized Poisson bracket on \( \Xi \) (generalized because it need not satisfy the Jacobi-identity [15]).

From (1) and (2), we immediately obtain the power-balance

\[ \frac{d}{dt} H = u^T y \]

with \( u^T y \) the power externally supplied to the system. Energy-dissipation is included by terminating some of the ports by resistive elements, see e.g. [16], [5], [4]. Indeed, consider instead of \( g(x)u \) in (1) a term

\[ \begin{bmatrix} u \\ u_R \end{bmatrix} = g(x)u + g_R(x)u_R \]

and extend correspondingly \( y = g^T(x) \frac{\partial H}{\partial x}(x) \) to

\[ \begin{bmatrix} y \\ y_R \end{bmatrix} = \begin{bmatrix} g^T(x) \frac{\partial H}{\partial x}(x) \\ g_R(x) \frac{\partial H}{\partial x}(x) \end{bmatrix} \]

Here \( u_R, y_R \) denote the power variables at the ports which are terminated by (linear) resistive elements

\[ u_R = -Sg_R(x) \]

for some non-negative definite symmetric matrix \( S \). Substitution in (1) leads to models of the form

\[ \Sigma : \begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\
y &= g^T(x) \frac{\partial H}{\partial x}(x)
\end{align*} \]

where

\[ R(x) \triangleq g_R(x)Sg_R^T(x) \]

which is a non-negative symmetric matrix depending smoothly on \( x \), i.e.

\[ R(x) = R^T(x) \geq 0, \quad \forall x \in \Xi \]

The matrix \( R(x) \) actually defines a symmetric bracket on the state manifold in the same way as the skew-symmetric matrix \( J(x) \) defines a generalized Poisson bracket.
For systems \( \Sigma \) given by (4) the power-balance (3) extends to
\[
\frac{d}{dt} H = - \frac{\partial^T H}{\partial x}(x) R(x) \frac{\partial H}{\partial x}(x) + u^T y \tag{6}
\]
where the first term on the right-hand (which is non-positive by (5)) represents the energy-dissipation due to the resistive elements in the system.

Now let us approach the stability analysis of the system \( \Sigma \). Towards this end, we recall here the following [7]

**Definition 1.1** Consider the system \( \dot{x} = f(x) \), with an equilibrium point \( \bar{x} \), i.e., \( f(\bar{x}) = 0 \). We say that a function \( V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) is a Lyapunov function for the equilibrium \( \bar{x} \) if:

1. it has a local minimum at \( \bar{x} \)
2. satisfies \( \frac{d}{dt} V \leq 0 \).

Let us first consider the case \( u = 0 \) (the unforced or uncontrolled case). If \( x^* \) is a minimum of the energy \( H(x) \), then necessarily \( \frac{\partial H}{\partial x}(x^*) = 0 \), and thus \( x^* \) is an equilibrium of the unforced dynamics
\[
\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) \tag{7}
\]
Furthermore by (5), (6) for \( u = 0 \)
\[
\frac{d}{dt} H = - \frac{\partial^T H}{\partial x}(x) R(x) \frac{\partial H}{\partial x}(x) \leq 0 \tag{8}
\]
and thus the total stored energy \( H \) is a Lyapunov function for investigating the stability of the equilibrium \( x^* \). Of course, this is common practice in the stability analysis of physical systems, and in some sense is the origin of Lyapunov stability theory.

On the other hand, in many application areas one wants to investigate the stability of \( \Sigma \) for a constant, but non-zero, input \( \bar{u} \in \mathbb{R}^m \). (An example where this scenario arises, which actually motivated our research, is in studies of transient stability of synchronous generators in power systems [8], see also [14], [13].) Corresponding to \( u = \bar{u} \) one considers forced equilibria \( \bar{x} \), which are solutions of
\[
[J(\bar{x}) - R(\bar{x})] \frac{\partial H}{\partial x}(\bar{x}) + g(\bar{x})\bar{u} = 0 \tag{9}
\]
Now, let \( \bar{x} \) be a forced equilibrium. Then in general, \( \bar{x} \) will need not be a minimum (or an extremum) of \( H \). Furthermore, inserting \( u = \bar{u} \) in (6) yields
\[
\frac{d}{dt} H = - \frac{\partial^T H}{\partial x}(x) R(x) \frac{\partial H}{\partial x}(x) + \bar{u}^T g^T(x) \frac{\partial H}{\partial x}(x) \tag{10}
\]
and in general the right-hand side of (10) will not be non-positive. Thus, in general, the total stored energy can not be used as a Lyapunov function for investigating the stability of \( x \). Hence the question comes up if, and how, we can construct physically-based Lyapunov functions for equilibria of forced physical systems \( \Sigma \). Providing some (partial) answers to this question are the main contributions of our work.

**Remark 1.1** Note that we may take \( J(x) \) and \( R(x) \) together in a single matrix \( I(x) \), \( \dot{J}(x) = -R(x) \), and that conversely every \( n \times n \) matrix \( I(x) \) can be decomposed as the difference of a skew-symmetric matrix \( J(x) \) and a symmetric matrix \( R(x) \) (not necessarily non-negative).

**Remark 1.2** The question above is of interest only for nonlinear systems. Since, as it is well-known, for linear systems (with \( J, R \) and \( g \) constant) with quadratic energy function \( H(x) = \frac{1}{2} x^T Q x \), we can simply shift the coordinates to obtain the so-called incremental Lyapunov function \( H(x) = \frac{1}{2} (x - x^\ast)^T Q(x - x^\ast) \). This procedure, of course, will not generate a Lyapunov function for nonlinear systems in general.

2 A Lyapunov function based on energy-balance

One way of approaching this question is to start from the power balance of the forced system (10), and to bring the second term on the right-hand side to the left-hand side as
\[
\frac{d}{dt} \left( H(x(t)) - \bar{u}^T \int_0^t g^T(x(\tau)) \frac{\partial H}{\partial x}(x(\tau)) d\tau \right) = - \frac{\partial^T H}{\partial x}(x(t)) R(x(t)) \frac{\partial H}{\partial x}(x(t))
\]
suggesting as candidate Lyapunov "function"
\[
H(x(t)) - \bar{u}^T \int_0^t g^T(x(\tau)) \frac{\partial H}{\partial x}(x(\tau)) d\tau = H(x(t)) - \bar{u}^T \int_0^t y(\tau) d\tau \tag{11}
\]
where we have replaced \( y \) from (1). Notice that the term \( \bar{u}^T \int_0^t y(\tau) d\tau \) is precisely the energy externally supplied to the system \( \Sigma \). Hence the new function (11) that we propose is exactly the difference between the energy of the system and the supplied energy.

To check whether (11) can be used as a Lyapunov function, the first basic question is, of course, if we can write \( \bar{u}^T \int_0^t y(\tau) d\tau \) as a function of the state \( x(t) \). The main emphasis in this paper will be in trying to answer this question. Condition 2 of the Definition 1.1 is clearly satisfied by construction. And there will only remain condition 1 to be verified.

From a control theoretic point of view the question posed above suggests to consider a cascade of \( \Sigma \) with input \( \bar{u} \), followed by the integration of \( y \), as depicted in Fig. 1, and to look for Lyapunov functions of the composed system
\[
\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x) \bar{u}
\]
\[
\dot{\xi} = g^T(x) \frac{\partial H}{\partial x}(x), \; \xi \in \mathbb{R}^m \tag{12}
\]
Assumption A \([J(x) - R(x)]\) is invertible for every \(x \in \mathbb{R}\).

We now consider the equation (9) in \(v = \frac{\partial H_s}{\partial x}(x)\), that is \([J(x) - R(x)]v + g(x)\dot{u} = 0\) (17)

By Assumption A, (17) has a unique solution \(v = K(x)\dot{u}\), with \(K(x) = -[J(x) - R(x)]^{-1}g(x)\) (18)

Let us now define, as an extension to (13), the system

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{\zeta}
\end{bmatrix} = \begin{bmatrix}
    J(x) & R(x)K(x) - g(x) \\
    -(R(x)K(x) - g(x))^T & J_s(x)
\end{bmatrix} \begin{bmatrix}
    \frac{\partial H_s}{\partial x} \\
    \frac{\partial H_s}{\partial \zeta}
\end{bmatrix}
\]

with \(H_s(x, \zeta)\) defined by (14), and with \(J_s(x) = -J_s^T(x)\), and \(R_s(x) = R_s^T(x)\) yet to determined. Note that the \(x\)-dynamics are unaffected, thus the \(x\)-dynamics of \(\Sigma\) for \(u = \bar{u}\) have been embedded in the dynamics (19).

By (18) we have

\[R(x)K(x) = J(x)K(x) + g(x)\]

and hence we can rewrite (19) into the simpler form

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{\zeta}
\end{bmatrix} = \begin{bmatrix}
    J_s(x) - R_s(x)
\end{bmatrix} \begin{bmatrix}
    \frac{\partial H_s}{\partial x} \\
    \frac{\partial H_s}{\partial \zeta}
\end{bmatrix}
\]

where \((x, \zeta) \in \mathbb{R} \times \mathbb{R}^m\), and we have defined

\[J_s(x) = \begin{bmatrix}
    J(x) \\
    -(J(x)K(x))^T
\end{bmatrix}\]

\[R_s(x) = \begin{bmatrix}
    R(x)K(x) \\
    (R(x)K(x))^T
\end{bmatrix}\]

Remark 3.1

Note that (20) is of the same form as (4), with \(J_s\) defining a generalized Poisson bracket and \(R_s\) a symmetric bracket on the augmented state space \(\mathbb{R} \times \mathbb{R}^m\).

Remark 3.2

The precise form of the proposed embedding (19) was motivated by a network analysis of (1) interconnected with the "source system" (15), together with the coequilibrium equation (17). See Remark 2.1. It is based on modifying, on the one hand, the dynamics of the source system using a non zero bracket; and, on the other hand, by choosing a more general interconnection than (16). Further details on this perspective of the problem, as well as its application to controller synthesis, may be found in [11].

Remark 3.3

In some cases it is convenient to allow for feedback terms.
in the description of physical systems and to consider instead of (4) models of the form
\begin{equation}
\dot{y} = g^T(x) \frac{\partial H}{\partial z}(x) + D(x)u
\end{equation}
with \(D(x) = -D^T(x)\). Because of the latter these models satisfy the power balance (6). In this case the composed system (12) is replaced by
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial c}(x) + g(x)u \\
\dot{c} &= g^T(x) \frac{\partial H}{\partial c}(x) + D(x)u
\end{align*}
and therefore (13) is replaced by
\begin{equation}
\begin{bmatrix} \dot{x} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} J(x) & -g(x) \\ g^T(x) & D(x) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial c} \\ \frac{\partial H}{\partial c} \end{bmatrix}
\end{equation}
This suggests a way to interpret the skew-symmetric matrix \(J,(x)\) in the general construction (20). Indeed, we may regard \(J,(x)\) as an additional feedthrough term, which does not disturb the power balance of the system.

4 Construction of the Lyapunov function

Next question is how we determine \(J,(x) = -J^T,(x)\), and \(R,(x) = R^T,(x)\). This is guided by (17), (18). Indeed, the \(m\)-dimensional linear spaces
\begin{equation}
P(x) = \left\{ \begin{bmatrix} -K(x)u \\ v \end{bmatrix} \mid v \in \mathbb{R}^m \right\}
\end{equation}
are, by construction, in the kernel of the matrix \([J(x), J(x)K(x)]\) defined by the first \(n\) rows of \(J,(x)\). We now define \(J,(x)\) in such manner that \(P(x)\) is in the kernel of the whole matrix \(J,(x)\), by setting
\begin{equation}
J,(x) = K^T(x)J(x)K(x)
\end{equation}
Clearly \(J,(x)\) satisfies \(J,(x) = -J^T,(x)\). In the same way, we note that \(P(x)\) is in the kernel of the first \(n\) rows of \(R,(x)\). We now define \(R,(x)\) in such manner that \(P(x)\) is in the kernel of the whole matrix \(R,(x)\), if we choose
\begin{equation}
R,(x) = K^T(x)R(x)K(x)
\end{equation}
Now, \(R,(x) = R^T,(x)\), and thus, since by assumption \(R(x) \geq 0\), also \(R,(x) \geq 0\).

We are ready to deliver the coup de grâce.

Assume that there exist smooth functions \(C_j : \mathbb{R} \to \mathbb{R}, j \in \mathbb{N} \cup \{1, \ldots, m\}\), such that
\begin{equation}
K_{ij}(x) = \frac{\partial C_j}{\partial x_i}(x), \quad i \in \mathbb{N} \cup \{1, \ldots, n\}, \quad j \in \mathbb{N}
\end{equation}
Then it immediately follows that the functions
\begin{equation}
\zeta_j - C_j(x), \quad j \in \mathbb{N}
\end{equation}
are constant along the trajectories of (20), with \(J,(x)\) and \(R,(x)\) as defined in (22), respectively (23). Indeed, we can write
\begin{equation}
\frac{d}{dt} \left[ \zeta_j - C_j(x) \right] = \left[ -\frac{\partial C_j}{\partial x}(x), e_j \right] \left[ J(x) - R(x) \right] \frac{\partial H}{\partial c}
\end{equation}
with \(e_j\) the \(j\)-th basis vector in \(\mathbb{R}^m\). Since the \((n + m)\)-dimensional column vector \([e_j]^T, -e_j^T\) is by (24) contained in \(P(x)\), it is by construction and definition of \(J,(x)\), and \(R,(x)\) contained in the kernels of \(J,(x)\), and \(R,(x)\) respectively. Thus the expression in (26) is zero. Hence, along trajectories of (20) we can express
\begin{equation}
\zeta_j = C_j(x) + c_j, \quad j \in \mathbb{N}
\end{equation}
where the constants \(c_1, \ldots, c_m\) depend on the initial conditions of \(\zeta\) (and can be set to zero). Thus the dynamics of
\begin{equation}
\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial c}(x) + g(x)u
\end{equation}
is copied on every submanifold of \(\mathbb{R} \times \mathbb{R}^m\) defined by (27).

The total energy of the augmented system
\begin{equation}
H,(x, c) = H(x) - \sum_{j=1}^m u_j C_j(x) + c_j
\end{equation}
restricted to such a submanifold is given as
\begin{equation}
H,(x, c) = H(x) - \sum_{j=1}^m u_j C_j(x) + c_j
\end{equation}
while the dynamics restricted to such a submanifold is given by
\begin{equation}
\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial c}(x) + g(x)u
\end{equation}
Note that by (24)
\begin{equation}
\frac{\partial H}{\partial c}(x) = \frac{\partial H}{\partial x}(x) - \sum_{j=1}^m u_j \frac{\partial C_j}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - K(x)u
\end{equation}
Hence, premultiplying by \([J(x) - R(x)]\), and using (18)
\begin{equation}
[J(x) - R(x)] \frac{\partial H}{\partial c}(x) = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u
\end{equation}
Consequently, by (9) and assumption A, the unique forced equilibrium \(\bar{u}\) corresponding to \(\bar{u}\) is an extremum of \(H,(x)\).

Remark 4.1

From the derivations above it follows that the functions \(\zeta_j - C_j(x)\) defined on the augmented state space \(\Xi \times \mathbb{R}^m\) are Casimirs of the generalized Poisson bracket defined by \(J,(x)\) (see for this notion [9], and the references quoted in there). Furthermore, the functions \(\zeta_j - C_j(x)\) are also “Casimirs” with respect to the symmetric bracket corresponding to \(R,(x)\), see [11].

5 Main result

Let us summarize the developments above in the following theorem.

Theorem 5.1 Consider \(\Sigma\) for constant \(u = \bar{u}\) that is,
\begin{equation}
\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial c}(x) + g(x)u
\end{equation}
with Assumption A. Define \( K(x) \) by (18) and assume its column vectors are closed 1-forms, that is, the functions \( K_i \) satisfy
\[
\frac{\partial K_{ij}}{\partial x_k} = \frac{\partial K_{ik}}{\partial x_j}, \quad i, k \in \mathbb{N} \tag{33}
\]
for \( j \in \mathbb{N} \). Then, there exist locally smooth functions \( C_1, \ldots, C_m \) satisfying (24), and the dynamics (32) can be alternatively represented by
\[
\dot{x} = [J(x) - R(x)] \frac{\partial H_r}{\partial x}(x) \tag{34}
\]
where \( H_r(x) \triangleq H(x) - \sum_{j=1}^m \delta_{ij}(C_j(x) + c_j) \). The function \( H_r(x) \) has an extremum at \( \bar{x} \), which is an equilibrium of (32). Further, we have
\[
\frac{d}{dt} H_r = -\frac{\partial^T H_r}{\partial x}(x) R(x) \frac{\partial H_r}{\partial x}(x) \leq 0 \tag{35}
\]
and thus \( H_r \) qualifies as Lyapunov function for the forced dynamics (32) provided we can show that \( H_r \) not only has an extremum at \( \bar{x} \) but even a minimum.

Proof
In view of the developments of the previous section, to complete the proof it only remains to show that, under the given conditions, there exist smooth functions \( C_1, \ldots, C_m \), satisfying (24). This follows immediately from (33) and Poincare's lemma.

The corollary below follows immediately from Theorem 5.1 and standard Lyapunov stability theory, e.g. [7].

Corollary 5.1 Assume that \( H_r \) has a strict local minimum at \( \bar{x} \), that is, there exists an open neighborhood \( B \) of \( \bar{x} \) such that \( H_r(x) > H_r(\bar{x}) \) for all \( x \in B \). Furthermore, assume that the largest invariant set under the dynamics (34) contained in
\[
\left\{ x \in \Xi \cap B \mid \frac{\partial^T H_r}{\partial x}(x) R(x) \frac{\partial H_r}{\partial x}(x) = 0 \right\}
\]
equals \( \{ \bar{x} \} \). Then, \( \bar{x} \) is a locally asymptotically stable equilibrium of the forced system (32).

Remark 5.1
Note that if \( \Xi \) is simply connected then \( C_1, \ldots, C_m \) exist globally.

Remark 5.2
An equivalent way to analyse the stability of the equilibrium \( \bar{x} \) of the forced system (32) by means of the Lyapunov function \( H_r \) is to look at the stability of the equilibrium \( (\bar{x}, \bar{\zeta}) \) with \( \bar{\zeta} = C_j(\bar{x}) \), \( j \in \mathbb{N} \) of the embedding system (20) by means of a candidate Lyapunov function of the form
\[
\dot{\bar{H}}(x, \zeta) \triangleq H(x) - \bar{u}^T \zeta + \Phi(\zeta - C_1(x), \ldots, \zeta_m - C_m(x))
\]
where the function \( \Phi \), depending on the Casimirs \( \zeta_j - C_j(x) \), \( j \in \mathbb{N} \), is still to be determined. This approach is similar to what is called the Energy-Casimir method in mechanics (see e.g. [9]). Note that restricted to any submanifold given by (27) the function \( \bar{H}(x, \zeta) \) reduces to the function \( H_r(x) \).

6 Examples

6.1 Linear systems
If \( J, R \) and \( g \) are constant matrices, then also \( K \) is a constant matrix, and the existence of functions \( C_1, \ldots, C_m \) satisfying (24) is automatic. (In fact \( C_j(x) \) is given as the linear function \( K_{ij} x_j + \cdots + K_{nj} x_n \).) In particular, for linear systems \( \Sigma \) with
\[
H(x) = \frac{1}{2} x^T Q x, \quad Q = Q^T
\]
Theorem 5.1 results in a linear forced dynamics.
\[
\dot{x} = (J - R) \frac{\partial H_r}{\partial x}(x)
\]
with (since \( K \bar{u} = \bar{Q} \bar{e} \))
\[
H_r(x) = \frac{1}{2} x^T Q x - x^T K \bar{u} + c = \frac{1}{2} (x - \bar{x})^T Q(x - \bar{x}) + c \tag{36}
\]
Hence we have recovered in this special case the incremental Lyapunov function which is normally used. More importantly, we have given an energy interpretation for it!

6.2 Mechanical systems
Consider a mechanical system with damping \( D(q) = D^T(q) \geq 0 \) and acted by external forces \( u \)
\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} =
\begin{bmatrix}
0 & I_k \\
-I_k & D(q)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q} \\
-\frac{\partial H}{\partial p}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
B(q)
\end{bmatrix} u \tag{37}
\]
with \( y = B^T(q) \frac{\partial H}{\partial p} \), \( y \) the generalized configuration coordinates \( q = [q_1, \ldots, q_k]^T \), and generalized momenta \( p = [p_1, \ldots, p_k]^T \). The outputs \( y \in \mathbb{R}^m \) are the generalized velocities corresponding to the generalized external forces \( u \in \mathbb{R}^m \). Let \( \bar{u} \) be a constant actuating force. It follows that
\[
K(q, p) = \begin{bmatrix}
0 & I_k \\
-I_k & -D(q)
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
B(q)
\end{bmatrix} = \begin{bmatrix}
B(q) \\
0
\end{bmatrix}
\tag{38}
\]
and hence
\[
J_r(q, p) = [B^T(q), 0] \begin{bmatrix}
0 \\
-I_k \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
B(q) \\
0
\end{bmatrix} = 0 \tag{39}
\]
\[
R_r(q, p) = [B^T(q), 0] \begin{bmatrix}
0 \\
0 \\
0 \\
-D(q)
\end{bmatrix}
\begin{bmatrix}
B(q) \\
0
\end{bmatrix} = 0 \tag{40}
\]
Hence the embedding system (19) or (20) is given as in (13):
\[
\begin{bmatrix}
\dot{q} \\
\dot{\bar{p}} \\
\dot{\bar{\zeta}}
\end{bmatrix} =
\begin{bmatrix}
0 & I_k & 0 \\
-I_k & D(q) & -B(q) \\
0 & B^T(q) & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q} \\
-\frac{\partial H}{\partial p} \\
\frac{\partial H}{\partial \bar{\zeta}}
\end{bmatrix} \tag{41}
\]
with \( H(q,p,\zeta) = H(q,p) - \tilde{u}^T\zeta \). Furthermore, the integrability conditions (24) boil down to the existence of functions \( C_1, \ldots, C_m \) such that

\[
B_{ij}(q) = \frac{\partial C_i}{\partial q_j}(q, p), \quad i, j \in \mathfrak{m}
\]  

(42)

Condition (42) means that the input vector fields formed by the columns of

\[
\begin{bmatrix}
0 \\ B(q)
\end{bmatrix}
\]

are actually Hamiltonian vector fields with Hamiltonians \( \mathcal{C}_1, \ldots, \mathcal{C}_m \).

Considering the output equation of the model and the dynamics, this simply means that the outputs are geometric functions of the generalized coordinates:

\[
y_j = \frac{\partial C_j}{\partial \dot{q}_i}(q) \frac{\partial H}{\partial p} = \frac{\partial C_j}{\partial \dot{q}_i} \dot{q} = \frac{d}{dt} C_j
\]

Assume moreover that the number of generalized forces is exactly the number of generalized momenta: \( m = k \). Then one may choose \( y \) as generalized coordinates and their conjugated momenta as generalized momenta and the dynamics has the form (37) with \( B(q) \) the identity matrix. Assume moreover that the mechanical system is a simple mechanical system with energy:

\[
H(q,p) = \frac{1}{2} p^T Y(q) p + V(q)
\]

where \( Y(q) \) is the mobility tensor of the system (which is positive definite) and \( V(q) \) is the potential energy of the system. The stability of the equilibria (\( \dot{p} = 0 \) and \( \frac{\partial V}{\partial q}(q) = \tilde{u} \)) of the forced system comes then down to analyze the positive definiteness of the function \( V(q) - \tilde{u}^T \dot{q} \).

7 Conclusions

In this paper we have shown that, under certain integrability conditions on the input vector fields, we can construct a Lyapunov function candidate to study the stability of forced Hamiltonian systems with dissipation. The main feature of this construction is that the Lyapunov function is directly derived from the Hamiltonian function of the system and the energy associated with the source. An interpretation which is possible provided the system is embedded in a higher dimensional system where the infinite energy reservoir is connected in a particular way to the forced system.

In a subsequent paper [11] we show how this construction can be used to derive stabilizing controllers for a large class of physical systems including mechanical, electrical and electromechanical dynamics. In some cases, the resulting controllers are new, while in others we find again some well-known schemes that have been derived using a Lagrangian formalism. In all cases, however, the proposed formulation is quite systematic and provides deep physical insight into the obstacles for stabilization with energy-shaping methods, features whose importance can hardly be overestimated and which are conspicuous by their absence in Lagrangian derivations.

References


