Balancing relations between the normalized left and right coprime factorizations of a nonlinear system

Jacquelin M.A. Scherpen

Delft Center for Systems and Control, Delft University of Technology
Mekelweg 2, 2628 CD Delft, The Netherlands.
Email: j.m.a.scherpen@desc.tudelft.nl

Abstract—This paper considers the relation between the singular value functions of the nonlinear normalized left coprime factorization (NLCF) and the nonlinear normalized right coprime factorization (NRCF). In previous work a new duality notion gave rise to a relation between the controllability, observability, future and past energy functions of the original system and its NLCF and NRCF. However, some linear parameter varying notions are still missing and are included in this paper. Both the NLCF and NRCF can be used for model reduction based on balanced realizations for an unstable nonlinear system. For linear systems model reduction based on balancing of the NLCF or NRCF yields the same reduced order model.

I. INTRODUCTION

In linear systems theory the Gramians of a system play an important role in many studies, and in especially when the study is dealing with balanced realizations. For unstable linear systems there exists balancing methods that are based on normalized coprime factorizations (e.g. [11], [8]). In those studies a relation between the Gramians of the right and left coprime factorizations and the solutions of the Control and Filter Algebraic Riccati Equation (CARE and FARE, respectively) are given.

A generalization for nonlinear systems is given in [16], where expressions for the nonlinear left and right coprime factorizations (NLCF and NRCF, respectively) are obtained. Other research, such as [1], [12] further developed coprime factorizations. In [16] the focus is mainly on balanced realizations for the NLCF and NRCF, and their relation with the HJB balanced representation.

In the case of NRCF [16] presents a similar relation as in the linear case for the observability and controllability function of the nonlinear NRCF, and the future and past energy function of the original nonlinear system (in the linear case, they correspond to the solutions of the CARE and FARE). A similar relation for the NLCF is only recently established with a new duality notion, e.g., [14]. The dual system as presented in [14] is inspired by the results in [6], where an adjoint state-space representation for the the nonlinear Hilbert adjoint, [15], is developed. This duality notion helps to establish relations that are similar to the ones for the NRCF in the dual coordinates. Also, a relation between the different energy functions of the original system, the NRCF and NLCF is established.

However, this relation in [14] is still missing some essential linear parameter varying interpretations that should be included in the analysis.

The new duality notion now opens the possibility to study a relation between the singular value functions of the NLCF and NRCF, and its input-normal output-diagonal representation as given in [13], [5]. This is important for determining the relation between the reduction order model based on the balanced realization of the NLCF or the NRCF.

In Section II we present some preliminaries about the NLCF of nonlinear systems. Then in Section III we give relations for the dual system of the NLCF between the various energy functions that are important for balanced realizations. In Section IV we present some issues regarding the singular value functions. Finally, in Section V, we present some conclusions and open issues.

Notation: We denote \( L_2(-\infty,0) \) by \( L^- \) and \( L_2(0,\infty) \) by \( L^+ \). Furthermore, by \( \frac{\partial}{\partial t}(x) \) we denote the row vector with partial derivative of a function \( K(x) \).

II. PRELIMINARIES

Consider a smooth nonlinear system of the form

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x)
\end{align*}
\]

where \( u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m \), \( y = (y_1, \ldots, y_p)^T \in \mathbb{R}^p \), and \( x = (x_1, \ldots, x_n)^T \) are coordinates for a smooth state space manifold denoted by \( M \). Furthermore, \( f, g_1, \ldots, g_m \) are smooth vectorfields on \( M \), where \( g = (g_1, \ldots, g_m) \), and \( h = (h_1, \ldots, h_p)^T \) is the smooth output map of the system. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium in \( 0 \), i.e. \( f(0) = 0 \). We also take \( h(0) = 0 \).

We can relate several energy functions with system (1). This is done in the next definition.

Definition 2.1: The controllability and observability function of a nonlinear system (1) are given by

\[
L_c(x_0) = \min_{u \in L^+} \frac{1}{2} \int_{-\infty}^{0} \| u(t) \|^2 dt,
\]

where \( x \in L^+ \), and \( x(0) = x_0 \).
and

\[ L_o(x_0) = \frac{1}{2} \int_0^\infty \| y(t) \|^2 \, dt, \]

where \( x(0) = x_0, \ u(t) \equiv 0, \ 0 \leq t < \infty, \)
respectively.

The past and future energy functions of a nonlinear system are defined as

\[ K^-(x_0) = \min_{u \in L^2_+} \left\{ \frac{1}{2} \int_{-\infty}^0 \| y(t) \|^2 + \| u(t) \|^2 \, dt \right\}, \]

\[ K^+(x_0) = \min_{u \in L^2_+} \left\{ \frac{1}{2} \int_0^\infty \| y(t) \|^2 + \| u(t) \|^2 \, dt \right\}, \]

respectively.

The above energy functions are related to some Hamilton-Jacobi-Bellman type of equations, stemming from Optimal Control theory. First we give the equations for the observability and controllability function.

**Theorem 2.2**: [13] Assume that \( f(x) \) is asymptotically stable on a neighborhood \( W \) of 0. Then

\[ \frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2} h^T(x)h(x) = 0, \quad L_o(0) = 0. \]

has a smooth solution \( \bar{L}_o \) for all \( x \in W \) if and only if \( L_o \) exists. Then \( L_o \) is the unique smooth solution of (6) for all \( x \in W \).

Furthermore, the Hamilton-Jacobi equation

\[ \frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2} \frac{\partial L_o}{\partial x}(x)g(x)g^T(x)\frac{\partial L_o}{\partial x}(x) = 0, \]

\( \bar{L}_o(0) = 0 \) has a smooth solution \( \bar{L}_o \) for all \( x \in W \) such that

\[ -(f(x) + g(x)g^T(x)\frac{\partial \bar{L}_o}{\partial x}(x)) \]

is asymptotically stable on \( W \) if and only if \( L_o(x) \) exists. Then \( L_o(x) \) is the unique smooth solution of (7), such that (8) is asymptotically stable for all \( x \in W \).

**Theorem 2.3**: e.g. [16] The Hamilton-Jacobi-Bellman equation

\[ \frac{\partial K^+}{\partial x}(x)f(x) - \frac{1}{2} \frac{\partial K^+}{\partial x}(x)g(x)g^T(x)\frac{\partial K^+}{\partial x}(x) \]

\[ + \frac{1}{2} h^T(x)h(x) = 0 \]

with \( K^+(0) = 0 \), has a smooth non-negative solution on a neighborhood \( Y \) of 0, such that

\[ f(x) - g(x)g^T(x)\frac{\partial K^+}{\partial x}(x) \]

is asymptotically stable, if and only if \( K^+ \) exists. Then \( K^+ \) is that solution.

Furthermore, the Hamilton-Jacobi-Bellman equation

\[ \frac{\partial K^-}{\partial x}(x)f(x) + \frac{1}{2} \frac{\partial K^-}{\partial x}(x)g(x)g^T(x)\frac{\partial K^-}{\partial x}(x) \]

\[ - \frac{1}{2} h^T(x)h(x) = 0 \]

with \( K^-(0) = 0 \), has a smooth non-negative solution on a neighborhood \( Y \) of 0, such that

\[ -(f(x) + g(x)g^T(x)\frac{\partial K^-}{\partial x}(x)) \]

is asymptotically stable, if and only if \( K^- \) exists on \( Y \). Then \( K^- \) is that solution.

We assume the system (1) to be zero-state observable. Furthermore, we assume that (11) has a smooth non-negative solution \( K^- \) on a coordinate neighborhood \( Y \) of 0. It follows from (11) that \( \frac{\partial K^-}{\partial x}(0) = 0 \) and we can write (see [9])

\[ \frac{\partial K^-}{\partial x}(x) = x^T M(x), \]

where \( M(x) \) is an \( n \times n \) matrix with all entries \( m_{ij}(x) \), \( i, j = 1, \ldots, n \), smooth functions of \( x \) and \( M(0) = \frac{\partial K^-}{\partial x}(0) \). We assume that

\[ \frac{\partial^2 K^-}{\partial x^2}(0) > 0 \]

and therefore there exists a neighborhood \( U \) of 0 for which \( M(x) \) is nonsingular and thus is invertible on \( U \). Furthermore, since \( h(0) = 0 \), we can write \( h(x) = C(x) \) where \( C(x) \) is a \( p \times n \) matrix with entries that are smooth functions of \( x \) and \( C(0) = \frac{\partial h}{\partial x}(0) \). Now consider for \( x \in U \)

\[ \dot{x} = \left( f(x) - (M(x))^{-1} C(x)^T h(x) \right) + \left( g(x) (M(x))^{-1} C(x)^T \right) \dot{w} \]

\[ z = h(x) + \left( 0 \quad -I \right) \dot{w} \]

This system is asymptotically stable on \( U \) under the assumption that \( K^- \) is proper on \( U \). \( K^- \) then serves as a Lyapunov function for (14). The system (14) is a representation of the normalized left coprime factorization (NLCF) of (1), see [16], [12].

**Remark 2.4**: It can be shown (see [16]) that linearizing the above system yields the corresponding linear NLCF. Since the linear NLCF is asymptotically stable, (14) is exponentially stable. Hence, there exists a neighborhood of 0 where all eigenvalues of \( A(x) - (M(x))^{-1} C(x)^T C(x) \) are in the left half plane as well.

**III. The NLCF and Duality**

For model reduction of nonlinear systems based on balanced realizations, see e.g. [13], [5], [3], the system has to be asymptotically stable. If this is not the case, we could consider to balance the normalized coprime factorization that is asymptotically stable. In [16] this is considered for the normalized right coprime factorization (NRCF), as well.

836
as for the nonlinear version of the linear LQG balancing (e.g., [7]), the so-called HJB balancing. For linear systems it does not matter if the NRCF or the NLCF is considered for balancing; the singular values are equal for the two factorizations. However, such relation is not established yet for nonlinear systems. Furthermore, the relation between the future and past energy functions and the controllability and observability functions of the NLCF is not established yet, whereas for NRCF this is already established in [16].

Now consider $K^-$ and $K^+$ for the system (1) and the controllability and observability functions $\tilde{L}_c$ and $\tilde{L}_o$ for the NLCF given by (14). Then it is straightforwardly obtained that

$$ K^-(x) = \tilde{L}_c(x) $$

If we assume that system (1) is linear, and minimal, and thus also (14) is a linear system. Then we can write

$$ \tilde{L}_c(x) = K^-(x) = \frac{1}{2} x^T Z x, $$

$$ \tilde{L}_o(x) = \frac{1}{2} x^T X x, \quad K^+(x) = \frac{1}{2} x^T P x, $$

where $Z$, $X$, and $P$ are positive definite matrices, and for equation (13) we obtain $M(x) = Z$. Then $Z^{-1}$ and $X$ are the controllability and observability Gramian of the NLCF, respectively. $Z^{-1} := S$ and $P$ are the stabilizing solutions of the FARE (Filter Algebraic Riccati Equation) and CARE (Control Algebraic Riccati Equation), respectively, e.g., [11], [8]. Furthermore, it can be proved that those matrices are related via (e.g. [11])

$$ Z^{-1} = X^{-1} - P^{-1} \quad (15) $$

Clearly, equation (15) is dealing with the inverses of the matrices that appear in the quadratic forms. This implies that (15) is not straightforwardly extended to the nonlinear case. In order to establish a relation like (15) for the nonlinear NLCF we first need to establish an appropriate notion of duality for nonlinear systems.

Now, we drop the assumption that the systems are linear, and we will consider a dual system that is inspired by the nonlinear Hilbert adjoint notion, [15] for which we have obtained state-space realizations in [6]. In [6] we mention duality “in the sense of Young”, using the Legendre transformation of the controllability and observability functions. Here, we will use this notion related to the nonlinear Hilbert adjoint descriptions of [6].

Consider $f(x) = A(x)x$ and $h(x) = C(x)x$ as before, where $A(x)$, $C(x)$ are an $n \times n$ and $p \times n$ matrix with elements depending smoothly on $x$, with $A(0) = \frac{\partial A}{\partial x}(0)$ and $C(0) = \frac{\partial C}{\partial x}(0)$. Assume that we have done this factorization such that the parameter varying system $(A(x), g(x), C(x))$ is minimal.

Now consider (1) in combination with the following dual system.

$$ \dot{p} = A(x)^T p + C(x)^T u_d $$

$$ y_d = g(x)^T p \quad (16) $$

and consider the Legendre transform of $K^+(x)$ as follows:

$$ \hat{K}^+(p) = -K^+(x) + p^T x $$

then we can state the following lemma.

**Lemma 3.1:** $\hat{K}^+(p)$ fulfills the Hamilton-Jacobi-Bellman equation (11) for the past energy function of system (16).

**Proof:** The result is straightforwardly obtained by considering equation (9) for system (1) and equation (11) for system (16) with $p = \frac{\partial K^+}{\partial x}(x)$. \hfill \square

**Remark 3.2:** Note that (16) is linear in $p$. \hfill \square

The dual system of the NLCF (14) is given by

$$ \dot{p} = \left( A(x)^T - C(x)^T C(x) (M(x))^{-1} \right) p $$

$$ + C(x)^T \tilde{w}_d $$

$$ z_d = \left( g(x)^T C(x) (M(x))^{-1} \right) p + \left( 0 - I \right) \tilde{w}_d $$

where $x$ is a solution of (14).

If we consider the controllability function $\tilde{L}_c(x)$ of (14), and its Legendre transform

$$ \tilde{L}_c(p) = -\tilde{L}_c(x) + p^T x, $$

then the corresponding dual coordinates are given by $p = \frac{\partial L_c}{\partial x}(x) = \frac{\partial L_c}{\partial x}(x) = M(x) x$, and thus $x = M(x)^{-1} p$.

**Lemma 3.3:** The Legendre transform of $\tilde{L}_c(x)$, $\tilde{L}_c(p)$, fulfills the Hamilton-Jacobi-Bellman equation for the controllability function of system (17). Furthermore, $\tilde{L}_c(p)$ is the observability function of system (17).

**Proof:** This follows immediately by considering the equations. \hfill \square

Now, by fixing $p = M(x) x = \frac{\partial K^+}{\partial x}(x)$, we should consider system (16) as a linear parameter varying system. Keeping this in mind, we are now able to establish the nonlinear counterpart of (15), i.e.,

**Theorem 3.4:** With $p = M(x) x$ and with $\tilde{L}_c(x, p)$ the observability function of (17), $\tilde{L}_o(x, p)$ the controllability function of (17) and $\hat{K}^+ = (x, p)$ the past energy function of (16), we have that $\tilde{L}_c(x, p) = \tilde{L}_o(x, p) - \hat{K}^+ (x, p)$.

**Proof:** Consider the corresponding equations, i.e.,

$$ \frac{\partial \tilde{L}_c}{\partial x}(x, p) \left( A(x)^T - C(x)^T C(x) (M(x))^{-1} \right) p $$

$$ + \frac{1}{2} p^T g(x) g(x)^T p + \frac{1}{2} p^T M(x)^{-1} C(x)^T C(x) M(x)^{-1} p $$

$$ = \frac{\partial \tilde{L}_c}{\partial x}(x, p) f(x) + \frac{\partial \tilde{L}_c}{\partial x}(x, p) g(x) u, $$

$$ \frac{\partial \tilde{L}_o}{\partial x}(x, p) \left( A(x)^T - C(x)^T C(x) (M(x))^{-1} \right) p $$

$$ + \frac{1}{2} \frac{\partial \tilde{L}_o}{\partial x}(x, p) C(x)^T C(x) \frac{\partial \tilde{L}_o}{\partial p}(x, p) $$

$$ = \frac{\partial \tilde{L}_o}{\partial x}(x, p) f(x) + \frac{\partial \tilde{L}_o}{\partial x}(x, p) g(x) u.$$
\[
\frac{\partial \hat{K}^+}{\partial p}(x,p)A(x)^T p + \frac{1}{2} \frac{\partial \hat{K}^+}{\partial p}(x,p)C(x)^T C(x) \frac{\partial \hat{K}^+}{\partial p}(x,p) \quad - \quad \frac{1}{2} p^T g(x)g(x)^T p \\
= \frac{\partial \hat{K}^+}{\partial x}(x,p)f(x) + \frac{\partial \hat{K}^+}{\partial x}(x,p)g(x)u
\]

Subtracting the equation for \( \hat{K}^+(x,p) \) from the equation for \( \hat{L}_c(x,p) \), where \( p = M(x) \), and thus \( x = M(x)^{-1} p = \frac{\partial L_c}{\partial p}(x,p) \), the relation is established.

**Remark 3.5:** The solutions of the equations used in the proof of Theorem 3.4 being the stabilizing solutions is equivalent to the solutions being positive definite, e.g., [16].

**Remark 3.6:** For a linear system Theorem 3.4 results in (15).

Due to linearity in \( p \) we can now easily write \( \frac{\partial \hat{K}^+}{\partial p}(p) = Y(x)p \) and \( \frac{\partial \hat{K}^+}{\partial p}(p) = W(x)p \), where \( Y(x) \) and \( W(x) \) are positive definite matrices on \( x \in U \).

**Corollary 3.7:** The linearity in \( p \) yields
\[
\frac{\partial \hat{L}_o}{\partial x}(x) = W(x) \frac{\partial \hat{L}_o}{\partial x}(x)
\]

**Proof:** Since
\[
x = \frac{\partial \hat{L}_o}{\partial x}(p) = \frac{\partial \hat{L}_o}{\partial p}(p) - \frac{\partial \hat{K}^+}{\partial p}(p)
\]
and \( p = \frac{\partial L_c}{\partial x}(x) \), we obtain the result.

**Remark 3.8:** For linear systems we have that \( Y(x) = X^{-1} \) and \( W(x) = P^{-1} \), and thus Corollary 3.7 yields
\[
x = X^{-1} Z x - P^{-1} Z x,
\]
which results in (15).

**IV. SINGULAR VALUE FUNCTIONS**

The Hankel singular values for the linear NRCF and NLCF (image and kernel representations, respectively) are the same. This is easily established by checking Table I. For nonlinear systems, we now establish a relation between the singular value functions by checking Table II, where the NRCF relations have been established in [16].

In [5], [4] we have developed a unique balancing transformation for nonlinear systems. Now consider the NLCF system (14), and its corresponding controllability and observability operators \( C \) and \( O \) (see e.g., [5]). As in the linear case, the Hankel operator \( H \) of the system (14) is given by the composition of the observability and controllability operators \( H = O \circ C \).

Now consider the solution pair \( \lambda \in \mathbb{R} \) and \( v \in L^*_2 \) of
\[
(dH(v))^\tau \circ H(v) = \lambda v.
\]
This structure is called the differential singular value structure of the Hankel operator. Then, [5], there exist \( n \) independent solution curves in the form
\[
\lambda = \lambda_i(s), \quad v = v_i(s), \quad i = 1, 2, \ldots, n, \quad s \in \mathbb{R},
\]
\[
||v||_{L^2} = |s|
\]
which are parametrized by \( s \). The related input-output ratio of the Hankel operator defined by
\[
\rho_i(s) := \frac{||H(v_i(s))||_{L^2}}{||v_i(s)||_{L^2}}
\]
\[
\min[\rho_i(s), \rho_i(-s)] > \max[\rho_{i+1}(s), \rho_{i+1}(-s)]
\]
called axis singular value functions. They have a closer relation to the Hankel operator than the original singular value functions of [13] because it satisfies
\[
||\Sigma||_H = \sup_{s \in \mathbb{R}} \rho_1(s)
\]
in a similar way as in the linear case. Also, the \( \rho_i \)'s are uniquely determined since they are defined only using the input-output property of the Hankel operator.

**Assumption A1** Consider the NLCF given in (14) and suppose that there exist \( n \) neighborhoods of the origin where the operators \( O, C, C' \) exist and are smooth. Here \( O \) denotes the observability operator of system (14), \( C \) denotes the controllability operator of system (14) and \( C' \) denotes the pseudo-inverse of \( C \).

**Assumption A2** Suppose that the Hankel singular values of the Jacobian linearization of the system (14) are nonzero and distinct.

**Theorem 4.1:** [5] Consider the system (14). Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood \( U \) of 0 and a coordinate transformation \( x = \Phi(z) \) on \( U \) converting the system an input-normal/output-diagonal form satisfying the following properties:
\[
z_i = 0 \iff \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0 \iff \frac{\partial L_o(\Phi(z))}{\partial z_i} = 0
\]
holds for all $i \in \{1, 2, \ldots, n\}$ on $U$. $
abla$

Given the above assumptions, we now have the following useful model reduction result which follows straightforwardly from [4].

**Theorem 4.2:** Consider the state-space realization of the NLCF given in (14). Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood $U$ of the origin and a coordinate transformation $x = \Phi(z)$ on $U$ converting the system into the following form

$$
L_{c}(\Phi(z)) = \frac{1}{2} z^T \zeta
$$

$$
L_{o}(\Phi(z)) = \frac{1}{2} \sum_{i=1}^{n} (z_i \rho_i(z_i))^2.
$$

We can even go one step further and obtain a fully balanced representation, i.e.,

**Theorem 4.3:** Consider the system (14). Suppose that Assumptions A1 and A2 hold. Then there exist a neighborhood $U$ of the origin and a coordinate transformation $x = \Phi(z)$ on $U$ converting the system into the following form

$$
L_{c}(\Phi(z)) = \frac{1}{2} \sum_{i=1}^{n} \frac{z_i^2}{\sigma_i(z_i)}
$$

$$
L_{o}(\Phi(z)) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \sigma_i(z_i).
$$

In particular, if $U = \mathbb{R}^n$, then

$$
\|\Sigma\|_H = \sup_{z_i \in \mathbb{R}} \sigma_i(z_i),
$$

with $\Sigma$ the input-output system given by (14) $
abla$

The above results make it possible to apply model reduction based on the NLCF of a nonlinear system. A similar procedure can be followed for model reduction based on the NRCF of a system.

**V. CONCLUSIONS AND OPEN ISSUES**

In this paper we have modified [14] in order to include the linear parameter varying interpretation that should be used when applying the duality notion of [14], [6].

Future work includes the use of the relations of Table II to study the relation between the singular value functions of the NLCF and NRCE. Clearly, the non-uniqueness of the dual system due to the factorizations of $f(x)$, $h(x)$, and $\frac{\partial K}{\partial x}(x)$, results in non-uniqueness for the $Y(x)$ and $W(x)$ in Corollary 3.7. This non-uniqueness and its exploitation is also a topic of future research. Finally, the differential singular value structure of the Hankel operator is closely related to the partials of $L_o$ and $L_c$, which also appear in the result of Corollary 3.7. It can be expected that that relation is crucial for the singular value analysis of the coprime factorizations.

---

**ACKNOWLEDGEMENTS**

The author gratefully acknowledges the discussion with Hakan Koroglu about linear parameter varying systems.

**REFERENCES**


