Production, Storage, and Futures Hedging Under Uncertainty

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Abstract
This paper provides an integrative survey of literature on commodity futures markets, on storage and production decisions, and on joint spot and futures price formation under uncertainty. The paper focuses on the risk reallocation role of futures markets. Basic models of futures trading by a competitive, risk-averse firm are enhanced with production and storage decisions in a static and dynamic setting. Through inventories and futures hedging both risk premia and an endogenous convenience yield play a role in the firm’s optimisation process. We thus provide a microeconomic foundation for the complementary relationship between the theory of normal backwardation and the theory of supply of storage. Because most futures users operate in an incomplete market setting, special attention is paid to the interaction of production, storage, and hedging determinants, the existence of risk premia and opportunities for diversification of portfolios.

Keywords: Incomplete Markets, Uncertainty, Futures Pricing, Storage
JEL-Codes: C61, D52, D81, G10, G13

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1 Introduction

The behaviour of commodity prices enjoys a renewed interest in the economic literature. This is partly due to the dramatic growth of commodity markets in volume, variety of contracts traded, and number of underlying commodities involved. Furthermore, it is due to the increased popularity of modelling techniques for investment and valuation based on stochastic price behaviour of commodities and commodity contingent claims (see e.g. Dixit and Pindyck, 1994; Trigeorgis, 1995). The recent interest in pricing of crude oil, heating oil and diesel fuel to explain inflationary pressures illustrates the macroeconomic relevance of commodity pricing (see e.g. Stock and Watson, 1999; Hamilton, 2000). Due to physical limitations and a nonnegativity constraint on inventories, pricing of real assets is more complex than pricing of financial assets. Commonly used one-factor asset pricing models, therefore, fail to capture essential commodity pricing characteristics as e.g. backwardation, basis risk, skewness, serial correlation, risk premia, and mean reversion to a satisfactory extent (see for example Pindyck and Rotemberg, 1988; Pindyck, 1993; Deaton and Laroque, 1992, 1996; Chambers and Bailey, 1996; Ng, 1996; Buehler Korn, and Schoebel, 2000). Other models incorporate some of these factors but take them as exogenous stochastic processes with constant distributions rather than varying endogenous components resulting from microeconomic dynamics of commodity markets (e.g. Brennan, 1991; Schwartz, 1997).

In this paper we review and integrate the most important contributions that explicitly analyse the microeconomics of production, storage, and hedging of commodity quantities and prices with the aim to better understand commodity price movements and forward price structures. We build on earlier contributions due to Williams (1987), Hirshleifer (1989b), Routledge et al. (2000) and Buehler et al. (2000) in that we allow for both risk premia and a convenience yield as arguments in the producer’s optimisation framework. Since most commodity markets are incomplete, we pay special attention to the futures price bias and determinants of the optimal futures hedge.

Traditionally futures markets are of key importance in explaining commodity price behaviour through two fundamental functions: they provide liquid
markets for hedging or risk shifting and the opportunity of price discovery (Black, 1976). Keynes (1930) was among the first to describe these two functions in *The Applied Theory of Money*. When used for hedging, the distinctive nature of futures markets, as compared to forward or other insurance markets, is that contracts are standardised, transaction costs are minimised, and liquidity is high, so that contracts are many times bought and sold during their lifetime. The price discovery function of futures markets supports firms in making production and storage decisions, and it is through this function that futures markets affect the efficiency of intertemporal resource allocation. Futures markets provide better price discovery services than forward markets due to their transparency and liquidity.² If we assume agents have rational expectations and share common information in the sense of Muth (1961), we can almost ignore the price discovery role of futures markets and focus on the function of risk transfer.³

² Many authors have studied futures market equilibrium to analyse how prices aggregate private information. In these studies, information is either only partially revealed in prices (Bray, 1981) or private information is discounted in the public price but no single individual has all information when making a trading decision (e.g. Grossman and Stiglitz, 1980). In both instances markets prove at least weakly efficient. However, incentives are left for information collection since prices alone are almost never sufficient statistics for strong-form efficiency. See also Hirshleifer (1975, 1977), Danthine (1978), Grossman (1977), Gale and Stiglitz (1989), and Hirshleifer (1991).

³ An additional advantage of the rational expectations assumption is that there is little conceptual difference between futures and forward markets. In most models in this paper they can be treated as one. For many commodities the forward market is as important as the futures market, since many primary producers trade on the forward market rather than futures market (e.g. agricultural products, energy). In theory forwards and futures bear the same price when interest rates are non-stochastic (Cox, Ingerssol, and Ross, 1981; Jarrow and Oldfield, 1981). Empirical evidence shows that arbitrage ensures high correlation between forward and futures prices (e.g. French, 1983,1986). However, there are some practical differences between exchange traded futures and ‘over-the-counter’ traded forwards, most notable of which are the daily cash flows resulting from clearing houses calculating mark-to-market value of futures contracts and making margin calls with those having open contracts.
Earlier and more extensive reviews generally neglect storage and inventories as important factors of commodity price formation (Peck, 1985; Streit, 1983; Stein, 1986; Weller, 1992; Moschini and Hennessy, 1999). Whilst Peck reviews the institutional history of futures markets, Streit concentrates on the functioning and efficiency of purely financial as opposed to commodity futures markets. Stein and Weller provide an overview of recent contributions to the risk shifting and the information function of futures markets but storage or production decisions are not incorporated in the optimisation process of the rational agent. In a recent survey, Moschini and Hennessy (1999) include production but leave out the storage decision. Thus the opportunities of connecting the theories of normal backwardation and supply of storage as well as analysing effects of joint optimisation in presence of futures trading are foregone. This review attempts to fill this gap.

The models discussed in this paper have a few common features, which we mention here to avoid unnecessary repetition. First, the models are mostly based on agricultural commodities whilst the conclusions are generally applicable to all storable commodities. Second, all models are based on a risk averse, competitive firm under uncertainty as formally described in Sandmo (1971). Besides the firm optimising the utility of profit, rather than profit itself, we explain risk averse behaviour from technological factors such as technological concavity, asymmetric information, and asymmetric adjustment costs due to possible ruinous losses and irreversibility of investments. These factors induce managers or owner-managers of commodity producing firms to behave as if they are risk averse, even if their subjective attitude towards risk is not (see also Williams, 1987; Aiginger, 1987). Risk averse behaviour gets reinforced as markets are incomplete and participants cannot diversify their full income risk away. Opportunities for production insurance or sharing (e.g. sharecropping or farm-outs), or issuance of company shares are limited or prohibitively expensive (Stiglitz, 1974,1983). Furthermore, all models are based on either constant absolute risk aversion or constant relative risk aversion, and, since uncertain variables as price and output are assumed to be (log-) normally distributed or jointly normally distributed, most models are based on a standard mean-variance framework. Finally, for ease of readability we have harmonised the meaning of
symbols throughout the paper, a list of which can be found in Appendix A. Mathematical derivations in the main text are kept to a minimum, relevant elaborations are provided in appendices.

The paper is organised as follows: Section 2 summarises traditional theories of hedging, speculation, and futures price formation. For the remaining sections we follow an incremental approach. Section 3 focuses on the optimal futures hedging decision under output uncertainty. Section 4 integrates the production decision into these models, taking examples where output is a certain decision variable and a stochastic decision variable. Section 5 provides an overview of traditional and modern contributions to the theory of storage and elaborates on and analyses a model of futures trading, storage, and production decisions in equilibrium. Section 6 presents the problem of incomplete markets, futures risk premia and alternative diversification opportunities. Section 7 summarises the paper, and concludes by outlining potentially rewarding directions for future research.

2 Traditional Theories of Hedging, Speculation, and Futures Price Formation

In this section we develop general functional forms of commodity futures pricing based on two traditional theoretical approaches to hedging, speculation, and commodity inventories. The theory of ‘normal backwardation’ explains futures price formation from the net short hedging pressure that exists on most futures markets. The shape of the futures curve depends on expected changes in prices as well as a risk premium. The theory of ‘supply of storage’ focuses on spot price formation in relation to futures based on cost-of-carry and the so-called ‘convenience yield’, that gives rise to a premium on commodity inventories in periods of limited spot availability. The theories are complementary in their explanation of commodity price formation and are foundations of modern commodity pricing theories as formulated by Brennan (1991), Schwartz (1997), and Routledge et al. (2000) for example.

2.1 The theory of normal backwardation
The theory of normal backwardation (Keynes, 1930) is the most famous theory on futures pricing, hedging, and speculation. This theory emphasises the risk reallocation role of futures markets. Whilst facing uncertainty regarding future spot prices risk averse producers are able to shift price risk on planned output to speculators by selling futures. Futures prices show a discount versus expected ‘actuals’, ‘cash’, or ‘spot’ prices because producers pay a premium to speculators to assume risk, even if supply and demand of commodities are balanced and price expectations are unbiased. Due to price uncertainty a hedging imbalance arises, and:

“...the spot price must exceed the forward price by the amount which the producer is ready to sacrifice in order to ‘hedge’ himself, i.e. to avoid the risk of price fluctuations during his production period. Thus, in normal conditions the spot price exceeds the forward price, i.e. there is a backwardation.” (Keynes, 1930, p.128).

The Theory of Normal Backwardation can thus be formalised as the difference between the current futures price, $p_f$, and the current spot price, $p$, also known as ‘the basis’, to be equal to the expected change in the spot price between today and the expiry date of the futures contract, indicated by subscript $T$, minus the applicable risk premium paid by producers to speculators, $\pi^*$,

$$p_f - p = E[p_t - p] - \pi^*,$$

(1)

where a tilde indicates random behaviour of the variable. According to equation (1),

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4 The term backwardation often leads to confusion in economic literature. Whilst the theory of normal backwardation compares futures prices to expected spot prices at maturity, in commodity markets the term ‘backwardation’ has been and still is used to indicate a situation where futures prices are below current spot prices (see e.g. Litzenberger and Rabinowitz, 1995; Gabillon, 1995). In this paper we use backwardation to indicate the situation where futures prices are below expected spot prices.
if $E[\hat{p}_T - p] = 0$, the basis will be equal to the risk premium, which positively depends on time to maturity. The ‘hedging imbalance’ caused by producers and the distribution of the uncertain price will jointly determine the size of $\pi^*$. We should note that even if backwardation is minimal, producers rarely hedge their entire expected output. On the one hand this is due to output uncertainty and thus the risk of being overhedged. On the other hand, part of the producer’s revenues is often automatically hedged as for most commodities output and price are inversely correlated. In effect if price elasticity of demand were equal to one, producers would have stable income regardless of their output. Consequently futures markets would only have a hedging function for processors but not for the typical commodity producers described in the theory of normal backwardation.

Hicks (1939), Kaldor (1939), Dow (1941), Blau (1944), and Telser (1967) supported and refined Keynes’ views on the causes of normal backwardation. Hicks (1939) explained net short hedging pressure from the organisation of the production process rather than from price uncertainty alone. Telser (1967) explained net short hedging pressure and resulting normal backwardation from the fact that sellers of futures contracts acquire an option with respect to timing, quality, and location of delivery of the commodity.

According to Keynes due to net short hedging pressure the risk premium will exist regardless of supply conditions. In case ‘discretionary’ or surplus stocks exist, these “…must cause the forward price to rise above the current spot price, i.e. to establish, in the language of the market a contango. …the quoted forward price, though above the present spot price, must fall below the anticipated future spot price by at least the amount of normal backwardation.”(1930, p. 144). This ‘contango’ is caused by the cost-of-carry, the latter being defined by Keynes as an allowance for deterioration of quality, warehouse and insurance charges, interest charges, and a “remuneration against the changes in the money-value of the commodity during which it has to be carried.”(1930, p. 135). Thus, in a world of price uncertainty and risk averse agents carrying charges include a risk premium on excess stocks, and, with
futures trading, two risk premia play a role: the risk premium on the stored commodity’s price, $\pi^x$, and the risk premium on the commodity futures price, $\pi^f$.

Working (1942), Johnson (1960), and Telser (1958, 1960) criticised the theory of normal backwardation for its implicit assumption of market imperfections. Their criticism focused on the risk premium, which due to arbitrage, free entry and competition would not exist. Furthermore, these authors challenged the strict dichotomy between hedgers and speculators. First, hedging pressure varies over a typical crop year from net short after a harvest to net long just before a new harvest, as processors want to secure supply (Cootner, 1960). Second, hedgers carry risk both from ‘routine hedging’, where they assume price difference risk or ‘basis risk’, and through ‘selective hedging’, where some positions are left unhedged (see also Gray, 1961). Consequently, they act on basis of speculative and arbitrage motives as much as traditional speculators do (Working, 1953, 1960, 1962). Their most important criticism, however, was that the theory did not sufficiently explain the role of inventories in futures price formation. Through the theory of supply of storage these critics offered an alternative explanation of futures backwardation.5

### 2.2 The theory of supply of storage
The theory of supply of storage is due to Kaldor (1939) who, in his famous article *Speculation and Economic Stability*, was the first to emphasise the significance of the ‘convenience yield’ in storage decisions.6

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5 Econometric tests of backwardation on futures markets in terms of a risk premium are inconclusive. Early supportive results were found by Houthakker (1957) and Cootner (1960), whilst Working (1942), Brennan (1958) and Telser (1958) did not find significant risk premia. Dusak (1973) and Carter, Rausser, and Schmitz (1983) examine the risk premium within the context of CAPM, and come to opposite conclusions on the existence of a risk premium. More recent publications include Chang (1985), Fama and French (1987), Hartzmark (1987), Fort and Quirk (1988), Bessembinder (1992), and Litzenberger and Rabinowitz (1995), who all find mixed evidence of a risk premium as explanation for normal backwardation. Roon, Nijman, and Veld (2000) find strong evidence for ‘cross-hedging’ pressure and risk premia.

6 Whilst Kaldor was the first to use the convenience yield in the context of futures markets,
“Stocks of goods … also have a yield, *qua* stocks, by enabling the producer to lay hands on them the moment they are wanted, and thus saving the cost or trouble of ordering frequent deliveries, or of waiting for deliveries. But the amount of stocks which can thus be ‘useful’ is, in given circumstances, strictly limited: their marginal yield falls sharply with an increase in stock above ‘requirements’ and may rise very sharply with a reduction of stocks below ‘requirements’. When redundant stocks exist their marginal convenience yield is zero.” (1939, p. 21).

Working (1942, 1949) supported Kaldor’s theory and emphasised the role of futures markets in storage decisions. At all times the price difference between prices for delivery at two different dates must equal the market-determined price of carrying the commodity between these dates. Therefore, arbitrage ensures that the amount of contango in the futures price curve will be limited by the marginal cost of storing one additional unit of the commodity. If, however, supply is relatively small, the market’s price for storage will be smaller than the costs-of storage or even negative due to a rise in the convenience yield. Brennan (1958), Weymar (1966) and Paul (1970) further formalised the theory of supply of storage along these lines, by showing that, assuming market equilibrium, the marginal return to storage, which is equal to the futures basis, is based on the marginal net cost-of-carry, $c_s$, the marginal convenience yield, $\lambda$, and a risk premium, $\pi^+$, that rises sharply with excess availability, $Q_r$,

$$p^f_T - p = c_s + \pi^+ - \lambda, \quad \text{with } \lambda, c_s, \pi^+ \geq 0$$

(2)

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Keynes was the first to suggest its existence. In *The General Theory of Employment, Interest and Money* (1937, Ch. 17) Keynes defines the convenience yield when presenting the concept of ‘own-rates of interest’ on real assets: “…potential convenience or security given by this power of disposal (exclusive of yield or carrying cost attaching to the asset), we shall call its liquidity premium, $l$.” (1937, p.226).
A graph of equation (2) for various levels of $Q_t$ is provided in Figure 1.

*Figure 1. Supply-of-Storage curve*

As can be seen in Figure 1, if $Q_t$ is large, the return on storing a commodity may exceed the total cost of storage, $c_s + \pi^*$, in other words the contango will be larger than the cost-of-carry. In that case, producers and traders will be encouraged to store, because hedging the commodity on the futures market ensures a return to storage that covers storage costs. Consequently, surplus of the commodity will be reduced on spot markets and storage supports what would otherwise be a very low price. Producers and processors will carry more stock if they expect the price to rise and vice versa. In case of shortage of supply, however, the convenience yield, $\lambda$, increases, the marginal return to storage becomes negative, and backwardation arises. In this case, individuals holding stocks will reduce their stocks by which temporary shortage on spot markets will resolve.\(^7\) Furthermore, as also argued by Keynes

\[^7\] From the above it intuitively follows that the opportunities for arbitrage and hedged storage offered by futures markets have a stabilising impact on price levels and supplies over time. For
(1930), a representative firm will not be willing to hold stocks beyond a ‘normal’ threshold, so the risk premium, $\pi^*$, increases with the amount of stocks held. However, this risk on spot market storable commodities can partly or entirely be transferred to speculators on the futures market by selling futures.

The theory of normal backwardation and the theory of supply of storage are alternative explanations of the futures basis. However, the theories are clearly interrelated through the phenomenon of ‘carrying charge’ or ‘arbitrage’ hedging. Seasonally driven changes in inventories as well as changes in supply and demand will determine which model predominates. Modern theorists subscribe to this view (e.g. Newbery and Stiglitz, 1981; Williams, 1986; Hirshleifer, 1989b; Buehler et al., 2000; Routledge et al., 2000). If inventories are high and the spot price is low, futures prices will match simple cost-of-carry based calculations including a risk premium for the commodity stocks. If however, inventories are close to zero and the spot price is high, the risk premium on stocks will approach zero and the futures price will be based on market expectations including a risk premium depending on market imperfections as e.g. short-selling constraints. Thus whilst the theory of supply of storage, through the absence of price expectations, assumes stable supply-demand conditions, the theory of normal backwardation incorporates investor anticipation of changes in market conditions but excludes the embedded timing option in the spot commodity’s price. The complementary relationship between the theory of normal backwardation and the theory of supply of storage is illustrated in Figure 2.

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a formal proof see inter alia Kawai (1983) and Turnovsky (1983).
In the remainder of this paper, the risk premium, the convenience yield, and the cost of carry will be relevant when explaining futures price formation and production, storage, and hedging under uncertainty.

3 Hedging and Speculation under Uncertainty

Whilst classical theorists provided intuitive explanations how risk was reallocated and commodity prices were formed, little formal work was done to analyse the market conditions under which backwardation or contango occurs. More importantly, the role of futures markets in agents’ decisions under uncertainty was left largely unexplored. Advances in portfolio theory (Markowitz, 1959) and development of the mean-variance rule (Sharpe, 1964) led to a considerable amount of papers on the role of futures markets in risk reallocation.8 In this section we outline the basic theoretical framework from this literature and analyse a few comparative static results.

8 Most analyses concerning futures markets hedging decisions stay close to the Capital Asset Pricing Model due to Markowitz (1959) in that the optimal futures hedging decision is based on mean and variance. As long as the mean and variance are known, one can estimate the
McKinnon (1967) was the first author to present a model of a commodity producer who minimises variance of income through futures hedging. Newbery and Stiglitz (1981), Kawai (1983), Anderson and Danthine (1983a), Stiglitz (1983), Britto (1984), and Duffie (1989) extended McKinnon’s model with very similar results. The mean variance framework in these models generally relies on constant absolute risk aversion. Duffie (1989) assumes constant relative risk aversion based on a quadratic utility function of expected value and variance of income. Both approaches are convenient because they lead to closed-form solutions. Rolfo (1980) presents a logarithmic utility function and derives a numeric solution that produces similar results as the mean-variance framework. Furthermore, most authors distinguish between quantity (output) and price uncertainty, although price uncertainty is mainly dependent on variations in output and the correlation between price and output. The risk-averse producer can hedge his income variability on the futures markets by buying or selling futures.

To set the framework we analyse a model of futures hedging based on Newbery and Stiglitz (1981). This model supports an integrative approach to the theories as described in the previous section, in that futures trading by producers results from a mixture of hedging and speculative motives. By deriving a supply and demand function for futures contracts, the model explains how risk aversion and uncertainty impact on the hedging and speculative decision, and can lead to normal backwardation. Since the model does not allow for storage, the convenience yield is expected value of the one-period utility function. One of the shortcomings of any mean-variance approximation is that an investor becomes increasingly risk-averse in an absolute sense as his wealth increases (Pratt, 1964). Levy and Markowitz (1979), however, provide plausible arguments why the mean-variance is an attractive approach once its limitations are understood. The main advantage for the optimisation models in this review is that it results in linear futures hedging functions and permits closed-form solutions. As Samuelson (1970) shows, higher moments than the second do not matter as long as the distribution of returns is ‘compact’: returns are normally distributed and investment decisions can be reversed instantaneously without cost. These assumptions are strong when applied to commodity futures markets but provide an analytically convenient starting point.
excluded as a source of backwardation, but will be included in Section 5 of this paper.

The producer has the following general expected utility function with constant absolute risk aversion:

$$E[U(y)] = -Ee^{-Ay},$$  
(3)

with $E$ being the expectations operator, $A = -U''/U'$ being the Arrow-Pratt coefficient of absolute risk aversion, and $y$ representing income. As $y$ has a multivariate normal distribution, the exponential utility function acts as a moment-generating function.

With $y \sim N(\bar{y}, \nu^2)$, one can define the expected utility as:

$$E[U(y)] = U(\hat{y}),$$  
(4)

with

$$\hat{y} = \bar{y} - \frac{1}{2}Av^2,$$  
(5)

being the utility certainty equivalent.

The producer trades on an unbiased futures market, where the futures price $p^f$ is equal to $\bar{E}p$, the expected future spot price in the next period, which, because $\bar{p}$ is normally distributed, in turn is equal to the average price, $\bar{p}$:

$$p^f = \bar{E}p = \bar{p}.$$  
(6)
As we discuss a one-period model we omit time subscripts. For ease of notation we will also leave out a discount rate by assuming it either constant or non-stochastic (Cox, Ingersoll and Ross, 1981). If \( z \) is the amount sold forward on the futures market, and \( \bar{q} \) represents average output with a multiplicative risk, \( \tilde{\theta} \), expected income from production and futures trading amounts to:

\[
E\tilde{y} = E\tilde{p}\tilde{\theta}\bar{q} - z(\bar{p} - \bar{p}^f),
\]

(7)

with \( E\tilde{\theta} = 1 \). Use of the multiplicative risk factor \( \theta \) will avoid scale problems in the analysis further below. Inserting (7) into (5), and taking into account approximate joint-normality of \( \tilde{p} \) and \( \tilde{p}\tilde{\theta} \), in line with standard portfolio theory this is equivalent to maximising:

\[
\max z = \bar{q}E(\tilde{p}\tilde{\theta}) - z(\bar{p} - \bar{p}^f) - \frac{1}{2} A[\bar{q}^2 \operatorname{var}(\tilde{p}\tilde{\theta}) - 2\bar{q}z \operatorname{cov}(\tilde{p}, \tilde{p}\tilde{\theta}) + z^2 \operatorname{var}\tilde{p}],
\]

(8)

with respect to \( z \). The first order condition for \( z \) is equal to:

\[
z = \frac{\bar{q} \operatorname{cov}(\tilde{p}, \tilde{p}\tilde{\theta})}{\operatorname{var}\tilde{p}} - \frac{\bar{p} - \bar{p}^f}{A \operatorname{var}\tilde{p}},
\]

(9)

\( ^9 \) When price and production are jointly normally distributed, producer revenues are not normally distributed. Newbery (1988) shows resulting errors from a mean-variance analysis stay small as long as homogeneous producers are assumed. See Honda and Ohta (1992) who present a Taylor approximation to analyse speculation and hedging.
allowing for $z$ to be positive or negative, i.e. the model allows for futures sales and purchases. Note that this is a single period model: the model contains no storage, i.e. all $q$ is being sold in the same period, and there is no convenience yield.

The risk-averse speculator, indicated by superscript $B$, to who futures trading is the sole source of revenue, maximises income, $y$, according to:

$$\max_{z^B} y^B = z^B (p^f - \bar{p}) - \frac{1}{2} A^B z^B \text{ var } \tilde{p},$$

(10)

which results in the first order condition:

$$z^B = -\frac{\bar{p} - p^f}{A^B \text{ var } \tilde{p}},$$

(11)

with the difference $\bar{p} - p^f$ being called ‘normal backwardation’ if $\bar{p} > p^f$, created by a risk premium paid by the producer who wants to hedge income risk. The speculator has no other risky income and can only be persuaded to take a long position in the futures market if $p^f$ is smaller than $\bar{p}$ at moment of trade. Equation (11) and equation (9) can be seen as futures demand and supply functions, that together determine futures market equilibrium. Assuming $A$ and $A^B$ have similar functional form, equation (11) is also a component of (9), and as it is independent of any production outcome, this means that the second component of (9) can also be called the ‘speculative element’ for the producer, whilst the first component can be seen as the production ‘hedging component’. As shown in (9) the hedging component is independent from the individual producer’s risk preferences, which means the hedging decision is solely taken on basis of the variance of price and covariance between price and production uncertainty, a point also made by Benninga, Eldor, and Zilcha (1983). Furthermore, (9) shows the speculative component will be equal to
zero if \( \tilde{p} \) is equal to \( p' \), or if \( A \), the level of risk aversion, becomes infinitely large.

In the latter case, the producer will determine \( z \) independently from the futures bias and will use futures only to hedge income uncertainty. Similarly, from (11) we learn that, at a given risk premium, the amount of futures a speculator will be willing to hold is determined by both \( A^\theta \) and \( \text{var} \tilde{p} \).

Through calculation of the moments of \( \tilde{p} \) and \( \tilde{q} \) (shown in the Appendix B), we define the standard deviation of quantity and price as \( \sigma_q \) and \( \sigma_p \), respectively. If \( \rho \) is the correlation coefficient between price and quantity, equation (9) can be rewritten as:

\[
z = \tilde{q}(1 + \rho \frac{\sigma_q}{\sigma_p} ) - \frac{\tilde{p} - p'}{A \tilde{p}^2 \sigma_p^2}.
\]

(12)

If the only source of risk were demand risk and \( \sigma_q = 0 \), and there would be no bias in the futures market so that \( p' = \tilde{p} \), the farmer would sell his entire production \( q \) forward. As soon as backwardation arises \( p' < \tilde{p}, z < q \), and if contango would arise, \( p' > \tilde{p}, z > q \). Also, as soon as \( \sigma_q > 0 \), assuming a negative correlation, the producer will hedge less than expected output to avoid non-delivery on part of his futures contracts. If, however, the only source of risk is output risk, \( \rho = -1 \), and \( \eta \) is the elasticity of demand defined as \( \eta = \sigma_q / \sigma_p \), equation (12) becomes:

\[
\frac{z}{\tilde{q}} = 1 - \eta.
\]

(13)

From this equation it is easily seen that if \( \eta = 1 \), the farmer is already perfectly
hedged against income risk and therefore he would not engage in any futures trade for hedging purposes (he may do for speculative purposes if \( \bar{p} \neq p' \)). If \( 0 \leq \eta < 1 \), the farmer will sell futures. In this case, demand is relatively inelastic and prices and output are so negatively correlated that income decreases when output increases and vice-versa. The individual producer will reduce his income variability under both the low output and the high output states through futures sales, as shown in Figure 3. However, in case \( \eta > 1 \), demand is relatively elastic and price and output are negatively correlated but not so negatively correlated that price variability offsets output variability. In this case, the individual will not sell his crop in a high-income situation but he will buy forward, thus transferring income from high-income to low-income states. This is shown in Figure 4. The individual can of course increase his income in situations where the price is low (and output and income are high) by selling some output on the futures market. This, however, would result in income to be reduced for states in which output is low and income is low. Because marginal utility decreases with an increase in the individual’s income, the producer will be encouraged to buy futures in high-income situations (see also Britto, 1984). Note that under these circumstances the decision to buy futures or the decision to store goods leads to equivalent results, i.e. the reduction of income variability. This is an important result, which will be discussed further in Section 5.
On basis of the model above we conclude that futures markets allow speculators to bear some of the producer’s risks. The net futures trade of the producer will reflect the balance of the desire to insure and to earn returns from speculating. A producer can never be fully hedged if he faces output uncertainty. However, producers can be completely hedged if distribution of income is purely related to
demand (=price) risk. Whether a producer will actually fully hedge his position depends on the appetite for speculative risk as well as the futures price bias. The greater the agreement on expected spot price and the less risk averse are the speculators, the smaller will be the risk premium and the futures price bias and the larger will be the fraction of hedging to speculative sales. A summary of key results from the above analysis is presented in Table 1.

Table 1: Impact of futures trading determinants on the futures position of a producer.

<table>
<thead>
<tr>
<th>Impact on z</th>
<th>Hedging decision</th>
<th>Speculative decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_q &gt; 0$</td>
<td>$0 &lt; \eta &lt; 1$</td>
<td>$\eta = 1$</td>
</tr>
<tr>
<td>$-$</td>
<td>$&lt; 0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

"-" means a negative impact on absolute value of $z$, "+" means a positive impact on absolute value of $z$, "+ 0" implies a short position, "0" implies a zero position, "+ > 0" implies a long position.

4 Production, Hedging, and Speculation under Uncertainty

Section 3 describes the hedging decision under price and output uncertainty, whilst the production decision itself is exogenous. In this section we incorporate the production decision in the optimisation framework. A fair amount of literature is available on the joint decision problem of optimal hedging and production. Some models are based on the assumption that production decisions can be based solely on (certain) futures prices. Since the optimal production decision is independent of the producer’s risk preference and expectations and can be separated from the hedging decision, it is called ‘stochastically separable’. In Section 4.1 we present examples of static and dynamic stochastic separability. An essential condition for stochastic separability is a deterministic production function. As soon as production becomes stochastic, the producer can no longer hold a perfectly diversified portfolio and the joint production and hedging decision becomes more complicated. This will be illustrated in Section 4.2.
4.1 Production and hedging under price uncertainty

Danthine (1978), Holthausen (1979), and Feder, Just, and Schmitz (1980) are early examples of production and futures hedging decisions under price uncertainty. Holthausen (1979) introduces a futures market into the famous Sandmo (1971) model of optimal production under price uncertainty. To summarise Holthausen’s results, we amend the mean variance utility function in Section 3 to include production cost, and analyse optimal expected utility from profit, \( \Pi \), for a non-stochastic production function with certain output \( q \) and a convex production cost function, \( c(q) \):

\[
\max_{q,z} \Pi(q,z) = \tilde{p}(q - z) + p^T z - c(q) - \frac{1}{2} A(q - z)^2 \text{ var } \tilde{p}.
\]

(14)

First order conditions with respect to \( q \) and \( z \) give:

\[
\frac{\delta U}{\delta q} = E\tilde{p} - c'(q) - A \text{ var } \tilde{p}(q - z) = 0
\]

(15)

and

\[
\frac{\delta U}{\delta z} = -E\tilde{p} + p^T + A \text{ var } \tilde{p}(q - z) = 0
\]

(16)

Condition (15) excluding \( z \) is also found in Sandmo (1971), who concludes that the firm chooses a lower level of output when price risk increases in the sense of Rothschild and Stiglitz (1970), the firm’s risk aversion increases, or both. These conclusions are no longer valid in the presence of a futures market, as optimal
production is then determined independently from the degree of risk aversion or price risk. This ‘stochastic separability’ is obtained by adding condition (15) and (16) to give \( p^f = c'(q) \). As in the standard microeconomics textbook example of a competitive firm, the firm chooses the level of output where the marginal cost of production is equal to the (certain) forward price. Note also that as the production decision becomes independent of the firm’s perceived price distribution or the level of risk aversion, the same conclusion would hold for risk-neutral producers. Thus, on basis of stochastic separability we can conclude that the introduction of a futures market allows producers to behave as if they live in a certain world, by taking the expected output price equal to the certain futures price. With this result Holthausen formalises an idea already described by Keynes (1930) in his theory of the “Forward Market”.

As discussed in section 3, however, this does not mean that producers will hedge automatically all their output since the futures trading decision is a mixture of hedging and speculative motives and thus depends on risk aversion and the futures risk premium. From the first order condition for \( z \) and maintaining \( E\tilde{p} = \bar{p} \), we obtain:

\[
z = q - \frac{\bar{p} - p^f}{A \var{p}}.
\]

Although this solution is simpler than equation (9) due to the absence of output risk, we derive similar conclusions on the company’s hedging and speculative decisions as summarised in Table 1. One additional result from equation (17) is worth noting however: if \( p^f \) is substantially lower than the expected spot price, the firm is not prepared to pay a high risk premium but will actually start to buy futures. To see this, we note that in the absence of a forward market the firm would produce an amount where \( c'(q) \) is equivalent to the certainty equivalent price, \( \bar{p} - A \var{p} \) (see also
Baron, 1970). Hence, in case \( p' < \tilde{p} - \text{var } p \), the firm would produce less than in the absence of a futures market. The revenue the firm misses from lost production is compensated for by buying futures with the perspective of selling these at a higher price at expiry.

The studies by Holthausen and others provide useful insights into production and hedging decisions in a static world. Anderson and Danthine (1983b) and Kamara (1993) provide examples of dynamic models with deterministic production functions. In these models producers have the option to adjust production and hedging levels at intermediate stages in response to newly arrived information. Because adjustments are unknown at the beginning of the process, final optimal production and hedging levels become stochastic, even if the production process itself is deterministic. However, even in this case stochastic separability can be maintained, as long as production and factor pricing functions are intertemporally separable. See Appendix C for an outline of the model by Kamara (1993).

### 4.2 Production and hedging under price and output uncertainty

Many commodity producers, especially farmers, do not benefit from stochastic separability. Since futures contracts cover price risk but not production risk, a producer cannot take a perfectly offsetting futures position to hedge his revenues and will almost certainly be obliged to trade on the cash market at delivery time. Because the producer cannot diversify his risks away costlessly under output uncertainty, the producer’s optimal hedging decision depends on risk aversion, expected output, and the relation between the current futures price and expected spot price. Furthermore, contrary to the stochastic separability case, the production decision is no longer independent from the optimal hedge position. Below we discuss two examples of production and hedging under joint price and output uncertainty. For ease of exposition we first present a static example, after which we present a dynamic example.

Examples of static analyses are given by Anderson and Danthine (1981, 1983a), Marcus and Modest (1984), and Honda (1984). Anderson and Danthine
(1983a) extend the standard hedging model of Holthausen as described in Section 4.1 with the random production function, $\tilde{g} = g(\tilde{q}, \tilde{\theta})$:

$$\max_{q, z} EU(\Pi) = \tilde{p}g(\tilde{q}, \tilde{\theta}) - c(\tilde{q}) + z(p' - \tilde{p}).$$

(18)

with the first order conditions for the optimal output, $\tilde{q}$, and futures position, $z$:

$$E(\tilde{p}\tilde{g}_t) - c'(q) - \lambda \left[ \text{cov}(\tilde{p}, \tilde{g}_t) - z \text{cov}(\tilde{p}, \tilde{p}_t) \right] = 0.$$  

(19)

$$Eg(\tilde{q}) + \frac{\text{cov}(\tilde{p}, \tilde{p})}{\text{var} \tilde{p}} + \frac{p' - E\tilde{p}}{A \text{var} p} = z,$$

(20)

where $\tilde{g}_t = g_1(\tilde{q}, \tilde{\theta})$, the first derivative of $\tilde{g}$ with respect to $\tilde{q}$. Equation (20) is almost identical to equation (9) in Section 3, and we maintain the conclusion from equation (9) that the optimal futures position depends on expected output, the covariance between price and revenue, the futures price bias, and the producer’s risk aversion. Note also that it is possible in (20) to distinguish between the pure hedge and the pure speculative component. If we assume the futures price bias to be zero, the optimal hedge will depend on the sign of the covariance between $\tilde{p}$ and $\tilde{p}_{\theta}$. As analysed in Section 3, if all producers have identical production functions, it is the elasticity of demand that matters. If $0 \leq \eta \leq 1$, the farmer will sell futures, but for an amount less than expected output. In case $\eta > 1$, demand is relatively elastic and the producer will buy futures.

The production decision is no longer independent of risk preferences and the expected spot price and there is no longer a separation between the output decision
and hedging decision as was the case under a deterministic output function as described in 4.1. Clearly, the complexity of the optimal production decision is enhanced due to the interaction of random production and price. On basis of first-order condition (19), Anderson and Danthine analyse how the presence of a futures market influences the production decision. A summary of results is provided in Appendix D. They confirm earlier findings by *inter alia* Newbery and Stiglitz (1981) and Holthausen (1979), in concluding that a futures market, by providing the opportunity for risk shifting, increases production compared to the situation where there is no futures market, unless the futures price bias becomes large.

Models of optimal hedging and output decisions under price and output uncertainty in a dynamic setting are examined by Anderson and Danthine (1983b), and Hirshleifer (1991). Dynamic analysis can add to the foregoing in a number of respects: first, the firm has the possibility to adjust production and hedging levels as new information arrives and price uncertainty resolves. Second, the impact of time is better represented in a dynamic model, especially for production models with clear seasonality patterns as e.g. a single annual harvest (Hirshleifer, 1991). Third, contrary to one period models, dynamic models allow inclusion of relevant institutional features of futures markets as mark to market margin settlements, for example.

Due to limited space we focus on the hedging solution, in the knowledge that the production decision is no longer stochastically separable and will therefore depend on the optimal futures levels, and the intertemporal distribution and interaction of expected output and price levels. Anderson and Danthine (1983b) extend a three period mean-variance framework with a stochastic production function. Through recursive optimisation similar to Kamara (1993) the authors formulate optimal futures positions for period 1 and 2, which can be separated into a hedge and a speculative part. We provide the period 1 optimal hedge given output uncertainty:
with $z_2^s = \frac{p_2^f - E_2 \tilde{p}}{A \text{var}_2 \tilde{p}}$, the optimal speculative position adopted in period 2 and $z_2^h = \frac{\text{cov}_2(\tilde{p}, \tilde{p} g_1)}{\text{var}_2 \tilde{p}}$, the optimal hedge position of period 2. The attentive reader will note that equation (21) reads as a complicated version of equation (9) in Section 3. The first set of two components of (21) are speculative components, of which the first indicates the producer goes short if he expects the futures price to fall and he goes long if he expects the futures price to rise. The second component is more ambiguous in sign: although the sign of $z_2^s$ is determined in exactly the same way as the first component of (21), under expected backwardation $p_2^f - \tilde{p} < 0$, the producer may speculate by buying futures for settlement at a higher price at maturity. Thus, the overall covariance sign can be negative and allows the producer to go short in period 1 although the futures price is expected to rise. If this applies, the second component can best be seen as a hedge on period 3 revenues from period 2 speculation, which is not a satisfactory explanation. Depending on the relative weight of the first and the second component, an overall speculative position of equal sign in period 1 and 2 seems more likely. The second set of two components in equation (21) can be seen as the pure hedging part as they depend on the covariance between spot and futures price with uncertain output and revenues from output. Not surprisingly, the formula determining $z_2^h$ is identical to the first component of equation (9) in Section 3, since in period 2 the producer faces a one-period optimisation problem. It needs no further illustration to see that by adding further periods to this type of
dynamic analysis the solution becomes more complicated, therefore making the characterisation of signs tedious and partly inconclusive.

Alternative solutions for dynamic models of production and hedging are provided by Ho (1984) and Karp (1988) who analyse optimal production and hedging in a continuous-time, finite horizon framework. Very few contributions to futures markets and hedging are based on continuous time modelling. Application of continuous time models to markets that have strong seasonality (with often a single annual harvest) and lumpy contracts is often criticised because the underlying stochastic processes are restrictive in that the instantaneous moments are sufficient statistics for the entire probability distribution (Merton, 1973). However, as demonstrated by Ho and Katz use of continuous time can be analytically convenient and lead to attractive solutions. Since they reach similar conclusions on production and hedging as the discrete time models summarised above, we do not elaborate on their analyses here.

5 Storage, Production and Hedging under Uncertainty

In the models discussed so far, a representative risk-averse firm facing price and output uncertainty takes optimal hedging, production and speculative decisions under specific constraints. Surprisingly, in the majority of the contributions reviewed in this paper the firm’s storage decisions are not included in the firm’s optimisation framework, although most commodities on futures markets are storable. As discussed in Section 2, the relationship between storage and futures markets is widely known due to the work of Kaldor (1939), Working (1942, 1953), and Brennan (1958). Moreover, processors and storage firms form a majority of traders on futures markets and the importance of storage to them is undisputed (Williams, 1987). As we know from Section 2.2, instead of selling all production through spot or futures contracts, firms often take the conscious decision to store part of the product for later sale or use in manufacturing. This, to avoid stock-outs or delays in the production and marketing process when new orders come in. Producers and traders thus derive a convenience yield from the decision to store. Furthermore, storage and futures
trading are substitutive since they provide alternative methods for reducing income and price variability. Simultaneously, they are complements since futures contracts and futures prices facilitate storage and can thus contribute to optimal intertemporal resource allocation (see also Black, 1976; Peck, 1976). Therefore, in this section we integrate the storage decision into the firm’s optimisation process. In Section 5.1 we focus on the competitive storage decision in isolation, and use a few results in Section 5.2, where we analyse the role of futures markets in the joint optimisation of production, storage, and hedging.

5.1 Storage and its impact on production, consumption, and price formation
Since the theoretical work by Working (1949), Brennan (1958), and Weymar (1966), contributions gradually shifted from explaining storage decisions based on futures price information to the impact the storage decision has on commodity supply and demand. Whilst McKinnon (1967), Paul (1970), and Newbery and Stiglitz (1981) compare welfare effects of competitive storage to those of for example public buffer stock schemes for commodities, Fort and Quirk (1988), Williams and Wright (1991), Chambers and Bailey (1996), and Deaton and Laroque (1996) analyse specifically the impact of storage on supply, demand, and resulting prices.10

10 In parallel, over the past two decades a number of studies have been published analysing the interaction of inventories and production at the macroeconomic level. Frequently cited examples include Eichenbaum (1984, 1989), Blinder (1986), Miron and Zeldes (1988), and Ramey (1989). Although macroeconomic studies have different origins compared to microeconomic studies of storage, many principles on which they are based are similar. Most macroeconomic contributions focus on the presumed production-smoothing role of inventories. Industries holding stocks of finished goods would reduce production adjustment and marketing costs compared to industries having no buffer-stocks. Translated to the macroeconomic level, this would imply that aggregate production level would be less variable than the aggregate consumption level. However, most studies find the opposite to be the case for a number of industries (e.g. Blinder, 1986; West, 1986). The variance of production generally exceeds the variance of average sales. Alternative approaches focus on the role inventories play in
By rewriting equation (2) as a basic arbitrage relationship in spot commodity markets under risk aversion, we arrive at the basic storage model based on Working (1942, 1949) and Williams and Wright (1991), where a risk averse agent maximises the expected profits from storage under rational expectations. In equilibrium total storage equates the price of the current period, \( p_t \), plus the marginal physical costs of storage, \( c_s \), and the risk premium, \( \pi^t \), minus the convenience yield, \( \lambda \), and the expected spot price in the next period, \( E_t \left[ \tilde{p}_{t+1} \right] \). The arbitrage relationships read as follows:

\[
\begin{align*}
    p_t + c_s + \pi^t - \lambda - E_t \tilde{p}_{t+1} &< 0, \quad S_t > 0, \\
    p_t + c_s + \pi^t - \lambda - E_t \tilde{p}_{t+1} &\geq 0, \quad S_t = 0.
\end{align*}
\]

Note that this arbitrage relationship brings in a fundamental asymmetry into the system, since storage cannot be negative. This nonnegativity constraint means that there is an equilibrium spot price above which 'discretionary' or 'excess' storage is zero [Pindyck (1993)]. Note also that, because storage costs are certain (assuming \( c_s \), \( \lambda \), and \( \pi^t \) are known and fixed within the period), in presence of a futures market a storage firm could make riskless profits as long as the \( p_t^f \) is an unbiased estimator of smoothing production costs rather than production levels. Storage would thus help shifting production to periods of lower cost, or help reducing marketing and stock-out costs. These studies too have, at best, received mixed empirical support (Miron and Zeldes, 1988; Eichenbaum, 1989) although Pindyck (1994), based on a convenience yield for stocks, finds strong evidence on basis of a non-linear production adjustment cost function, that inventories are mainly held to reduce marketing rather than production costs. As Wright and Williams (1991) and Pindyck (1993, 1994) show, analytical results from both lines of theory can be integrated and microeconomic results with respect to production levels and the impact of responsive supply to storage can lead to better understanding of fluctuations in business cycles.
the future spot price, $\tilde{p}_{t+1}$, and the above arbitrage equilibrium has not been fulfilled (see also Sarris, 1984). With commodity traders this is the classic example of the ‘contango game’. If storers are sufficiently risk averse and are willing to pay a risk premium, this arbitrage activity could thus lead to net short hedging pressure and a movement from contango into backwardation or reinforce backwardation (Hirshleifer, 1989b).

Profit maximising storage for a firm that is also producing the commodity depends, apart from the physical costs of storage, on the particular specification of supply and demand for the commodity, functions that are potentially affected by storage. In a setting without possibility of storage, supply and demand is driven by the period’s output (e.g. harvest) and the demand for commodities for consumption and production. Under storage, however, based on (25) speculative demand and supply will arise in addition. Thus, total supply in any period will not only depend on current output but equally on precautionary and speculative carryover from the previous period. Similarly, total demand will consist of speculative demand in addition to regular demand that would arise in the absence of storage (see also Ng, 1996).

To analyse the effects of storage on the production decision, the current availability of the commodity plays a crucial role. If it is assumed that current period’s availability is price inelastic, i.e. harvests always take one period once the production decision is taken, storage and responsive supply are likely to have opposite effects because storage can be seen as a substitute for production (Scheinkman and Schechtman, 1983). In case of oversupply, it is rewarding to put commodities at a low price into storage and transfer them to a next period, which will lead a rational firm to reduce next period’s production. The same reasoning can be applied to a situation of relative shortage, which will induce a higher responsive supply next period. Under both circumstances, storage and responsive supply are likely to fluctuate whilst they stabilise any period’s supply. Of course, in the absence of storage the elasticity of supply is in fact irrelevant since a production disturbance in any period has no effect on subsequent periods. In that case planned production
has no reason to change from one period to the next. Both cases are illustrated in Figure 5.

**Figure 5. Storage and supply response**

Source: Williams and Wright (1991)

Due to speculative storage in relation to equation (22), beyond equilibrium availability $Q^{eq}$, the demand function will show a similar kink as the responsive supply curve. Normal consumer demand can be assumed linear with respect to price, but the possibility of transfer of a part of today’s consumption to the future through competitive storage brings a kink into the demand function, as illustrated in Figure 6.

**Figure 6. Demand curve with storage**

Source: Williams and Wright (1991)
The impact of storage on prices is also asymmetric: storage supports what would otherwise be very low prices in the absence of storage, but since storage cannot become negative (you cannot borrow from the future), very high prices in periods of low availability will occur and storage is not able to play a stabilising role. This asymmetry results in a positively skewed distribution of the spot price.\textsuperscript{11} Storage can also change the mean of the price distribution. In our basic analysis it is assumed that demand is linear and the supply elasticity is equal to zero, so the mean remains unchanged.\textsuperscript{12}

\textit{Figure 7. Price distribution curve}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{price_distribution_curve.png}
\caption{Price distribution curve}
\end{figure}

\textsuperscript{11} For empirical evidence on skewed futures price distributions see also Helms and Martell (1985).
\textsuperscript{12} See Williams and Wright (1991) for an extensive analysis of variations in same period supply and demand elasticity and their impact on other variables.
From the graphical analysis we conclude storage reduces the variance of price substantially, which confirms its role of price stabilisation mechanism, although the stabilisation properties are not symmetric. If we look at intertemporal price formation, the fact that we allow for speculative storage means that expected (discounted) future prices cannot be greater than current prices by more than the cost-of-carry, the risk premium, and the convenience yield. In effect, total supply determines availability and storage in period \( t \), thus determining carryover in \( t+1 \) and influencing the price in period \( t+1 \). Thus, a serial correlation arises which would not exist with non-storable commodities, the supply of which is subject to independent, identically distributed (i.i.d.) shocks (Deaton and Laroque, 1992, 1996). In summary, whether it comes to exogenous output, demand, or external price shocks, competitive storage can be seen as efficiently dispersing disturbances of shocks in time. However, through the responsive supply mechanism it can destabilise planned production. In the next section we will explore this feature in conjunction with production and hedging decisions.

### 5.2 Storage, Production, and Hedging under Uncertainty

This section integrates various results from previous sections by analysing the interaction between optimal storage, production, and hedging decisions in presence of futures markets. This interaction has been explored by a few authors only. Scheinkman and Schechtman (1983) formulate a partial-equilibrium model of production and storage, where the futures market is a source of price information (futures prices are considered unbiased forecasts of future spot prices), but futures trading itself is not included. Williams (1987) and Hirshleifer (1989b) take futures trading specifically into account but leave out the convenience yield as an important determinant of storage. We include the convenience yield in the model below based on Hirshleifer (1989b), that analyses storage, hedging and production by risk-averse

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13 For empirical evidence see also Taylor (1985).
producers, storage firms, speculators, and consumers. The model is multi-period and so allows for optimal carryover and responsive supply. We focus on the producer’s optimal decisions.

The producer optimises utility of consumption over time, where consumption is formulated as follows:

$$\bar{C}_t = \bar{y}_t - \bar{S}_t \bar{p}_t - (c_s - \lambda)\bar{S}_t - \bar{W}_t, \quad \text{with } S_t \geq 0, C_t > 0,$$

(23)

where $c_s$ is the function of storage costs (cost-of-carry), with $c'_s, c''_s > 0$, $\lambda$ is the function of convenience yield, with $\lambda', \lambda'' > 0$ if $S < S^{eq}$ and $\lambda = 0$ if $S \geq S^{eq}$ (see also Pindyck, 1994), and

$$\bar{y}_t = (\tilde{q}_t + \omega_{t-1} S_{t-1}) \bar{p}_t + (\tilde{p}_t - p'_t) z_{t-1} + (1 + R) W_{t-1}.$$

(24)

In (24), $S_{t-1}$ is storage decided upon in period $t-1$ with $0 < \omega < 1$ as the spoilage coefficient, and $W_{t-1}$ is the amount of wealth in the form of financial and real assets including the commodity invested at the constant rate $R$. Assuming a given output function for the moment, the decision problem that producers face at date 0 is to maximise a strictly concave utility function of total consumption, $C_t$, with respect to the futures position, $z_t$, storage, $S_t$, and risk-free financial and real asset investments, $W_t$. This results in the following first order conditions:

$$E_t[U'_{t+1} p_{t+1}] = p'_t E_t[U'_{t+1}],$$

(25)
\[ U_t^* = RE_t[U_{t+1}^*]. \]

(26)

\[ U_t^* (p_t + c_{st}-\lambda) \geq \omega_t E_t[U_{t+1}^*, p_{t+1}] \text{ if } S_t \geq 0. \]

(27)

Equation (25) is a basic arbitrage equation by showing that the futures position trades off the expected marginal disutility of paying the futures price at \( t+1 \) with the expected utility of obtaining the value of the spot commodity. Equation (26) is a standard Euler equation that compares the (dis-)utility of giving up a dollar today for an additional dollar at \( t+1 \). Equation (27) is similar to equation (22), by comparing the disutility of sacrificing the cost of one unit of the commodity including storage costs with the utility of receiving one unit of the commodity net of spoilage in period \( t+1 \). Combining (25), (26), and (27) gives:

\[ (p_t + c_{st}+\pi^s - \lambda) / \omega_t \geq p_{t+1} / R \text{ if } S_t \geq 0. \]

(28)

Similarly to the arbitrage condition in Section 5.1, equation (28) indicates that futures trading and storage decisions are guided by the ratio of the current futures price to the spot price, adjusting for marginal storage costs and the convenience yield. This condition also illustrates the substitutive character of storage and futures trading by showing that an optimising producer or storage firm will always make a choice between acquiring the good now for storage, thus enjoying a convenience yield against spoilage and cost-of-carry, or buy futures.

We will concentrate on the determinants of the storage and hedging decision. If we assume that the elasticity of demand is equal to one, \( \eta = 1 \), the producer has stable revenues regardless of the production outcome. As we concluded in Section 3, in the absence of storage this should neither lead to backwardation nor to contango as producers have no reason to trade futures for hedging purposes (they might have for
speculative purposes). However, if we assume producers do not store, but storage firms do, total commodities sold by producers, $G$, and storage firms, $H$, reads:

$$G \tilde{q}_t + H (\omega_{t-1} S_{t-1} - \tilde{S}_t) = G \left[ \tilde{q}_t + \left( \frac{H}{G} \right) (\omega_{t-1} S_{t-1} - \tilde{S}_t) \right] = \frac{D_t}{\tilde{p}_t},$$

(29)

with $\omega_{t-1} S_{t-1} - \tilde{S}_t$ representing the revenues of storage firms, and $D_t$ representing overall unhedged revenues. Now producers have an incentive to hedge as their intertemporal revenue stream risks to be distorted by storage firms, who change the effective elasticity of demand. If $S_t = 0$, the producer has stable revenues and there is no incentive to hedge. However, in case $S_t \geq 0$ and assuming storage costs are zero, the producer, who is not able to store himself, can simulate storage by, using (31), investing $(H/G) \omega_t S_t$ now and buying futures $\tilde{z}_t^c = (H/G) \omega_t S_t$. This strategy would stabilise income as long as the futures price is unbiased. If there is no futures price bias, storage firms can eliminate all risk by going short the net of spoilage value of stocks, $z_t^r = -\omega_t S_t$. Producers and storage firms are likely to clear the futures market together, since they have negatively correlated incomes and opposite hedging requirements. The fact that producers have a tendency to go long is supported by the conclusions from Section 3 on the impact of elasticity of demand on hedging and the impact of storage on the consumer demand curve as described in Section 5.1.

Without storage, revenue would be constant. However, with storage, the producer benefits in times of high output, since the price reduction is smaller than in the absence of storage. In other words, demand is made more elastic under storage, $\eta > 1$, and the producer goes long when the price is low to transfer income to times where the price is high and stocks are zero.

As illustrated in Section 5.1, in case of positive storage costs, $\psi_{t-1} (S_t) > 0$, the impact storage of the commodity has on demand elasticity might change. Even if consumer demand elasticity does not change, in periods of high uncertainty the
storage demand elasticity, $\eta_s$, may, due to a significantly positive convenience yield, be $0 < \eta_s < 1$, so that producer’s revenue falls in periods of high output by generating high storage costs against the embedded option in stocks. Thus, at $t=0$ with an expected high output at $t=1$, producers may have tendency to sell futures against their entire production, whilst storage firms will have tendency to go long in futures. Since part of the output will be sold on the market, storage firms will buy fewer futures than producers will sell. Hence, a downward hedging pressure arises, and, certainly when combined with a positive convenience yield, resulting in backwardation. In summary, we can conclude that if storage costs are close to or equal to zero, an ambiguous hedging pressure arises because storage firms change consumer demand elasticity which makes producers hedge long. As soon as storage costs become significantly positive, storage firms will store less and demand elasticity will not change significantly. However, producers will receive less revenue from storage firms because of high storage costs incurred. Thus a net short hedging pressure arises as described in Section 3.

For producers that are able to store, we conclude, unsurprisingly, that production and storage are inversely related. Furthermore, supply response creates hedging incentives that are the reverse of those caused by storage, although this depends on storage versus production costs. To explain this result, as a base case we assume that storage costs and production costs are equal per unit of $q$. If we now assume that $q_1$ is high, storage in period 1 will increase, lowering the expected price and futures price for period 2. A lower price in period 2 reduces $g(q_2)$. However, despite reduced production costs that we assume equal to increased storage costs including spoilage, $y_2$ is likely to decrease compared to $y_1$, due to an increased elasticity of demand, that causes the output decrease to dominate the price increase ($\eta > 1$). As a consequence, in period 1 producers have an incentive to buy futures cheaply to compensate for reduced $y_2$. If storage becomes large, the convenience yield may approach zero, and, combined with the long hedging by producers, contango may arise. This result depends can be offset by an eventual short hedging
pressure resulting from the desire to hedge stocks.

This outcome depends on our basic assumption of equal storage versus production costs of course. Assuming everything else equal, if storage costs significantly exceed production costs, which is likely in a high $q_1$ situation due to the convex properties of $c(q)$ and $c_\gamma$, and a reduced convenience yield, the long hedging incentive will increase since $y_2$ will be reduced even further. If, however, storage costs are insignificant compared to production costs, the elasticity of demand effect might be offset by much lower overall costs, leading to an increase in $y_2$ and thus to a short hedging pressure.

In order to distinguish the varying impact of several production, storage and hedging determinants on the futures and storage position, as well as on the futures price bias, the main results from both this section and previous sections are summarised in Table 2.

6 Futures Hedging, Incomplete Markets, and Uncertainty

As discussed in the foregoing sections, it is only under restrictive assumptions that risk-averse agents can take optimal production or storage decisions as if they have or can have a perfectly diversified portfolio. Therefore, the traditional ‘routine’ hedge, i.e. $z = -\tilde{q}$, is in general suboptimal. As demonstrated in the previous sections this is due to production uncertainty, the correlation between price and output, risk aversion and the impact of storage and consumption elasticities as well as relevant cost functions. Furthermore, although not a necessary condition the models above are all based on the assumption that the futures price is normally distributed, is an unbiased estimate of the future spot price and that there is no ‘basis’ risk, in other words $p_f \equiv p$ at expiry. These assumptions rarely apply in practice. Additional risk from futures contracts may not only be in the price basis, but also in quality, timing and location of the delivered product. In fact a perfect ‘routine’ futures hedge seldom exists, which explains why it is that only a small number of expired futures contracts
actually result in delivery (Stein, 1986). Furthermore, the absence of a perfect hedge make that stochastic separability, where real decisions get separated from hedging decisions, remain an exception. Consequently, many agents adopt a portfolio approach to hedging, whereby a frequently suboptimal risk reduction is achieved through dealing in a variety of assets and asset price contingent contracts.

The fundamental problem to producers not being able to obtain a perfect hedge is that the economy does not have a complete set of markets or, in other words, some assets are nonmarketable (Mayers, 1972). When markets are incomplete, existing markets serve different functions simultaneously and none of them quite satisfactorily. Commodity markets, for example, typically have the function of exchange of goods and exchange of risk. Because markets are incomplete, the intertemporal resource allocation may not be Pareto optimal (Newbery and Stiglitz, 1981).

Townsend (1978) demonstrates that if there are as many linearly independent spot prices as there are states of the world, unconditional contracts as futures contracts have the same spanning property as the classic Arrow Debreu securities. Breeden (1984) argues that as long as continuous and costless trading is possible in unconditional futures, no contingent commodity related contracts as options for example are needed. These results are interesting from a theoretical perspective, but leave open the interesting question why so few futures markets are successful. Analysis of market imperfections enables us to understand part of the answer.

First, some claims are non-marketable due to general information problems and in particular moral hazard and adverse selection. In Mayers’ CAPM, therefore, the risk premium of a security as a futures contract is composed of a term proportional to its covariance with traded assets, and a term proportional to its covariance with nonmarketable risks. This means that although investors may hold identical portfolios of marketable assets, they could have different probability distributions of wealth. An example of a non-marketable claim is the current and future production of a farmer. The risk of moral hazard with farm production explains why many risk sharing and risk spreading initiatives as e.g. crop sharing or
crop insurance by producers fail (Stiglitz, 1974; Hirshleifer and Subrahmanyam, 1993). Furthermore, most farmers cannot diversify out of their own risk by issuing equity because of information problems and set-up costs. Second, futures markets differ from securities (bond, stock) markets in one extremely important way: the production income from the producer is directly correlated with the return on futures contracts. This means automatically that the futures market provides limited income insurance. Furthermore, participation to the futures market, even if it were to provide good income insurance, is not open to all due to set-up and monitoring costs and imperfections as short-selling constraints, for example.
### Table 2: Impact of production, storage, and futures determinants on the futures and storage positions of a representative producer.

<table>
<thead>
<tr>
<th>Impact on</th>
<th>Section 2</th>
<th>Section 3</th>
<th>Section 4</th>
<th>Section 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_t$ (Working)</td>
<td>$\lambda$ (Keynes)</td>
<td>$\pi^*$ (Working)</td>
<td>$\pi^w$ (Working)</td>
<td>$\sigma_q \quad \sigma_p \quad 0 &lt; \eta &lt; 1 \quad \eta = 1 \quad \eta &gt; 1 \quad \bar{p} - p^f \quad A \quad g(q) \quad E[\bar{p}_{t+1}] \quad c(q) \quad c_s^1 \quad 0 &lt; \eta, &lt; 1$</td>
</tr>
<tr>
<td>$z_t$</td>
<td>0</td>
<td>0</td>
<td>&lt; 0</td>
<td>0</td>
</tr>
<tr>
<td>$S_t$</td>
<td>0</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>$\bar{p} - p^f$</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

"-" means a negative impact, "+" means a positive impact, "< 0" implies a short position, "0" implies a zero position or zero impact, "> 0" implies a long position. "n.a." means result not applicable or not available. $^1$ assumes $S > 0$. 

---

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The Theory of Normal Backwardation as discussed in Section 2 implicitly assumes that there is no other way for producers to diversify risk. Because claims on the producer’s profits are nonmarketable, a net short hedging pressure arises which thus leads to backwardation. Dusak (1973) and Grauer and Litzenberger (1979), however, follow the approach that all claims are marketable and that portfolios can be freely diversified. In line with the Capital Asset Pricing Model, in their models the futures risk premium is a function of nondiversifiable risk rather than hedging pressure. Thus, the futures risk premium depends on the covariance between futures prices and changes in economic state variables. Hedging pressure as such does not play a role. Because the Theory of Normal Backwardation and the CAPM approach are both based on risk aversion and have some empirical foundation, Stoll (1979) and Hirshleifer (1988b, 1989a) integrate the Theory of Normal Backwardation with the CAPM approach.

Hirshleifer (1989a) allows for hedging pressure by assuming nonmarketability of production risk as well as through fixed set-up costs that exclude certain producers from participating in the futures market. His representative farmer trades on the futures market with speculators, but both producers and speculators can simultaneously trade a public stock market portfolio, $M$. The constant risk-averse producer has the following consumption constraint whilst maximising utility from consumption in a mean-variance setting:

$$
\tilde{C} = \begin{cases} 
W - c_f + (\tilde{p} - c)\tilde{q} + (\tilde{p} - p^\prime)z + M\tilde{R}_m, z \neq 0 \\
W + (\tilde{p} - c)\tilde{q} + M\tilde{R}_m, z = 0 
\end{cases}
$$

(30)

where $c_f$ are the fixed set-up costs for trading futures, $c$ are production costs, and $\tilde{R}_m$ is the return on the stock market portfolio, $M$. According to (30), all producers face a non-marketability problem due to $\tilde{q}$ regardless whether they trade futures or are excluded from the futures market through $c$.  

Let the futures risk premium be $\pi^f = E[\tilde{p} - p']$ and let $\pi^f = \tilde{p} - p'$.

Now, the expected utility of a representative producer is maximised over futures and the stock portfolio. The first order conditions are, assuming $z \neq 0$:

\[
\pi^f = ACov(\tilde{\pi}^f, \tilde{C})
\]

(31)

\[
\bar{R}_m = Acov(\tilde{R}_m, \tilde{C})
\]

(32)

with $A$ being the coefficient of absolute risk aversion. As the futures position is included in (32) through $\tilde{C}$, we can solve for optimal $z$ using the consumption constraint (30):

\[
z = \frac{\pi^f - \text{cov}(\tilde{\pi}^f, (\tilde{p} - c)\tilde{q} + M\tilde{R}_m)}{\text{var}(\tilde{\pi}^f)}
\]

(33)

To analyse the futures position for speculators in this model to come to a supply-demand balance of futures, one assumes in (33) that $q = 0$. If the spot price is not correlated with the stock market return, the covariance term vanishes. As was seen in Section 3, speculators only trade futures if the risk premium $\pi^f \neq 0$. If however the spot price is positively correlated with the stock market return and $M > 0$, a short futures position is taken to diversify movements from the stock market. So whilst for speculators the position is determined by the risk premium and the correlation between the spot price and the stock market return, for the producer the covariance of total revenues from production and the stock market and the risk premium influence the futures position. For speculators and producers, solving for the risk premium gives:
\[ \pi' = A\bar{M}\text{cov}(\tilde{R}_m, \tilde{p}) + A\text{cov}(\tilde{p} - p', \tilde{p}) \]

where the first term relates to the premium for speculators and the second to producers. Thus, the futures risk premium will be determined by the covariance of the futures contract payoff with the return on the stock market portfolio and with net revenues from production. Whilst the first component can be seen as a typical CAPM component, through production revenues the second component brings in the hedging pressure as a determinant of futures risk premia. First empirical tests of (34) have been encouraging but fail to rule out the existence of hedging pressure. Hence, despite additional diversification opportunities through the stock market, the non-marketability problem remains.\textsuperscript{14}

In response to nonmarketability, several authors have identified opportunities for perfect diversification through use of various unconditional and contingent claims in addition to futures contracts. To start with commodity related claims, when dealing in a good for which no futures contract exists or production risk is substantial, a cross hedge with a related futures contract or a portfolio of related futures contracts may be appropriate (Anderson and Danthine, 1981). Furthermore, a few authors suggest the use of forward contracts in addition to futures contracts (e.g. Katz, 1984; Paroush and Wolf, 1986), whilst Moschini and Lapan (1992, 1995) suggest the use of commodity options in combination with futures. For most contributions that suggest hedging instruments other than the commodity-related futures or forward contracts, a few comments apply. First, it is unclear why more ‘distant’ diversification opportunities are better accessible than direct futures hedging opportunities and can thus resolve the matter of set-up costs and information costs. Second, the stability of moments is fundamental to the effectiveness of any hedge, and the stability of moments of a cross diversification opportunity is generally less than a directly related hedge, thereby only increasing rather than reducing basis risk. Third, even if the risk premium is

\textsuperscript{14} See Bessembinder (1992, 1993), Kamara (1993), and Roon, Nijman, and Veld (2000).
significant, linear instruments as futures may still be preferred to options because the former do not generate monitoring costs nor do they leave an element of uncertainty in the producer’s income (Battermann et al., 2000).

Other models assume production is a marketable asset, by allowing for issuance of publicly traded stocks (e.g. Black, 1976; Marcus and Modest, 1984). Black (1976) goes as far to say that the success of stock markets shows that for many commodities futures contracts are not needed as a means of transferring risk. Furthermore, share contracting, where processors take a stake in the primary production process with selected farmers, still occurs for a few commodities, especially in less developed countries where equity issuance is prohibitively expensive. Apparently, monitoring mechanisms and contractual arrangements can overcome market imperfections although at a substantial premium (Stiglitz, 1974; Hirshleifer and Subrahmanyam, 1993). An alternative that also appears regularly in the literature is vertical integration of activities as production and storage as means to diversify risk (for example Hirshleifer, 1989b). A more innovative example is presented by Ho (1984), who suggests the creation of tradable output indices to allow farmers to hedge their production risk separately from their price risk. Although these alternatives are theoretically attractive, their practical applicability remains limited.

7 Summary and Conclusions
In this paper we analysed the main literature on the microeconomics of production, storage, and hedging of commodity quantities and prices with the aim to better understand commodity price movements and forward price structures.

A review of futures markets literature is incomplete without a description of the classical dichotomy between the theory of normal backwardation and the theory of supply of storage as alternative explanations of the futures basis. A discussed these theories are also interrelated through the phenomenon of ‘carrying charge’ hedging. Seasonally driven changes in inventories as well as changes in supply and demand will determine which model predominates. Whilst the theory of supply of storage, through the absence of price expectations, assumes stable supply-demand conditions,
the theory of normal backwardation incorporates investor anticipation of changes in market conditions and market imperfections as e.g. short-selling constraints, but excludes the embedded timing option in the spot commodity’s price. Incorporation of both theories in a standard optimisation framework permits analysis of interaction of the various determinants for storage, production, and hedging and their impact on the futures price curve.

Therefore different motives to trade and the futures price bias were further explored by integration of the futures hedging and the production decision. Futures markets play a role in production planning due to their hedging function and their price information function. However, it was shown that in most cases producers cannot hedge price and output risk simultaneously, and they are therefore unlikely to hedge their entire production through a routine futures hedge. Production risk is nontradable on commodity futures markets.

Classical and modern contributions on futures markets have paid little attention to the influence of storage decisions on futures trading and on the forward price curve. Therefore, key contributions to the theory of storage were reviewed. Whilst storage is seen as a source of spot and futures price stabilisation, it is also considered as a main source of variability for planned production levels through the responsive supply mechanism. Storage and futures are substitutive in that they provide alternative methods for reducing income and price variability. The substitutive characteristics are underlined by first theoretical results on the joint impact of elasticities of storage, responsive supply, and of market demand. With incorporation of an endogenous convenience yield, it was demonstrated that these elasticities have an ambiguous impact on the net hedging pressure and the resulting futures curve.

Since users of futures generally face a non-marketable asset problem in that they cannot insure against output risks due to information asymmetries, moral hazard, set-up and monitoring costs, they operate in a system of incomplete markets in which the traditional ‘routine’ futures hedge is suboptimal or nonexistent. Consequently, the risk premia on futures contracts are composed of a systematic risk component based on its covariance with economic state variables and a hedging pressure component.
Although the literature suggests various opportunities for diversification, many opportunities have similar information issues and access restriction as futures markets have.

The literature as reviewed in this paper focuses heavily on the use of futures by producers and their impact on price formation, whilst in reality storage firms, processing firms, and financial institutions play an important role on these markets as well. Further theoretical research on the role of these agents and the interaction of storage, production, and hedging is necessary to complete the theoretical framework. Furthermore, numerous empirical papers are devoted to testing the theories of normal backwardation and the theory of storage as alternative theories. More attention needs to be paid to the role of varying inventories and time-varying, endogenous convenience yields as determinants of commodity price formation.
Appendix A

List of Symbols

\[ E[...] \text{ or } E \] Expectations operator
\[ U(...) \] Utility operator
\[ \cdot \gamma \text{ (or: } \cdot \gamma_{t-1,t+1}\text{)} \] Period \( t \) value of variable (or as subscript: previous value or next periods’ value of variable)
\[ T \text{ (or: } \cdot \tau \text{)} \] Terminal period of model (or as subscript: terminal value or expiry date in case of futures contract)
\[ p^f \] Price of a commodity futures contract
\[ p \] ‘Spot’ or ‘cash’ price of a commodity
\[ q \] Output of a commodity
\[ z \] Commodity futures contract
\[ \tilde{x} \] Stochastic outcome of variable ‘\( x \)’
\[ \bar{x} \] Statistical mean of variable ‘\( x \)’
\[ \pi^z \] Risk premium on futures contracts
\[ c_s \] Cost-of-carry or cost of storage
\[ \pi^x \] Risk premium on (excess) stocks
\[ \lambda \] Convenience yield
\[ A \] Arrow-Pratt risk aversion coefficient
\[ y \] Income
\[ x \sim N(\bar{x}, \sigma^2) \] Variable is normally distributed with mean \( \bar{x} \) and variance \( \sigma^2 \)
\[ \hat{y} \] Utility certainty equivalent of income
\( \theta \quad \text{Multiplicative risk factor} \)

\( \sigma_x \quad \text{Coefficient of variation or standard error of variable 'x'} \)

\( \rho \quad \text{Correlation coefficient} \)

\( \eta \quad \text{Elasticity of demand} \)

\( \Pi \quad \text{Profit} \)

\( g(\cdot) \quad \text{Production function} \)

\( c(q) \quad \text{Cost of production} \)

\( \omega \quad \text{Spoilage factor} \)

\( \eta_s \quad \text{Elasticity of storage demand} \)

\( R_m \quad \text{Return on stock market portfolio} \)

\( R \quad \text{Risk-free return on wealth} \)

\( W \quad \text{Total wealth} \)

\( Q \quad \text{Availability of commodities} \)

\( Q^{\text{eq}} \quad \text{Equilibrium availability} \)

\( S \quad \text{Commodities in storage, stocks} \)

\( S^{\text{eq}} \quad \text{Commodities in stocks at market equilibrium} \)

\( M \quad \text{Stock market portfolio} \)

\( D \quad \text{Revenues} \)

\( G \quad \text{Number of identical producers} \)

\( H \quad \text{Number of identical storage and trading firms} \)

\( z^G \quad \text{and} \quad z^H \quad \text{Futures contracts positions held by producers and storers} \)
respectively

\( c_f \)  
Cost of access to and monitoring of futures trading

\( \text{var}(..) \)  
Variance of variable or product of variables

\( \text{cov}(x, y) \) or \( \sigma_{xy} \)  
Covariance of variables ‘x’ and ‘y’

\( \max_x y \)  
Maximisation operator of \( y \) over \( x \)
Appendix B (to Section 3)

Equation (10) is obtained through the moments of \( p \) and \( \theta \). Aitchison and Brown (1957) describe properties of jointly normally distributed variables. Assume \( x \) and \( y \) are normally distributed with standard deviations \( \sigma_1, \sigma_2 \) and correlation coefficient, \( \rho \):

\[
x, y \sim N(0,0, \sigma_1^2, \sigma_2^2, \rho)
\]

The moment generating function is as follows:

\[
M(t_1, t_2) = \exp\left\{ \frac{1}{2} (\sigma_1^2 t_1^2 + 2 \rho \sigma_1 \sigma_2 t_1 t_2 + \sigma_2^2 t_2^2) \right\}, \tag{B1}
\]

and its moments are determined by:

\[
\mu_{rs} = \mathbb{E} x^r y^s = \text{coefficient of } \frac{t_1^r t_2^s}{r!s!} \tag{B2}
\]

in the expansion of the mean generating function. Therefore:

\[
\mu_{11} = \rho \sigma_1 \sigma_2 \tag{B3}
\]

\[
\mu_{12} = (1 + 2 \rho^2) \sigma_1^2 \sigma_2^2 \tag{B4}
\]

\[
\text{cov}(p, p \theta) = \bar{p}^2 (\sigma_1^2 + \rho \sigma_1 \sigma_2) \tag{B5}
\]
Appendix C (to Section 4.1)

Kamara (1993) formulates a three period model of a producer optimising production and futures trading under production certainty in a dynamic setting. The producer is risk averse and hence maximises expected value of a concave utility function. Let $E_t[U(\pi)]$ denote the expected utility conditional on the information available on date $t$. The profit function reads:

\[ \pi = \tilde{p} g(q_1, q_2) - c(q_1, q_2) + z_1(\tilde{p} - p_1^f) + z_2(\tilde{p} - p_2^f). \]  

(C1)

In equation (C1), $g(q_1, q_2)$ is the non-stochastic production function for output available on date 3, and $c(q_1, q_2)$ is the factor pricing function with $q_1$ and $q_2$ the levels of input at dates 1 and 2 respectively. $z_1$ and $z_2$ are futures contract positions for maturity on date 3 entered into on dates 1 and 2 at prices $p_1^f$ and $p_2^f$. $\tilde{p}$ represents the uncertain spot price at date 3.

The production function is said to be intertemporally separable if $g_1(q_1, q_2)$, the partial derivative of $g(\cdot)$ with respect to $q_1$, is independent of $q_2$ and $c_1(q_1, q_2)$, the partial derivative of $c(\cdot)$ with respect to $q_1$, is independent of $q_2$. If this intertemporal separability exists and there exists a futures contract with time to maturity that matches the production process, the optimal production rule is:

\[ c_1(q_1, q_2) = p_1^f g_1(q_1, q_2), \]

and

\[ c_2(q_1, q_2) = p_2^f g_2(q_1, q_2). \]  

(C2)

To achieve this result, the producer goes through a recursive optimisation, and starts with the optimal level of input on date 2. The first order conditions on date 2 are:
\[ c_s(q_1, q_2)E_2[U'(\pi)] = g_s(q_1, q_2)E_2[U'(\pi)\bar{p}] \]  
\hfill (C3)

and

\[ p_2' E_2[U'(\pi)] = E_2[U'(\pi)\bar{p}]. \]  
\hfill (C4)

Consequently,

\[ c_s(q_1, q_2) = p_2' g_s(q_1, q_2). \]  
\hfill (C5)

The optimal input level on date 2, \( q_2^* \), is a function of \( q_1 \) and \( p_2' \), and the optimal futures position on date 2, \( z_2^* \), is a function of \( q_1 \), \( z_1 \), and \( p_2' \). If we denote these functions by \( q_2^* = A(q_1, p_2') \) and \( z_2^* = B(q_1, z_1, p_2') \) and let the producer choose the levels of \( q_1 \) and \( z_1 \) that maximise \( E_1[U(\pi)] \), given \( q_2^* \) and \( z_2^* \), the first order conditions with respect to \( q_1 \) and \( z_1 \) yield:
\[ E_i \{ U'(\pi) \left[ \tilde{p} g_i(q_1, q_2) - c_i(q_1, q_2) \right] \} = -E_i \{ U'(\pi) B_i(q_1, z_i, p_i^\ell) (\tilde{p} - p_i^\ell) \}. \] 

(C6)

\[ -E_i \left\{ U'(\pi) A_i(q_1, p_i^\ell) \left[ \tilde{p} g_2(q_1, q_2) - c_2(q_1, q_2) \right] \right\} \]

and

\[ E_i \left\{ U'(\pi) (p_i^\ell - p_i^1) \right\} = -E_i \left\{ U'(\pi) B_i(q_1, z_i, p_i^\ell) (\tilde{p} - p_i^\ell) \right\}. \] 

(C7)

But

\[ E_i \left\{ U'(\pi) B_i(q_1, z_i, p_i^\ell) (\tilde{p} - p_i^\ell) \right\} = E_i \{ B_i(q_1, z_i, p_i^\ell) E_2 \left\{ U'(\pi) (\tilde{p} - p_i^\ell) \right\} \} = 0, \] 

(C8)

for \( i = 1, 2 \). The first equality follows from the fact that \( B_i(q_1, z_i, p_i^\ell) \) is nonstochastic at date 2 and from the properties of the marginal and conditional distributions. The second equality follows from (C4). Likewise, because \( A_i(q_1, p_i^\ell) \) is nonstochastic on date 2, using (C9) yields:

\[ E_i \{ U'(\pi) A_i(q_1, p_i^\ell) \left[ \tilde{p} g_2(q_1, q_2) - c_2(q_1, q_2) \right] \} = 0. \] 

(C9)

Hence, (C6) and (C7) become:

\[ E_i \{ U'(\pi) c_i(q_1, q_2) \} = E_i \{ U'(\pi) g_i(q_1, q_2) \tilde{p} \}, \] 

(C10)

and

\[ \tilde{p} E_i \{ U'(\pi) \} = E_i \{ U'(\pi) p_i^\ell \}. \] 

(C11)
When the production function and factor pricing functions are intertemporally separable, \( g_1(q_1, q_2) \) and \( c_1(q_1, q_2) \) are nonrandom on date 1. Equation (C9) then becomes

\[
c_1(q_1, q_2) E_1[U'(\pi)] = g_1(q_1, q_2) E_1[U'(\pi) \bar{p}] = g_1(q_1, q_2) E_1[U'(\pi) p_2^f],
\]

where the second equality uses \( E_1[U'(\pi) \bar{p}] = E_1\{E_2[U'(\pi) \bar{p}]\} \) and equation (C4). From (C11) it follows that \( g_1(q_1, q_2) = p_1^f g_1(q_1, q_2) \).

So, despite the initial uncertainty, stochastic separability is maintained in the dynamic model as long as the marginal productivity of the production decision at each intermediate stage is deterministic. Note that the same results are obtained if only one-period futures contracts would exist and the relationship between \( p_1^f \) and \( p_2^f \) would be exactly linear. This condition is fulfilled as long as interest rates are assumed non-stochastic and the convenience yield is non-existent or constant. In case of one-period futures contracts, the result on \( z_1 \) would not lead to a ‘mark-to-market’ settlement or ‘margin call’ with the futures clearing house at date 2, but to final cash settlement of equal size. The futures position could be rolled from stage 2 to stage 3.

As soon as the intertemporal separability falls apart and the production and factor pricing functions become intertemporally complementary, in other words \( g_1(q_1, q_2) \) and \( c_1(q_1, q_2) \) depend on \( q_2 \), the output decision at date 1 and date 2 would depend on the producer’s risk preferences and expectations. By substituting (C3) into (C4) the production decision on date 1 would become:

\[
E_1[U'(\pi)c_1(q_1, q_2)] = E_1[U'(\pi)g_1(q_1, q_2)p_2^f].
\]

### Appendix D (to Section 4.2)
Anderson and Danthine (1983a) derive the optimal production decision, assuming the producer takes the optimal futures decision, by inserting equation (20) into (19) to get:

\[
E(\tilde{p} \tilde{g}_1) + \frac{(p' - E\tilde{p}) \text{cov}(\tilde{p}, \tilde{p} \tilde{g}_1)}{\text{var } p} - c'(\tilde{q}) = \frac{A[\text{cov}(\tilde{p} \tilde{g}_1, \tilde{p} \tilde{g}_1) \text{var } \tilde{p} - \text{cov}(\tilde{p}, \tilde{p} \tilde{g}_1) \text{cov}(\tilde{p}, \tilde{p} \tilde{g}_1)]}{\text{var } \tilde{p}} = 0 \tag{D1}
\]

In contrast with the stochastic separability case as presented in Section 4.1, equation (20) shows that under output uncertainty the producer’s risk aversion starts to play a role in the production decision (with the final term on the left-hand side of (20) as a risk premium). Furthermore, the expectations of the futures spot price and the non-diversified output risk matter as well. Thus, the production decision is no longer independent of risk preferences and the expected spot price and there is no longer a separation between the output decision and hedging decision as was the case under a deterministic output function as described in 4.1. Clearly, the complexity of the optimal production decision is enhanced due to the interaction of random production and price.

Anderson and Danthine (1983a) demonstrate that the presence of a futures market increases production compared to the situation where there is no futures market. We repeat equation (19) that describes optimal \( q \) given optimal \( z \):

\[
E(\tilde{p} \tilde{g}_1) - c'(\tilde{q}) - A[\text{cov}(\tilde{p} \tilde{g}_1, \tilde{p} \tilde{g}_1) - z \text{cov}(\tilde{p}, \tilde{p} \tilde{g}_1)] = 0. \tag{19'}
\]

To see how a futures market impacts on \( q \) we let the left hand side of (19’) be denoted as \( \phi_q \).

We differentiate \( \phi_q \) with respect to \( q \) and \( z \) to obtain:

\[
\frac{\partial \phi_q}{\partial \tilde{q}} = \frac{-\phi_{\tilde{q}q}}{\phi_{qq}} \tag{D2}
\]

\[
\frac{\partial \phi_q}{\partial \tilde{z}} = \frac{-\phi_{\tilde{z}q}}{\phi_{qq}}
\]
where $\varphi_{qq} = A \text{cov}(\tilde{p}, \tilde{p}\tilde{g})$ and where $\varphi_{qq} < 0$. The $\text{cov}(\tilde{p}, \tilde{p}\tilde{g})$ can have either sign of course (as demonstrated in Section 3), but if we assume a negative covariance, $\frac{\partial q}{\partial \omega} > 0$, i.e. output increases with an increase in futures.
References


