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πK scattering in full QCD with domain-wall valence quarks

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We calculate the $\pi^+K^+$ scattering length in fully-dynamical lattice QCD with domain-wall valence quarks on MILC lattices with rooted staggered sea-quarks at a lattice spacing of $b = 0.125$ fm, lattice spatial size of $L = 2.5$ fm and at pion masses of $m_\pi \sim 290, 350, 490$ and 600 MeV. The lattice data, analyzed at next-to-leading order in chiral perturbation theory, allows an extraction of the full $\pi K$ scattering amplitude at threshold. Extrapolating to the physical point gives $m_{\pi d_{3/2}} = -0.0574 \pm 0.0016 - 0.0024$ and $m_{\pi d_{1/2}} = 0.1725 \pm 0.0017 - 0.0152$ for the $I = 3/2$ and $I = 1/2$ scattering lengths, respectively, where the first error is statistical and the second error is an estimate of the systematic due to truncation of the chiral expansion.

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I. INTRODUCTION

In hadronic atoms, nature has provided a relatively clean environment in which to explore the low-energy interactions of charged hadrons. The electromagnetic interaction allows for oppositely-charged, long-lived hadrons to form Coulomb bound states. The locations of the energy levels of these systems are perturbed by the strong interactions, while the lifetimes of the ground states are dictated by the strong interactions that couple the charged hadrons to lighter neutral ones.

Theoretically, the simplest hadronic atom to understand is composed of two pions: $\pi^+\pi^-$. Precision experiments have been performed—and are ongoing—to measure the lifetimes and energy levels of such atoms [1]. In the isospin limit, Bose statistics dictates that two pions interacting in an $s$-wave can be in either an isospin-0 or isospin-2 state. By measuring the decay width and energy levels of pionium, the $I = 0$ and $I = 2$ strong-interaction scattering lengths can be isolated. While the difference between energy levels, and hence deviations from the Coulomb spectrum, are relatively straightforward to measure, it is somewhat more challenging to determine the lifetime of these atoms. Recently, the DIRAC collaboration [1] at CERN has measured the lifetime to be $\tau_{\pi^+\pi^-} = 2.91^{+0.49}_{-0.42} \times 10^{-15}$ s, with the dominant decay mode being $\pi^+\pi^- \rightarrow 2\pi^0$. On the theoretical side, progress in lattice QCD has been quite rapid, with a recent fully-dynamical calculation of the $I = 2$ $\pi\pi$ scattering length at pion masses between $m_\pi \sim 290$ MeV and 500 MeV [2]. When combined with two-flavor chiral perturbation theory, a prediction of the scattering length at the physical point is found to have an uncertainty that is somewhat smaller than that from experiment. An up-to-date discussion of the status of $\pi\pi$-interactions can be found in Ref. [3].

Studying the low-energy interactions between kaons and pions with $\pi^-K^+$ bound-states allows for an explicit exploration of the three-flavor structure of low-energy hadronic interactions, an aspect that is not directly probed in $\pi\pi$ scattering. Experiments have been proposed by the DIRAC collaboration [4] to study $\pi K$ atoms at CERN, J-PARC and GSI, the results of which would provide direct measurements or constraints on combinations of the scattering lengths. In the isospin limit, there are two isospin channels available to the $\pi K$ system, $I = 1/2$ and $I = 3/2$. The width of a $\pi^-K^+$ atom depends upon the difference between scattering lengths in the two channels, $\Gamma \sim (a_{1/2} - a_{3/2})^2$, (where $a_{1/2}$ and $a_{3/2}$ are the $I = 1/2$ and $I = 3/2$ scattering lengths, respectively) while the shift of the ground state depends upon a different combination, $\Delta E_0 \sim 2a_{1/2} + a_{3/2}$. Recently, the Roy-Steiner equations (analyticity, unitarity and crossing-symmetry) have been used to extrapolate high-energy $\pi K$ data down to threshold [5], where it is found that
\[ m_\pi(a_{1/2} - a_{3/2}) = 0.269 \pm 0.015, \]
\[ m_\pi(a_{1/2} + 2a_{3/2}) = 0.134 \pm 0.037, \]

which can be decomposed to \( m_\pi a_{1/2} = 0.224 \pm 0.022 \) and \( m_\pi a_{3/2} = -0.0448 \pm 0.0077 \). (See also Ref. [6] for a similar approach.) In addition, three-flavor chiral perturbation theory (\( \chi PT \)) has been used to predict these scattering lengths out to next-to-next-to-leading order (NNLO) in the chiral expansion. At NLO [7–9],

\[ m_\pi(a_{1/2} - a_{3/2}) = 0.238 \pm 0.002, \]
\[ m_\pi(a_{1/2} + 2a_{3/2}) = 0.097 \pm 0.047, \]

while at NNLO [10] \( m_\pi a_{1/2} = 0.220 \) and \( m_\pi a_{3/2} = -0.047,^1 \) One must be cautious in assessing the uncertainties in these theoretical calculations, as one can only make estimates based on power-counting for the contribution of higher-order terms in the chiral expansion. There has been one determination of the \( \pi^+ K^- \) scattering length in quenched QCD [12], however, the chiral extrapolation of the scattering length did not include the non-analytic dependences on the light-quark masses that are predicted by chiral perturbation theory.

It is worth mentioning a novel motivation for accurate determinations of meson-meson scattering from lattice QCD calculations. Recent work has identified in a model-independent way the lowest-lying resonance in QCD which appears in \( \pi\pi \) scattering [13]. Crucial to this development has been the accurate determination of the low-energy \( \pi\pi \) scattering amplitude, including the recent lattice QCD determination of the \( I = 2 \) scattering length [2]. A similar analysis has very recently been carried out for \( \pi K \) scattering in the \( I = \frac{1}{2} \) s-wave in order to determine the lowest-lying strange resonance [14]. Improved accuracy in the low-energy \( \pi K \) scattering amplitude should be welcome to this endeavor.

In this work we present the results of a fully-dynamical lattice QCD calculation of \( \pi^+ K^- \) scattering. By calculating the \( m_\pi \) and \( m_K \) dependence of the \( \pi^+ K^- \) \( (I = \frac{1}{2}) \) scattering length, we are able to provide a determination of both the \( I = \frac{1}{2} \) and \( I = \frac{1}{2} \) scattering lengths at the physical point. We have performed a hybrid mixed-action calculation with domain-wall valence quarks tuned to the staggered sea-quark masses of the MILC configurations. As the computer resources do not presently exist to perform such calculations at or very near the physical value of the light-quark masses, these are performed at pion masses between \( m_\pi \sim 290 \text{ MeV} \) and \( \sim 600 \text{ MeV} \). These results are combined with calculations in continuum three-flavor \( \chi PT \) to extrapolate to the physical point.

\[ \Delta E_n = E_n - m_1 - m_2 \]
\[ = \sqrt{p_n^2 + m_1^2} + \sqrt{p_n^2 + m_2^2} - m_1 - m_2 \]
\[ = \frac{p_n^2}{2\mu_{12}} + \ldots, \]

where \( \mu_{12} \) is the reduced mass of the system. In the absence of interactions between the particles, \( |p\cot\delta| = \infty \), and the energy levels occur at momenta \( p = 2\pi j/L \), corresponding to single-particle modes in a cubic cavity. Expanding Eq. (3) about zero momenta, \( p \sim 0 \), one obtains the familiar relation

\[ \Delta E_0 = -\frac{2\pi a}{\mu_{12} L^3} \left[ 1 + c_1 \frac{a}{L} + c_2 \left( \frac{a}{L} \right)^2 \right] + \mathcal{O}\left( \frac{1}{L^5} \right), \]

with

\[ c_1 = \frac{1}{\pi} \sum_{j=1}^{N} \frac{1}{|j|^3} - 4\Lambda = -2.837297, \]
\[ c_2 = c_1^2 - \frac{1}{\pi^2} \sum_{j=1}^{N} \frac{1}{|j|^4} = 6.375183, \]

\[ ^1 \text{At tree level, Weinberg [11] determined that } m_\pi a_{1/2} = 0.137 \text{ and } m_\pi a_{3/2} = -0.0687. \]
and $a$ is the scattering length, defined by

$$a = \lim_{p \to 0} \frac{\tan \delta(p)}{p}. \quad (8)$$

For the $I = \frac{3}{2}$ $\pi K$ scattering length, $a_{3/2}$, that we compute in this work, the difference between the exact solution to Eq. (3) and the approximate solution in Eq. (6) is much less than 1%.

### III. DETAILS OF THE LATTICE CALCULATION

Our computation uses the mixed-action lattice QCD scheme developed by LHPC [17,18] which places domain-wall valence quarks from a smeared-source on $N_f = 2 + 1$ asqtad-improved [19,20] MILC configurations generated with rooted $^3$ staggered sea quarks [29] that are hypercubic-smeared (HYP-smeared) [30–33]. In the generation of the MILC configurations, the strange-quark mass was fixed near its physical value, $b m_s = 0.050$, (where $b = 0.125 \text{ fm}$ is the lattice spacing) determined by the mass of hadrons containing strange quarks. The two light quarks in the configurations are degenerate (isospin-symmetric). As was shown by LHPC [17,18], HYP-smearing allows for a significant reduction in the residual chiral symmetry breaking at a moderate extent $L_s = 16$ of the extra dimension and domain-wall height $M_s = 1.7$. Using Dirichlet boundary conditions we reduced the original time extent of 64 down to 32. This allowed us to recycle propagators computed for the nucleon structure function calculations performed by LHPC. For bare domain-wall fermion masses we used the tuned values that match the staggered Goldstone pion to few-percent precision. For details of the matching see Refs. [17,18]. The parameters used in the propagator calculation are summarized in Table I. All propagator calculations were performed using the Chroma software suite [34,35].

As it is the difference in the energy between interacting mesons and noninteracting mesons that provides the scattering amplitude, we computed the one-pion correlation function $C_{\pi^+}(t)$, the one-kaon correlation function $C_{K^+}(t)$, and the kaon-pion correlation function $C_{\pi^+ K^+}(p, t)$, where $t$ denotes the number of time slices between the hadronic-sink and the hadronic-source, and $p$ denotes the magnitude of the (equal and opposite) momentum of each meson. The single-pion correlation function is

$$C_{\pi^+}(t) = \sum_x \langle \pi^-(t, x) \pi^+(0, 0) \rangle, \quad (9)$$

where the summation over x corresponds to summing over all the spatial lattice sites, thereby projecting onto the momentum $p = 0$ state. The single-kaon correlation function has a similar form. The $\pi^+ K^+$ correlation function that projects onto the $s$-wave state in the continuum limit is

$$C_{\pi^+ K^+}(p, t) = \sum_{|p|=p} \sum_{x,y} e^{i p (x-y)} \langle \pi^-(t, x) K^-(t, y) \rangle \times K^+(0, 0) \pi^+(0, 0), \quad (10)$$

where, in Eqs. (9) and (10), $\pi^+(t, x) = \bar{u}(t, x) \gamma_5 d(t, x)$ is an interpolating field for the $\pi^+$, and $K^+(t, x) = \bar{u}(t, x) \gamma_5 s(t, x)$ is an interpolating field for the $K^+$. In the relatively large lattice volumes that we are using, the energy difference between the interacting and noninteracting two-meson states is a small fraction of the total energy, which is dominated by the masses of the mesons. In order to extract this energy difference we formed the ratio of correlation functions, $G_{\pi^+ K^+}(p, t)$, where

$$G_{\pi^+ K^+}(p, t) = \frac{C_{\pi^+ K^+}(p, t)}{C_{\pi^+}(t) C_{K^+}(t)} = \sum_{n=0}^{\infty} A_n e^{-\Delta E_n t}, \quad (11)$$

and the arrow becomes an equality in the limit of an infinite number of gauge configurations. In $G_{\pi^+ K^+}(p, t)$, some of the fluctuations that contribute to both the one- and two-meson correlation functions cancel, thereby improving the quality of the extraction of the energy difference beyond what we are able to achieve from an analysis of the individual correlation functions.

### IV. ANALYSIS AND CHIRAL EXTRAPOLATION

A convenient way to present the data is with “effective scattering length” plots, simple variants of effective mass

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1For recent discussions of the “legality” of the mixed-action and rooting procedures, see Ref. [21–28].
As can be seen from Fig. 1, there is a large systematic error associated with the m010 ensemble. There would appear to be two distinct plateaus. Rather than fitting to one of the plateaus, we chose to fit over a large range (7)–(15) which includes both plateaus and then assigned a systematic error which encompasses minima and maxima over the fit range as indicated by the effective scattering length plot. More statistics will have to be acquired on this ensemble before any conclusions can be drawn about this correlator.

In SU(3) chiral perturbation theory [36–38] at NLO, the expansion of the crossing even ($a^+$) and crossing odd ($a^-$) scattering length times the reduced mass is known to be [7–9]
TABLE II. Results from the lattice calculation. All errors are computed from jackknife. The uncertainty associated with the m010 ensemble πK energy shift and related quantities is dominated by the systematic error. The fitting ranges are shown in the square brackets.

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>m_π/f_π</th>
<th>m_K/f_π</th>
<th>μ_πK/f_π</th>
<th>δE_πK (MeV)</th>
<th>μ_πK μ_πK</th>
<th>Γ \times 10^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>m007</td>
<td>2.000(17)</td>
<td>3.980(25)</td>
<td>1.332(10)</td>
<td>11.89(81) [8–15]</td>
<td>11.89(81) [8–15]</td>
<td>0.01263(75)</td>
</tr>
<tr>
<td>m010</td>
<td>2.337(11)</td>
<td>3.958(16)</td>
<td>1.469(07)</td>
<td>11.40(50) [7–15]</td>
<td>11.40(50) [7–15]</td>
<td>0.01155(40)</td>
</tr>
<tr>
<td>m020</td>
<td>3.059(12)</td>
<td>3.988(15)</td>
<td>1.731(07)</td>
<td>10.15(69) [10–15]</td>
<td>10.15(69) [10–15]</td>
<td>0.00213(12)</td>
</tr>
<tr>
<td>m030</td>
<td>3.484(10)</td>
<td>4.004(12)</td>
<td>1.869(05)</td>
<td>10.06(54) [11–16]</td>
<td>10.06(54) [11–16]</td>
<td>0.00267(12)</td>
</tr>
</tbody>
</table>

\[
\mu_{\pi K} = \frac{\mu_{\pi K} a_{\pi K} \Lambda}{2 \pi f_\pi} \left[ 16 L_{\pi K}(\Lambda) + \frac{1}{16 \pi^2} \left( \frac{11 m_K^2}{2 (m_K^2 - m_\pi^2)} \log \frac{m_K}{\Lambda} - \frac{67 m_K^2 - 8 m_\pi^2}{9 (m_K^2 - m_\pi^2)} \log \frac{m_\pi}{\Lambda} + \frac{24 m_K^2 - 5 m_\pi^2}{18 (m_K^2 - m_\pi^2)} \log \frac{m_\eta}{\Lambda} \right) - \frac{4 \sqrt{(m_K - m_\pi)(2 m_K + m_\pi)}}{m_K + m_\pi} \log \frac{m_K - m_\pi}{2 m_K + m_\pi} \left( \frac{m_K + m_\pi}{m_K + 2 m_\pi} \log \frac{m_K + m_\pi}{2 m_K + m_\pi} + \frac{43}{9} \right) \right].
\] (14)

where the counterterm \( L_{\pi K}(\Lambda) \) is a renormalization scale, \( \Lambda \), dependent linear combination of the Gasser-Leutwyler counterterms

\[
L_{\pi K} = 2 L_1 + 2 L_2 + L_3 - 2 L_4 - \frac{L_5}{2} + 2 L_6 + L_8. \quad (15)
\]

It is important to note that the expressions in Eqs. (13) and (14) are written in terms of the full \( f_\pi \), and not the chiral limit value. The functions \( \chi^{(\text{NLO}+,\text{phys})}(m_\pi/\Lambda, m_K/\Lambda, m_\eta/\Lambda) \) and \( \chi^{(\text{NLO},-)}(m_\pi/\Lambda, m_K/\Lambda, m_\eta/\Lambda) \) clearly depend upon the renormalization scale \( \Lambda \). In the analysis that follows, it was found to be convenient to normalize the meson masses to \( f_\pi \), and therefore we can choose the renormalization scale to be \( \Lambda = f_\pi^{\text{phys}} = 132 \, \text{MeV} \), and use the values of \( m_\pi/f_\pi \) and \( m_K/f_\pi \) in Table II directly. Deviations between the \( \Lambda = f_\pi \) calculated on each lattice and \( \Lambda = f_\pi^{\text{phys}} \) are higher order in the chiral expansion.

The \( I = \frac{1}{2} \) and \( I = \frac{3}{2} \) scattering lengths are related to those in Eqs. (13) and (14) by

\[
a_{1/2} = a^+ + 2 a^- \quad a_{3/2} = a^+ - a^- = a_{\pi K}. \quad (16)
\]

It is convenient to define the function \( \Gamma \) via a subtraction of the tree-level and one-loop contributions in order to isolate the counterterms,

\[
\Gamma(m_\pi, m_K) = \frac{f_\pi^2}{16 m_\pi^2} \left( \frac{4 \pi f_\pi^2}{\mu_{\pi K} a_{\pi K} K} \right) + 1 \chi^{(\text{NLO,+})} - \frac{2 m_K m_\pi}{f_\pi^2} \chi^{(\text{NLO},-)} \right). \quad (17)
\]

where we use the Gell-Mann-Okubo mass-relation among the mesons to determine the \( \eta \)-mass, which we do not measure in this lattice calculation. At NLO this becomes

\[
\Gamma = L_5(f_\pi^{\text{phys}}) - \frac{2 m_K m_\pi}{m_\pi} L_{\pi K}(f_\pi^{\text{phys}}). \quad (18)
\]

It is clear that the dependence of \( \Gamma \) on \( m_\pi \) and \( m_K \) determines \( L_5 \) and \( L_{\pi K} \) and, in turn, allows an extraction of \( a_{3/2} \) and \( a_{1/2} \). The numerical values of \( \Gamma \) and their jackknife errors calculated on each ensemble of lattices are given in Table II, and are plotted in Fig. 3. By fitting a straight line to the values of \( \Gamma \) as a function of \( m_K/m_\pi \) the counterterms

\[
\mu_{\pi K} a_{\pi K} = \frac{\mu_{\pi K} a_{\pi K}}{2 \pi f_\pi} \left[ 16 L_{\pi K}(\Lambda) + \frac{1}{16 \pi^2} \left( \frac{11 m_K^2}{2 (m_K^2 - m_\pi^2)} \log \frac{m_K}{\Lambda} - \frac{67 m_K^2 - 8 m_\pi^2}{9 (m_K^2 - m_\pi^2)} \log \frac{m_\pi}{\Lambda} + \frac{24 m_K^2 - 5 m_\pi^2}{18 (m_K^2 - m_\pi^2)} \log \frac{m_\eta}{\Lambda} \right) - \frac{4 \sqrt{(m_K - m_\pi)(2 m_K + m_\pi)}}{m_K + m_\pi} \log \frac{m_K - m_\pi}{2 m_K + m_\pi} \left( \frac{m_K + m_\pi}{m_K + 2 m_\pi} \log \frac{m_K + m_\pi}{2 m_K + m_\pi} + \frac{43}{9} \right) \right].
\]

\[
\mu_{\pi K} a_{\pi K} = \frac{\mu_{\pi K} a_{\pi K}}{2 \pi f_\pi} \left[ 16 L_{\pi K}(\Lambda) + \frac{1}{16 \pi^2} \left( \frac{11 m_K^2}{2 (m_K^2 - m_\pi^2)} \log \frac{m_K}{\Lambda} - \frac{67 m_K^2 - 8 m_\pi^2}{9 (m_K^2 - m_\pi^2)} \log \frac{m_\pi}{\Lambda} + \frac{24 m_K^2 - 5 m_\pi^2}{18 (m_K^2 - m_\pi^2)} \log \frac{m_\eta}{\Lambda} \right) - \frac{4 \sqrt{(m_K - m_\pi)(2 m_K + m_\pi)}}{m_K + m_\pi} \log \frac{m_K - m_\pi}{2 m_K + m_\pi} \left( \frac{m_K + m_\pi}{m_K + 2 m_\pi} \log \frac{m_K + m_\pi}{2 m_K + m_\pi} + \frac{43}{9} \right) \right].
\]
uncertainty in the values of the scattering lengths extrapolated to the physical point that is introduced by the truncation of the chiral expansion at NLO. In our work on $f_k/f_\pi$ [39] we extracted a value of $L_5$ as it is the only NLO counterterm that contributes. The numerical value obtained is only perturbatively close to its true value, as it is contaminated by higher-order contributions. Therefore, by fixing the $L_5$ that appears in Eq. (18) to the value of $L_5$ extracted from $f_k/f_\pi$, an estimate of the uncertainty in both $L_{\pi K}$ and in the extrapolated values of the scattering lengths due to the truncation of the chiral expansion can be estimated. Specifically, we sampled $L_5$ from a Gaussian distribution for a range of $f_k/f_\pi$ values [39] and then fit $L_{\pi K}$ using $\chi^2$-minimization. We then generated a value of $L_{\pi K}$ from a normal distribution formed from its mean and standard error. This fit is denoted “fit C”, and the same fit but with the m030 data pruned is denoted “fit D”. The results of the four fits are given in Table III and plotted in Fig. 3. These fits lead to an extraction of

$$L_{\pi K} = 4.16 \pm 0.18 \pm 0.26_{-0.91}$$

and a prediction of the scattering lengths extrapolated to the physical point of

$$m_\pi a_{3/2} = -0.0574 \pm 0.0016 \pm 0.0024 \pm 0.0008$$

$$m_\pi a_{1/2} = 0.1725 \pm 0.0017 \pm 0.0023 \pm 0.0016$$

We have chosen to take the central values and statistical errors from fit D and have set the systematic error due to truncation of the chiral expansion by taking the range of the various quantities allowed by the four fits, including statistical and systematic errors. In Fig. 4 we plot the 68% and 95% confidence-level error ellipses for the four fits given in Table III in the $L_5$-$L_{\pi K}$ plane. In Fig. 5 we plot the 95% confidence-level error ellipses associated with the four fits in the $m_\pi a_{1/2}$-$m_\pi a_{3/2}$ plane. For purposes of comparison we have included the current-algebra point [11] on the plot as well as 1-σ error ellipses from analyses

### TABLE III. Results of the NLO fits. The values of $m_\pi a_{3/2}$ and $m_\pi a_{1/2}$ correspond to their extrapolated values at the physical point, where the error ellipses in the $L_5$-$L_{\pi K}$ plane have been explored at 68% confidence level (see Fig. 4).

<table>
<thead>
<tr>
<th>FIT</th>
<th>$L_5 \times 10^3$</th>
<th>$L_{\pi K} \times 10^3$</th>
<th>$m_\pi a_{3/2}$</th>
<th>$m_\pi a_{1/2}$</th>
<th>$\chi^2$/dof</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3.83 ± 0.49</td>
<td>3.55 ± 0.20</td>
<td>-0.0607 ± 0.0025</td>
<td>0.1631 ± 0.0062</td>
<td>0.17</td>
</tr>
<tr>
<td>B</td>
<td>2.94 ± 0.07</td>
<td>3.27 ± 0.02</td>
<td>-0.0620 ± 0.0004</td>
<td>0.1585 ± 0.0011</td>
<td>0.001</td>
</tr>
<tr>
<td>C</td>
<td>$5.65 \pm 0.02^{+0.18}_{-0.54}$</td>
<td>$4.24 \pm 0.17$</td>
<td>$-0.0567 \pm 0.0017$</td>
<td>$0.1731 \pm 0.0017$</td>
<td>0.84</td>
</tr>
<tr>
<td>D</td>
<td>$5.65 \pm 0.02^{+0.18}_{-0.54}$</td>
<td>$4.16 \pm 0.18$</td>
<td>$-0.0574 \pm 0.0016$</td>
<td>$0.1725 \pm 0.0017$</td>
<td>0.90</td>
</tr>
</tbody>
</table>

*Input from $f_k/f_\pi$ [39].

*In Mathematica format, the 95% confidence-level error ellipses in the $m_\pi a_{1/2}$-$m_\pi a_{3/2}$ plane are:

fit A: Ellipsoid [[0.1631, -0.0607], [0.0197, 0.0007], [0.9283, 0.3719], [-0.3719, 0.9283]]
fit B: Ellipsoid [[0.1585, -0.0620], [0.0076, 0.0004], [0.9461, 0.3239], [-0.3239, 0.9461]]
fit C: Ellipsoid [[0.1731, -0.0567], [0.0042, 0.0016], [0.7534, 0.6576], [-0.6576, 0.7534]]
fit D: Ellipsoid [[0.1725, -0.0574], [0.0046, 0.0027], [0.7881, 0.6156], [-0.6156, 0.7881]].

FIG. 3 (color online). $\Gamma$ vs $m_K/m_\pi$. The dark error bar on the data points is statistical, while the lighter error bar corresponds to the systematic error. The lines correspond to the four linear fits (A,B,C,D). The bars on the $y$ axis represent the 1-σ errors in the determinations of $L_5$ = $\Gamma(m_K/m_\pi = 0)$ as given in Table III. (At 95% confidence level, these determinations are in agreement.)

$L_5$ and $L_{\pi K}$ (renormalized at $f_\pi^{phys}$) can be determined.

Ideally, one would fit to lattice data at the lightest accessible values of the quark masses in order to ensure convergence of the chiral expansion. While we only have four different quark masses in our data set, with pion masses ranging from $m_\pi \sim 290$ MeV to 600 MeV, fitting all four data sets and then “pruning” the heaviest data set and refitting provides a useful measure of the convergence of the chiral expansion. Hence, in “fit A”, we fit the data from all four lattice ensembles (m007, m010, m020 and m030), while in “fit B”, we fit the data from the lightest three lattice ensembles (m007, m010 and m020).

With the limited data set presently at our disposal, it is not practical to fit to the NNLO expression [10] for the scattering length. However, it is important to estimate the
of the scattering lengths. It would be interesting to see the corresponding to 39% confidence level, one should be careful at NLO [7] (denoted Ref. [11]. We also display Steiner equations [5].

The four fits (A,B,C,D) at 68% (dotted lines) and 95% (solid lines) confidence level.

FIG. 4 (color online). Error ellipses for the four fits (A,B,C,D) at 68% (dotted lines) and 95% (solid lines) confidence level. The star corresponds to the current-algebra predictions ($\chi$PT $p^3$) from Ref. [11].

FIG. 5 (color online). Error ellipses for the four fits (A,B,C,D) at 95% confidence level. (Note that these results are derived from lattice data on a single lattice spacing of $b = 0.125$ fm). The star corresponds to the current-algebra predictions ($\chi$PT $p^3$) and from a fit using the Roy-Steiner equations [5].

Based on fitting experimental data using $\chi$PT at NLO [7] and using Roy-Steiner equations [5]. As 1-$\sigma$ error ellipses correspond to 39% confidence level, one should be careful in finding discrepancy between the various determinations of the scattering lengths. It would be interesting to see the NLO $\chi$PT and Roy-Steiner error ellipses at higher confidence levels.

For the sake of completeness, we also quote numbers for the crossing odd ($a^-$) and crossing even ($a^+$) scattering lengths extrapolated to the physical point:

$$m_{\pi} a^- = 0.0766 \pm 0.0005^{+0.0000}_{-0.0003}$$
$$m_{\pi} a^+ = 0.0193 \pm 0.0016^{+0.0021}_{-0.0008}$$

where the procedure for choosing the central value and determining the systematic error is the same as above. Error ellipses at 68% and 95% confidence levels are shown in Fig. 6. The crossing-odd scattering length is of special interest as its corrections are protected by SU(2) chiral symmetry and are therefore of order $m_{\pi}^4$ and expected to be small [40,41].

Given how well our lattice data fit the NLO continuum $\chi$PT formulas, it would seem that the $O(b^2)$ discretization errors are comparable or smaller than the systematic error due to omitted $O(m_{\pi}^3)$ effects in the chiral expansion. However, one should keep in mind that our determinations of, for instance, the low-energy constants $L_5$ and $L_{\pi K}$ are subject to $O(b^2)$ scattering shifts. In contrast with the $\pi^+ \pi^+$ and $K^+ K^+$ scattering lengths, the mixed-action quantity $\Delta_{\text{Mix}}$ makes an explicit contribution to the $K^+ \pi^+$ scattering length [42,43]. While this adds an additional unknown contribution to this process, a mixed-action $\chi$PT analysis

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5In Mathematica format, the 95% confidence-level error ellipses in the $m_{\pi} a^--m_{\pi} a^+$ plane are:

- fit A: Ellipsoid [{0.0746, 0.0139}, {0.0116, 0.0004}, {0.3150, 0.9491}, {0.9491, -0.3150}]
- fit B: Ellipsoid [{0.0736, 0.0115}, {0.0043, 0.0002}, {0.3260, 0.9322}, {0.9322, -0.3260}]
- fit C: Ellipsoid [{0.0766, 0.0200}, {0.0029, 0.0009}, {0.0014, 1.0000}, {1.0000, -0.0014}]
- fit D: Ellipsoid [{0.0766, 0.0193}, {0.0033, 0.0011}, {0.0017, 1.0000}, {1.0000, 0.0017}].

6We thank Heiri Leutwyler for emphasizing this point to us.
of \( \pi K \) scattering, including lattice data from the fine MILC lattices \((b \sim 0.09 \, \text{fm})\), will be able to address this source of systematic error quantitatively. We continue to search for the computational resources to accomplish this task.

V. CONCLUSIONS

In this paper we have computed the \( \pi^+ K^+ \) scattering length in fully-dynamical lattice QCD at pion masses ranging between \( m_\pi \sim 290 \, \text{MeV} \) and \( 600 \, \text{MeV} \). We have used the continuum expressions for the scattering lengths in SU(3) chiral perturbation theory, together with lattice data for \( f_K / f_\pi \), to predict the physical \( I = 3/2 \) and \( I = 1/2 \) \( \pi K \) scattering lengths with unprecedented accuracy. Naively one would expect that \( \pi^+ K^+ \) scattering would give information about \( I = 3/2 \) scattering only. However, the lattice data, when combined with chiral perturbation theory, implies a constraint on \( I = 1/2 \) scattering as well. We anticipate that with improved statistics, together with calculations on lattices with smaller lattice spacings, the theoretically-predicted regions for \( m_\pi a_{3/2} \) and \( m_\pi a_{1/2} \) can be further reduced beyond those shown in Fig. 5. These regions can then be compared with the expected measurements from \( K^+ \pi^- \) atoms, to provide an exciting test of hadronic theory.

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