First-order transitions for n-vector models in two and more dimensions
Enter, Aernout C.D. van; Shlosman, Senya B.

Published in:
Physical Review Letters

DOI:
10.1103/PhysRevLett.89.285702

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2002

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
https://doi.org/10.1103/PhysRevLett.89.285702

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment.

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Download date: 27-09-2023
First-Order Transitions for $n$-Vector Models in Two and More Dimensions: Rigorous Proof

Aernout C. D. van Enter

Institute for Theoretical Physics, Rijksuniversiteit Groningen, P.O. Box 800, 9747 AG Groningen, The Netherlands

Senya B. Shlosman

CPT, CNRS Luminy, Case 907, F13288 Marseille, Cedex 9 France

(Received 22 May 2002; published 30 December 2002)

We prove that various SO(n)-invariant $n$-vector models with interactions which have a deep and narrow enough minimum have a first-order transition in the temperature. The result holds in dimensions two or more and is independent of the nature of the low-temperature phase.

DOI: 10.1103/PhysRevLett.89.285702 PACS numbers: 64.60.Cn, 05.50.+q, 75.10.Hk

Recently Bl"ote, Guo, and Hilhorst [1], extending earlier work by Domany, Schick, and Swendsen [2] on two-dimensional classical $XY$ models, performed a numerical study of two-dimensional $n$-vector models with nonlinear interactions. For sufficiently strong values of the nonlinearity, they found the presence of a first-order transition in temperature. In [2] a heuristic explanation of this first-order behavior, based on a similarity with the high-$q$ Potts model, was suggested, explaining the numerical results. A further confirmation of this transition was found by Caracciolo and Pelissetto [3], who considered the $n \to \infty$ (spherical limit) of the model and found the same first-order transition.

On the other hand, various studies, mostly based on renormalization-group analyses or Kosterlitz-Thouless-type arguments based on the picture of binding/unbinding of vortices, have contested this first-order behavior and/or the Potts model analogy (e.g., [4–6]).

Here we settle the issue by presenting a rigorous proof of the existence of this first-order transition. It may seem somewhat surprising that two-dimensional $n$-vector models, whose magnetization by the Mermin-Wagner theorem [7] is always zero, can have such a phase transition. The reason is that the transition we are talking about here is manifested by the long-range order in higher-order correlation functions. Such transitions were discovered by one of us some time ago; see [8]. But the results of [8] were related to the fact that there the symmetry group was the (disconnected) group $O(2)$, and at the transition point the symmetry group was that of the bond configurations, and $\langle \cdot \rangle^\beta$, at $\beta = \beta_c$, corresponding to the Hamiltonian (1).

H = -J \sum_{(i,j) \in \mathbb{Z}^2} \left( 1 + \cos(\phi_i - \phi_j) \right)^p \tag{1}

To formulate our result we have to introduce for every nn bond $b = (i, j)$ the following bond observables:

\[ P_b^c(\phi_i, \phi_j) = \begin{cases} 1 & \text{if } |\phi_i - \phi_j| < \varepsilon/2, \\ 0 & \text{if } |\phi_i - \phi_j| > \varepsilon/2, \end{cases} \tag{2} \]

which project on the ordered bond configurations, and $P_b^\gamma(\phi_i, \phi_j) = 1 - P_b^c(\phi_i, \phi_j)$. Our main result is contained in the following:

Theorem 1: Suppose the parameter $p$ is large enough. Then there exists a transition temperature $\beta_c = \beta_c(J, p)$, such that there are two different Gibbs states, $\langle \cdot \rangle^c$ and $\langle \cdot \rangle^\gamma$, at $\beta = \beta_c$, corresponding to the Hamiltonian (1).
For some specific choice of $e = e(p)$ in (2), we have for the “ordered” state $\langle \gamma \rangle^<$ that
$$
\langle P^< \rangle^< > \kappa(p),
$$
while in the “disordered” phase $\langle \gamma \rangle^>$
$$
\langle P^> \rangle^> > \kappa(p),
$$
for each bond $b$, with $\kappa(p) \to 1$ as $p \to \infty$.

Before analyzing the model (1), we present an even simpler toy model, which already displays the mechanism, and which is even closer to the Potts model. The single-spin space is the circle, $S^1$; the free measure is the Lebesgue measure, normalized such that $S^1$ has measure one, so $S^1 = [-1,1]$. (One can take here any sphere $\mathbb{S}^n$ instead.) The toy Hamiltonian is
$$
H = -J \sum_{\langle i,j \rangle \in \mathbb{Z}^d} U(\phi_i, \phi_j).
$$
(3)
The rotation-invariant nearest-neighbor interaction $U(\phi_1, \phi_2) = U(|\phi_1 - \phi_2|)$ is given by
$$
U(\phi) = \begin{cases} -1 & \text{if } |\phi| \leq \frac{\xi}{2}, \\ 0 & \text{otherwise.} \end{cases}
$$
Here $e$ plays a similar role to $\frac{1}{2}$ in the $q$-state Potts model. The Hamiltonian is RP under reflections in coordinate planes. For the case $\mathbb{Z}^2$ one has also RP under reflections in lines at $45^\circ$, passing through the lattice sites. That case is the easiest.

One has to show that $\langle P^< \rangle^< \beta$ is small for large $\beta$ (the ordered, typical low-temperature phase bonds); $\langle P^< \rangle^> \beta$ is small for small $\beta$ (the disordered, typical high-temperature-phase bonds); $\langle P^< \rangle^> \beta$ is small for all $\beta$, provided $\beta$ is small enough. Here $\langle \gamma \rangle^<\beta$ is the state with periodic boundary conditions in the box $\Lambda$ of size $L$, and $b', b''$ are two orthogonal bonds sharing the same site. The estimates have to be uniform in $L$, for $L$ large. The first two are straightforward applications of RP and the chessboard estimate. So let us get the last one. By the chessboard estimate,
$$
\langle P^< \rangle^< \beta \leq \langle P \rangle^< |\Lambda|^1/|\Lambda|,
$$
where the observable $P$ is the indicator (of the “universal contour”)
$$
P = \prod_{b \in E_{01}} P^< \prod_{b \in E_{23}} P^>. 
$$
Here $E_{01}$, $E_{23}$ is the partition of all the bonds in $\Lambda$ into two halves; $E_{01}$ consists of all bonds $(x, x + e_1)$ and $(x, x + e_2)$, for which $x^2 + x^2 = 0$ or $1 (\mod 4)$, while $E_{23}$ is the other half; $e_1$ and $e_2$ are the two coordinate vectors.

To proceed with the estimate (4) we need the estimate on the partition function. We have
$$
Z(\beta, e) \geq \left( \frac{e}{2} \right)^{|\Lambda|} e^{2\beta |\Lambda|} + (1 - 4e)^{|\Lambda|/2}.
$$
(5)
(The first summand is obtained by integrating over all configurations $\phi$, such that $|\phi_x| \leq \frac{\xi}{2}$ for all $x \in \Lambda$. For the second one we take all configurations $\phi$ which are arbitrary on the even sublattice and which satisfy $|\phi_x - \phi_y| > e/2$ for every pair of nn, for every $y$ on the odd sublattice that leaves the spins to be free in a set of measure $\geq 1 - 4e$.) Solving
$$
\left( \frac{e}{2} \right)^{|\Lambda|} e^{2\beta |\Lambda|} = (1 - 4e)^{|\Lambda|/2}
$$
for $\beta$, we find
$$
e^{4\beta} = (1 - 4e) \left( \frac{e}{2} \right)^{-2},
$$
so for $\beta \geq \beta_0$ the first term in (5) dominates, while for $\beta \leq \beta_0$ the second term dominates. Similarly, the partition function $Z(\beta, e)$, taken over all configurations $\phi$ with $P(\phi) = 1$, satisfies
$$
Z(\beta, e) \leq e^{\beta |\Lambda|} e^{\frac{3}{4}|\Lambda|} + O(\sqrt{|\Lambda|}).
$$
(7)

If $\beta \geq \beta_0$, we write, using (6),
$$
\langle P^< \rangle^< \beta \leq \frac{e^{\beta |\Lambda|}}{\sqrt{e}} = \frac{2}{\sqrt{e}} \frac{1}{e^{1/2} e^\beta} \leq \frac{1}{[e - (1 - 4e)(\frac{\xi}{2})^{-2}]^{1/4}} \leq C e^{1/4}.
$$

If $\beta \leq \beta_0$, similarly have
$$
\langle P^< \rangle^< \beta \leq \frac{e^{\beta |\Lambda|}}{\sqrt{e}} \leq \frac{1}{[e - (1 - 4e)(\frac{\xi}{2})^{-2}]^{1/4}} \leq C e^{1/4}.
$$
So we are done.

For the nonlinear models, we employ the fact that for small difference angles $\cos(\phi_i - \phi_j)$ is approximately $1 - O((\phi_i - \phi_j)^2)$ and furthermore that $\lim_{p \to 1}(1 - \frac{1}{p})^p = e^{-1}$. This suggests to choose $e(p) = 1/\sqrt{p}$.

Because the separation between ordered and disordered bonds is somewhat arbitrary, to obtain an inequality similar to (5) we make a slightly different choice. We consider a bond $(i, j)$ disordered if $|\phi_i - \phi_j| \leq C/\sqrt{p}$ for some large $C$. So first we choose a sufficiently large constant $C$. For the estimate of the ordered partition function we integrate only over the much smaller intervals of “strongly ordered” configurations: $|\phi_i| \leq C^{-1}/\sqrt{p}$ to obtain a lower bound:
$$
Z(\beta, p) \geq \left( \frac{1}{C \sqrt{p}} \right)^{|\Lambda|} e^{[2(1 - O(1/C^2))]|\Lambda|} + \left( 1 - \frac{4C}{\sqrt{p}} \right)^{|\Lambda|/2}.
$$
(8)

This makes use of the fact that the strongly ordered bonds all have energy almost equal to $-J$, whereas the disordered partition function is bounded by that of the toy model, but with $e$ replaced by $C/\sqrt{p}$. 

For the estimate which shows that ordered and disordered bonds tend not to neighbor each other, we obtain

\[ Z(\beta, p) \leq e^{\beta|\lambda| (1 + O(e^{-C}))} |C/\sqrt{p}| (3/4) |\lambda| + O(1/\lambda). \quad (9) \]

The rest of the argument is essentially unchanged. We first choose \( C \) big enough (such that \( 1/C \) is small with respect to 1), and we can still choose \( p \) big enough for the argument to go through.

We have not tried to minimize the value of \( p \) for which our proof works. Experience with the high-\( q \)-state Potts model suggests that, even if we tried, we would still be rather far off the actual value where the first-order transition appears.

To summarize, we have proved the existence of first-order transitions for a wide class of nonlinear vector models. An important consequence of this result is that the occurrence of such first-order transitions for sufficiently steep and narrow interactions limits the validity of strong universality claims which would suggest that knowing the symmetry and the dimension of the interaction suffices for determining the order of the transition.

After submitting this Letter, we learned that a similar result was obtained by Chayes [12].

---

[7] The Mermin-Wagner theorem, which excludes the breaking of a continuous symmetry in two dimensions at finite temperatures, was first proved in N.D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966). Since then various more precise and more general versions have been proved; for a recent result, see, e.g., D. Ioffe, S.B. Shlosman, and Y. Velenik, Commun. Math. Phys. 226, 433 (2002). The fact that the zero magnetism which is enforced by the Mermin-Wagner theorem is compatible with various types of phase transitions was noted, for example, by J.M. Kosterlitz and D.J. Thouless J. Phys. C 6, 1181 (1973); J.M. Kosterlitz, J. Phys. C 7, 1046 (1974); V.L. Berezinskii, Sov. Phys. JETP 32, 493 (1971), and also in [8].
[10] In a series of papers, A. Patrascioiu and E. Seiler, see, e.g., Phys. Lett. B 430, 314 (1998), have claimed that a Kosterlitz-Thouless phase occurs at low temperatures in \( n \)-vector models also with \( n \geq 3 \). This claim is not widely believed, but the question is still open.
[11] Reflection positivity is by now a fairly standard technique that was introduced into statistical mechanics from field theory to prove the existence of various types of phase transitions in a series of papers by F.J. Dyson, J. Fröhlich, R.B. Israel, E.H. Lieb, B. Simon, and T. Spencer. The application to Potts models was performed by R. Kotecký and S.B. Shlosman, Commun. Math. Phys. 83, 493 (1982). The method is reviewed, for example, in the last four chapters of H.-O. Georgii, Gibbs Measures and Phase Transitions (de Gruter, Berlin, New York, 1988) and in S.B. Shlosman, Russ. Math. Surveys 41, 83 (1986). Our proof directly follows this last source. One can think of the method here as a kind of generalized, symmetrized Peierls contour argument, where the symmetrization requires the RP property.